SQUARE FUNCTION AND HEAT FLOW ESTIMATES ON DOMAINS

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Abstract. The first purpose of this note is to provide a proof of the usual square function estimate on $L^p(\Omega)$. It turns out to follow directly from a generic Mikhlin multiplier theorem obtained by Alexopoulos, and we provide a sketch of its proof in the Appendix for the reader's convenience. We also relate such bounds to a weaker version of the square function estimate which is enough in most instances involving dispersive PDEs and relies on Gaussian bounds on the heat kernel (such bounds are the key to Alexopoulos'result as well). Moreover, we obtain several useful $L^p(\Omega; H)$ bounds for (the derivatives of) the heat flow with values in a given Hilbert space H.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Let Δ_D denote the Laplace operator on Ω with Dirichlet boundary conditions, acting on $L^2(\Omega)$, with domain $H^2(\Omega) \cap H^1_0(\Omega)$.

The first result reads as follows:

Theorem 1.1. *Let $f \in C^\infty(\Omega)$ and $\Psi \in C_0^\infty(\mathbb{R}_+^*)$ such that*

$$\sum_{j\in\mathbb{Z}} \Psi(2^{-2j}\lambda) = 1, \quad \lambda \in \mathbb{R}_+. \tag{1.1}$$

Then for all $p \in (1,\infty)$ we have

$$\|f\|_{L^p(\Omega)} \approx C_p \Big\| \Big(\sum_{j\in\mathbb{Z}} |\Psi(-2^{-2j}\Delta_D) f|^2 \Big)^{1/2} \Big\|_{L^p(\Omega)}, \tag{1.2}$$

where the operator $\Psi(-2^{-2j}\Delta_D)$ is defined by (3.20) *below.*

Readers who are familiar with functional spaces' theory will have recognized the equivalence $\dot{F}^{0,2}_p \approx L^p$, where the Triebel-Lizorkin space is defined using the right hand-side of (1.2) as a norm. In other words, $L^p(\Omega)$ and the Triebel-Lizorkin space $\dot{F}^{0,2}_p(\Omega)$ coïncide. Such an equivalence (and much more !) is proven in [29, 30, 31], though one has to reconstruct it from several different sections (functional spaces are defined differently, only the inhomogeneous ones are treated, among other things). As such, the casual user with mostly a PDE background might find it difficult to reconstruct the argument for his own sake without digesting the whole theory. It turns out that the proof of (1.2) follows directly from the classical argument (in $\mathbb{R}^n$) involving Rademacher functions, provided that an appropriate Mikhlin-Hörmander multiplier theorem is available. We will provide details below.

A weaker version of Theorem 1.1 is often used in the context of dispersive PDEs:

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Theorem 1.2. *Let $f \in C^\infty(\Omega)$, then for all $p \in [2, \infty)$ we have*

$$\|f\|_{L^p(\Omega)} \leq C_p \Big(\sum_{j\in\mathbb{Z}} \|\Psi(-2^{-2j}\Delta_D) f\|^2_{L^p(\Omega)} \Big)^{1/2}. \tag{1.3}$$

The second part of the present note aims at giving a self-contained proof of (1.3) (or its inhomogeneous version), with "acceptable" black boxes, namely complex interpolation, spectral calculus and known bounds on the heat flow on domains. Our strategy to prove Theorem 1.2 is indeed to reduce matters to an estimate involving the heat flow, by proving almost orthogonality between spectral projectors and heat flow localization; this only requires parabolic estimates in $L^p(\Omega)$, together with a little help from spectral calculus.

We now state several estimates involving the heat flow, which will be proved by direct arguments. It should be noted that for nonlinear applications, it is quite convenient to have bounds on derivatives of spectral multipliers, and such bounds do not follow immediately from the multiplier theorem from [1, 2]. We consider the linear heat equation on Ω with Dirichlet boundary conditions and initial data f

$$\partial_t u - \Delta_D u = 0, \text{ on } \Omega \times \mathbb{R}_+; \quad u|_{t=0} = f \in C^\infty(\Omega); \quad u|_{\partial\Omega} = 0. \tag{1.4}$$

We denote the solution $u(t, x) = S(t)f(x)$, where we set $S(t) = e^{t\Delta_D}$. For the sake of simplicity Δ_D has constant coefficients, but the same method applies in the case when the coefficients belong to a bounded set of C^∞ and the principal part is uniformly elliptic (one may lower the regularity requirements on both the coefficients and the boundary, and a nice feature of the proofs which follow is that counting derivatives is relatively straightforward).

Let us define two operators which are suitable heat flow versions of $\Psi(-2^{-2j}\Delta_D)$:

$$Q_t = \sqrt{t}\nabla S(t) \quad \text{and} \quad \mathbf{Q}_t \overset{\text{def}}{=} t\partial_t S(t). \tag{1.5}$$

Theorem 1.3. *Let $2 \leq p < +\infty$, then we have*

$$\|f\|_{L^p(\Omega)} \leq c_p \Big\| \Big(\int_0^{+\infty} |Q_t f|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^p(\Omega)} \lesssim \Big(\int_0^\infty \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \Big)^{1/2}, \tag{1.6}$$

as well as

$$\|f\|_{L^p(\Omega)} \leq c_p \left(\|S(1)f\|_{L^p(\Omega)} + \Big\| \Big(\int_0^1 |Q_t f|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^p(\Omega)} \right), \tag{1.7}$$

which implies

$$\|f\|_{L^p(\Omega)} \leq C_p \left(\|S(1)f\|_{L^p(\Omega)} + \Big(\int_0^1 \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \Big)^{1/2} \right). \tag{1.8}$$

Moreover, Q_t may be replaced by $\mathbf{Q}_t$ in the last two inequalities.

In fact, the last two estimates are inhomogeneous versions of Theorems 1.1 and 1.2 ; when Q_t is replaced by $\mathbf{Q}_t$, both estimates hold, with an equivalence rather than inequality, and also in the homogeneous setting, without the $S(1)f$ term on the righthand side and with an integral over $(0, +\infty)$; but they require more involved arguments (essentially using both Theorem 1.1 and equivalence of norms for Triebel or Besov spaces on domains).

Notice that there is no difficulty to define $Q_t f$ or $\mathbf{Q}_t f$ as distributional derivatives for $f \in L^p(\Omega)$, while simply defining $\Psi(-2^{-2j}\Delta_D)$ on $L^p(\Omega)$ is already a non trivial task. The purpose of the next Proposition is to prove that both operators are in fact bounded on $L^p(\Omega)$.

Proposition 1.4. Let $1 < p < +\infty$. The operator Q_t is bounded on $L^p(\Omega)$, uniformly for $0 \leq t \leq 1$, while the operator $\mathbf{Q}_t$ is bounded on $L^p(\Omega)$, uniformly in $0 \leq t < +\infty$. Moreover the operator $\mathbf{Q}_t$ s bounded on $L^1(\Omega)$ and $L^\infty(\Omega)$.

For practical applications, one may need a vector valued version of Proposition 1.4. Let H be any separable Hilbert space and $L^p(\Omega; H)$ be the Hilbert valued Lebesgue space. Then we have

Proposition 1.5. Let $1 < p < +\infty$. The operator Q_t is bounded on $L^p(\Omega; H)$, uniformly for $0 \leq t \leq 1$. Moreover, $\mathbf{Q}_t$ is bounded on $L^p(\Omega; H)$, uniformly for $0 \leq t < +\infty$ as well as on $L^1(\Omega; H)$ and $L^\infty(\Omega; H)$.

Remark 1.6. The typical setting would be to consider the solution u to the heat equation with initial data $f(x, \theta) \in L^2_\theta = H$. Notice that the Hilbert valued bound does not follow from the previous scalar bound; however the argument will essentially be the same, replacing $|\cdot|$ norms by Hilbert norms.

Remark 1.7. A straightforward consequence of Propositions 1.5 and 1.4 is that the Riesz transforms $\partial_j(-\Delta_D)^{-\frac{1}{2}}$ are continuous on Besov spaces defined by the RHS of (1.8); when Q_t is replaced by $\mathbf{Q}_t$ in the norm, these spaces (or rather, their homogeneous version) are equivalent to the ones defined by the RHS of (1.3), see Remark 3.4 later on.

2. From a Mikhlin multiplier theorem to the square function

The following "Fourier multiplier" theorem is obtained in [1] under very weak hypothesis on the underlying manifold. Since [1] is not widely available, we provide in the Appendix a sketch of its proof, mainly following the discussion in [2], which is a specific application to Markov chains. There are other references providing equivalent statements: e.g. [27] for a version closer to the sharp Hörmander's multiplier theorem, under suitable additional hypothesis, all of which are verified on domains. Since completion of the original version of the present work, we have learned of more recent results: [19], Theorem 7.9, provides an independent proof, with different arguments related to a systematic use of wavelets to analyze fonctional spaces. The setting of [19] is general enough to cover domains. Finally, it should be mentioned that [20] also provides a proof of such a theorem, essentially following Alexopoulos' arguments like we do here.

For $m \in L^\infty(\mathbb{R}^+)$, one usually defines the operator $m(-\Delta_D)$ on $L^2(\Omega)$ through the spectral measure dE_λ:

$$m(-\Delta_D) = \int_0^{+\infty} m(\lambda)\, dE_\lambda, \tag{2.1}$$

and $m(-\Delta_D)$ is bounded on L^2.

Theorem 2.1 ([1],[19]). *Let $m \in C^N(\mathbb{R}^+)$, $N \in \mathbb{N}$ and $N \geq n/2 + 1$, such that*

$$\sup_{\lambda, k \leq N} \lambda^k |\partial^k_\lambda m(\lambda)| < +\infty. \tag{2.2}$$

Then the operator defined by (2.1) *extends to a continuous operator on $L^p(\Omega)$, and sends $L^1(\Omega)$ to weak $L^1(\Omega)$.*

In order to use the argument of [1], we need the Gaussian upper bound on the heat kernel, which is provided in our case by [9]. Once we have Theorem 2.1, all we need to do to prove Theorem 1.1 is to follow Stein's classical proof from [26][1], and we recall it briefly for the convenience of the reader. Let us introduce the Rademacher functions, which are defined as follows:

- the function $r_0(t)$ is defined by $r_0(t) = 1$ on $[0, 1/2]$ and $r_0(t) = -1$ on $(1/2, 1)$, and then extended to $\mathbb{R}$ by periodicity;
- for $m \in \mathbb{N} \setminus \{0\}$, $r_m(t) = r_0(2^m t)$.

Their importance is outlined by the following inequalities (see the Appendix in [26]),

$$c_p \int_0^1 |\sum_m a_m r_m(t)|^p \, dt \leq (\sum_m |a_m|^2)^{\frac{p}{2}} \leq C_p \int_0^1 |\sum_m a_m r_m(t)|^p \, dt \,. \tag{2.3}$$

Now, define

$$m^{\pm}(t, \lambda) = \sum_{j=0}^{+\infty} r_j(t) \Psi_{\pm j}(\lambda),$$

where $\Psi_j = \Psi(2^{-j}\lambda)$ and Ψ was defined in the introduction. A straightforward computation proves that the bound (2.2) holds for $m^{\pm}(t, \lambda)$; such bound is of course dependent of our choice of Ψ (but one may change Ψ within a large class of smooth, compactly supported functions without affecting the bound). Therefore, applying Theorem 2.1, we have, uniformly in $t \in (0, 1)$,

$$\|m^{\pm}(t, -\Delta_D) f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)};$$

On the other hand, using (2.3) with $a_j = \Psi_{\pm j}(\Delta_D) f$, we get

$$\int_0^1 |\sum_j r_j(t) \Psi_{\pm j}(\Delta_D) f|^p \, dt \approx (\sum_j |\Psi_{\pm j}(\Delta_D) f|^2)^{\frac{p}{2}} \,, \tag{2.4}$$

and, taking $L^1(\Omega)$ norms,

$$\int_0^1 \| \sum_j r_j(t) \Psi_{\pm j}(\Delta_D) f\|^p_{L^p(\Omega)} \, dt \approx \|(\sum_j |\Psi_{\pm j}(\Delta_D) f|^2)^{\frac{1}{2}}\|_{L^p(\Omega)} \,, \tag{2.5}$$

and therefore, we obtain one side of (1.2) in Theorem 1.1

$$\left(\int_0^1 \|m^{\pm}(t, -\Delta_D) f\|^p_{L^p(\Omega)} dt\right)^{1/p} \approx \|(\sum_{j=0}^{\pm\infty} |\Psi(-2^{-2j}\Delta_D) f|^2)^{\frac{1}{2}}\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.$$

This proves one side of the equivalence in (1.2): the other side follows from duality, once we see the above estimate as an estimate from $L^p(\Omega)$ to $L^p(\Omega; l^2)$, which maps f to $(\Psi(-2^{-2j}\Delta_D) f)_{j \in \mathbb{Z}}$.

Remark 2.2. If one wishes to "ignore" Theorem 2.1, one may alternatively use the Dynkin-Helffer-Sjöstrand formula as in [7] to define operators $\Psi(-\Delta_D)$, and both definitions are known to coincide on $L^2(\Omega)$. However, the Dynkin-Helffer-Sjöstrand formula seems to be restricted to defining $m(-\Delta_D)$ for functions m which exhibit slightly more decay than required in Theorem 2.1, see the Appendix.

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[1] we thank Hart Smith for bringing this to our attention

3. Heat flow estimates

The results and proofs contained in this section should be thought as not relying on Theorem 2.1, but rather on simple integration by parts arguments.

In order to prove Proposition 1.3 we need the following lemma.

Lemma 3.1. *For all* $1 \leq p \leq +\infty$*, we have*

$$\|S(t)f\|_{L^p(\Omega)} \to_{t\to\infty} 0, \tag{3.1}$$

$$\sup_{t\geq 0} \|S(t)f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} . \tag{3.2}$$

Moreover,

$$\| \sup_{t\geq 0} |S(t)f| \|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} . \tag{3.3}$$

Proof: The estimate (3.2) clearly follows from (3.3), which in turn is a direct consequence of the Gaussian nature of the Dirichlet heat kernel, see [9]. The same Gaussian estimate implies (3.1). However we do not need such a strong fact to prove (3.2), which will follow from the next computation as well (see (3.4)) when $1 < p < +\infty$. Estimate (3.1) can also be obtained through elementary arguments. We defer such a proof to the end of the section.

3.1. **Proof of Theorem 1.3.** If $p = 2$ the proof is nothing more than the energy inequality, combined with (3.1). In fact, for $p = 2$, we have equality in (1.6) with $C_2 = 2$. We now take $p = 2m$ where $m \in \mathbb{N}$ and $m \geq 2$. Multiplying equation (1.4) by $\bar{u}|u|^{p-1}$ and taking the integral over Ω and $[0,T]$, $T > 0$ yields, taking advantage of the Dirichlet boundary condition,

$$\frac{1}{p}\int_0^T \partial_t \|u\|^p_{L^p(\Omega)} dt + \int_0^T \int_\Omega |\nabla u|^2 |u|^{p-2} dxdt + \\ + \frac{(p-2)}{2} \int_0^T \int_\Omega |\nabla(|u|^2)|^2 |u|^{p-4} dxdt = 0, \tag{3.4}$$

from which we can estimate either $\|u\|^p_{L^p(\Omega)}(T) \leq \|f\|^p_{L^p(\Omega)}$ (which is (3.2)) or

$$\|f\|^p_{L^p(\Omega)} \leq \|u\|^p_{L^p(\Omega)}(T) + p(p/2+1) \int_0^T \int_\Omega |\nabla u|^2 |u|^{p-2} dxdt.$$

Letting T go to infinity and using (3.1) from Lemma 3.1 and Hölder inequality we find

$$\|f\|^p_{L^p(\Omega)} \leq p(p-1) \Big(\int_\Omega \Big(\int_0^\infty |\nabla u|^2 \, dt \Big)^{\frac{p}{2}} dx \Big)^{\frac{2}{p}} \Big(\int_\Omega (\sup_t |u|^{p-2})^{\frac{p}{p-2}} dx \Big)^{\frac{p-2}{p}} .$$

The proof follows using again Lemma 3.1, as

$$\|f\|^p_{L^p(\Omega)} \leq C_p \| \Big(\int_0^\infty |\nabla u|^2 dt \Big)^{\frac{1}{2}} \|_{L^p(\Omega)} \left(\| \sup_{t\geq 0} |u| \|_{L^p(\Omega)} \right)^{p-2} .$$

We are therefore done with proving (1.6); note that we may prove the weaker part (e.g., the inequality without the middle term) without assuming the maximal in time bound, by reversing the order of integration in our argument. This would keep the argument for heat square functions essentially self-contained, without any need for Gaussian bounds on the heat kernel. A simple modification of the previous argument yields the inhomogeneous version (1.7).

Remark 3.2. We do not claim novelty here: our argument follows closely (a dual version of) the proof of a classical square function bound for the Poisson kernel in the whole space, see [26].

This concludes the proof of Theorem 1.3 except for the replacement of Q_t by $\mathbf{Q}_t$ in the Besov norms on the righthand side of (1.8); we defer this to the end of the next subsection.

Notice that, at this point, we proved Theorem 1.2, but with the Ψ operator replaced by the gradient heat kernel and the discrete parameter 2^{-2j} by the continuous parameter t. The rest of this section is devoted to proving the equivalence between the Besov norms which are defined by the heat kernel or the spectral localization.

Lemma 3.3. *Let $1 \leq p \leq +\infty$. We have the following equivalence between dyadic and continuous versions of the Besov norm:*

$$\frac{3}{4}\sum_{k\in\mathbb{Z}} \|\mathbf{Q}_{2^{-2k}} f\|^2_{L^p(\Omega)} \leq \int_0^\infty \|\mathbf{Q}_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \leq 3\sum_{k\in\mathbb{Z}} \|\mathbf{Q}_{2^{-2k}} f\|^2_{L^p(\Omega)}.$$

The same equivalence holds for the inhomogeneous version of the norms.

This follows at once from factoring the semi-group: for $2^{-2j} \leq t \leq 2^{-2(j-1)}$, write $S(t) = S(t-2^{-2j})S(2^{-2j})$ and use (3.2). We now turn to the direct proof of the inhomogeneous version of Theorem 1.2 from its heat flow version (1.8). Let $\Psi \in C_0^\infty(\mathbb{R}^*)$ satisfying (1.1) and denote $\Delta_j f \overset{\text{def}}{=} \Psi(2^{-2j}\Delta_D) f$, where $\Psi(2^{-2j}\Delta_D)f$ is given by the Dynkin-Helffer-Sjöstrand formula (see [7]). From Theorem 1.3 and Lemma 3.3 we have

$$\|f\|_{L^p(\Omega)} \lesssim \|S(1)f\|_{L^p(\Omega)} + \Big(\sum_{k\in\mathbb{N}} \|\mathbf{Q}_{2^{-2k}} f\|^2_{L^p(\Omega)}\Big)^{1/2} \tag{3.5}$$

and we will show that (3.5) implies the inhomogeneous version of (1.2): it suffices to prove the following almost orthogonality property between localization operators Δ_j and $\mathbf{Q}_{2^{-2k}}$:

$$\forall k, j \in \mathbb{Z}, \qquad \|\mathbf{Q}_{2^{-2k}}\Delta_j f\|_{L^p(\Omega)} \lesssim 2^{-2|j-k|}\|\Delta_j f\|_{L^p(\Omega)}. \tag{3.6}$$

Then, from $(2^{-2|j-k|})_k \in l^1$ and $(\|\Delta_j f\|_{L^p(\Omega)})_j \in l^2$ we estimate

$$\sum_{k\in\mathbb{Z}} \|\mathbf{Q}_{2^{-2k}} f\|^2_{L^p(\Omega)} = \sum_{k\in\mathbb{Z}} \|\sum_{j\in\mathbb{Z}} \mathbf{Q}_{2^{-2k}}\Delta_j f\|^2_{L^p(\Omega)} \tag{3.7}$$

as an $l^1 * l^2$ convolution to conclude in one direction (and reverse roles of $\mathbf{Q}_k$ and Δ_j in the other). This proves equivalence of homogeneous Besov norms, and an easy adaptation of the argument provides the inhomogeneous version.

It remains to show (3.6):

- for $k < j$ we write

$$\mathbf{Q}_{2^{-2k}}\Delta_j f = 2^{-2(j-k)}2^{-2k}\Delta_D S(2^{-(2k+1)}\Delta_D)$$
$$2^{-2k}\Delta_D S(2^{-(2k+1)}\Delta_D)\breve{\Psi}(-2^{-2j}\Delta_D)\Psi(-2^{-2j}\Delta_D)f,$$

  where we set $\breve{\Psi}(\lambda) \overset{\text{def}}{=} \frac{1}{\lambda}\tilde{\Psi}(\lambda)$, and $\tilde{\Psi} \in C_0^\infty$, $\tilde{\Psi} = 1$ on suppΨ. By Lemma 1.4, the operator $\mathbf{Q}_{2^{-(2k+1)}} = 2^{-(2k+1)}\Delta_D S(2^{-(2k+1)})$ is bounded on $L^p(\Omega)$ and we obtain (3.6) using Corollary 3.8 for $\breve{\Psi}$.

- for $k \geq j$ we set $\Psi_1(\xi) = \tilde{\Psi}(\xi)\exp(\xi)$, $\Psi_2(\xi) = \Psi(\xi)$, and we use again Lemma 3.11 to write (slightly abusing the notation as $2^{-2k} - 2^{-2j} < 0$, e.g. we can apply $S(2^{-2k} - 2^{-2j})$ only to a function which is spectrally localized away from the zerof frequency, like $\Delta_j f$)

$$S(2^{-2k} - 2^{-2j})\Delta_j f = S(2^{-2k})\Psi_1(-2^{-2j}\Delta_D)\Psi_2(-2^{-2j}\Delta_D)f. \tag{3.8}$$

Then

$$\mathbf{Q}_{2^{-2k}}\Delta_j f = 2^{-2(k-j)}\Big(2^{-j}\Delta_D S(2^{-2j})\Big)\Big(S(2^{-2k} - 2^{-2j})\Delta_j f\Big),$$

and using again Lemma 1.4 we see that the operator $2^{-j}\Delta_D S(2^{-2j})$ is bounded while the remaining operator (3.8) is bounded by Corollary 3.8. This ends the proof.

Remark 3.4. One may prove a similar bound with $\mathbf{Q}_{2^{-2k}}$ and Δ_j reversed, either directly or by duality. Hence Besov norms based on Δ_j or $\mathbf{Q}_{2^{-2k}}$ are equivalent.

3.2. **Proof of Proposition 1.4.** For $\mathbf{Q}_t$, boundedness on all L^p spaces, including $p \in \{1, +\infty\}$, follows once again from a Gaussian upper bound on $\partial_t S(t)$ (see [10] or [12]). However the subsequent Gaussian bound on the gradient $\nabla_x S(t)$ in [10] is a direct consequence of the Li-Yau inequality, which holds only inside convex domains. We were unable to find a reference which would provide the desired bound for Q_t in the context of the exterior domain, but there are several available references dealing with the Stokes flow in an exterior domain, in the context of Navier-Stokes equations: see e.g. [22, 18]. Their arguments may easily adapted to the simpler case of the Dirichlet heat flow. We note that the restriction to $t \leq 1$ appears there, in connection with values of p which are greater than the space dimension. In the context of the exterior of a ball, one may provide counterexamples in the range $p > n$ to the continuity of Q_t for $t > 1$, see [20].

We do however provide an elementary proof for continuity of Q_t in the range $0 \leq t \leq 1$; we expect one could adapt the following argument to greater t, at least for values $1 < p < 2$, but elected not to do so here.

Set $v(x,t) = (v_1, .., v_n)(x,t) := Q_t f = t^{1/2}\nabla u(x,t)$ and assume without loss of generality that v_j are real: we multiply the equation satisfied by v by $v|v|^{p-2}$, where $|v|^2 = \sum_{j=1}^n v_j^2$, and integrate over Ω,

$$\partial_t\left(\frac{1}{p}\|v\|^p_{L^p(\Omega)}\right) - \sum_{j=1}^n \int_{\partial\Omega}((\overrightarrow{\nu}\cdot\nabla)v_j)\cdot v_j|v|^{p-2}d\sigma + \\ + \int_\Omega |\nabla v|^2|v|^{p-2}\,dx + \frac{(p-2)}{2}\int_\Omega \nabla(|v|^2)|v|^{p-4}\,dx = \frac{1}{2t}\|v\|^p_{L^p(\Omega)}, \tag{3.9}$$

where $\overrightarrow{\nu}$ is the outgoing unit normal vector to $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$. We claim that the second term in the left hand side is a lower order term: in fact we write

$$\sum_{j=1}^n \int_{\partial\Omega}(\overrightarrow{\nu}\cdot\nabla v_j)\cdot v_j|v|^{p-2}d\sigma = \\ = \frac{t^{p/2}}{2}\int_{\partial\Omega}\partial_\nu(|\partial_\nu u|^2 + |\nabla_{tang}u|^2)(|\partial_\nu u|^2 + |\nabla_{tang}u|^2)^{(p-2)/2}d\sigma, \tag{3.10}$$

and from $u|_{\partial\Omega} = 0$ the time and tangential derivative $(\partial_t, \nabla_{\text{tang}})u|_{\partial\Omega}$ vanishes; furthermore, using the equation in geodesic normal coordinates, $\partial_\nu^2 u = k(x)\partial_\nu u$ on $\partial\Omega$, hence

$$\sum_{j=1}^{n}\int_{\partial\Omega}(\overrightarrow{\nu}\cdot\nabla v_j)\cdot v_j|v|^{p-2}d\sigma = \frac{t^{p/2}}{2}\int_{\partial\Omega} k(x)|\partial_\nu u|^p\,d\sigma = \int_{\partial\Omega} k(x)|v|^p\,d\sigma\,. \tag{3.11}$$

Now, provided ∂_Ω is bounded, this trace term is controled by $\int_\Omega |\nabla(|v|^{p/2})|^2 + |v|^p\,dx$: in turn, this can be controled by the third term in (3.9) and the righthanside, provided $t<1$.

Furthermore, provided $1<p<2$, we may multiply by $\|v\|^{2-p}_{L^p(\Omega)}$ and integrate over $[0,T]$, to get

$$\|v\|^2_{L^p(\Omega)}(T) \lesssim \int_0^T \|Q_t f\|^2_{L^p(\Omega)}\frac{dt}{t} \lesssim \|f\|_p^2,$$

where the last inequality is the dual of (1.6). Hence we are done with $1<p<2$.

Remark 3.5. We ignored the issue of v vanishing in the third and fourth terms in (3.9). This is easily fixed by replacing $|v|^{p-2}$ by $(\sqrt{\varepsilon+|v|^2})^{p-2}$ and proceeding with the exact same computation. Then let ε go to 0 after dropping the positive term on the left handside of (3.9).

Now let $p=2^m$ with $m\ge 1$: we proceed directly by integrating (3.9) over $[0,T]$, discarding the boundary term as we restrict to $T<1$, to get

$$\begin{aligned}\frac{1}{p}\|v\|^p_{L^p(\Omega)}(T) &+ \int_0^T\int_\Omega |\nabla v|^2|v|^{p-2}\,dxdt+ \\ &+\frac{(p-2)}{2}\int_0^T\int_\Omega |\nabla(|v|^2)|^2|v|^{p-4}\,dxdt \lesssim \int_0^T \frac{1}{2t}\|v\|^p_{L^p(\Omega)}\,dt.\end{aligned}\tag{3.12}$$

On the other hand (recall (3.4)),

$$\frac{1}{p}\|u\|^p_{L^p(\Omega)}(T) + (p-1)\int_0^T\int_\Omega |\nabla u|^2|u|^{p-2}\,dxdt = \frac{1}{p}\|f\|^p_{L^p(\Omega)}. \tag{3.13}$$

If $p=2$ the estimates are trivial since from (3.12), (3.13),

$$\frac{1}{2}\|v\|^2_{L^2(\Omega)}(T) \le \int_0^T \frac{1}{2t}\|v\|^2_{L^2(\Omega)}dt = \frac{1}{2}\int_0^T \|\nabla u\|^2_{L^2(\Omega)}\,dt \le \frac{1}{4}\|f\|^2_{L^2(\Omega)}.$$

Now, let $p\ge 4$; for convenience, denote by J the second integral in the left hand-side of (3.12) (notice that the third integral is bounded from above by J), hence

$$J = \int_0^T\int_\Omega |\nabla^2 u|^2|\nabla u|^{p-2}t^{\frac{p}{2}}\,dxdt = \int_0^T\int_\Omega (\sum_{i,j}|\partial^2_{i,j}u|^2)(\sum_j|\partial_j u|^2)^{\frac{(p-2)}{2}}t^{\frac{p}{2}}\,dxdt,$$

and set

$$I_k = \int_0^T\int_\Omega |\nabla u|^{2k}|u|^{p-2k}t^{k-1}\,dxdt \text{ where } 2\le 2k\le p. \tag{3.14}$$

For our purposes, it suffices to estimate the right hand-side of (3.12), which rewrites

$$\frac{1}{2}\int_0^T t^{\frac{p}{2}-1}\|\nabla u\|^p_{L^p(\Omega)}\,dt = \frac{1}{2}I_{\frac{p}{2}}. \tag{3.15}$$

Integrate by parts the inner (space) integral in I_k, the boundary term vanishes and collecting terms,

(3.16)
$$\int_\Omega \nabla u \nabla u |\nabla u|^{2(k-1)} |u|^{p-2k}\, dx \ \le\ \frac{(2k-1)}{(p-2k+1)} \int_\Omega |\nabla^2 u||\nabla u|^{2k-2}|u|^{p-2k+1} dx.$$

By Cauchy-Schwarz the integral in the right hand side of (3.16) is bounded by

$$\Big(\int_\Omega |\nabla^2 u|^2 |\nabla u|^{p-2} dx\Big)^{1/2} \Big(\int_\Omega |\nabla u|^{4k-4-(p-2)} |u|^{2p+2-4k} dx\Big)^{1/2},$$

therefore for $k \ge \frac{p}{4}+1$ we have

$$I_k \lesssim \frac{(2k-1)}{(p-2k+1)} J^{\frac12} I^{\frac12}_{2k-\frac{p}{2}-1}.$$

We aim at controlling I_m by $J^{1-\eta} I_1^\eta$, for some $\eta > 0$ which depends on m (notice that when $p = 4$, which is $m = 2$, we are already done, using $k = 2$!). Set $k = \frac{p}{2} - (2^j - 1)$ with $j \le m-2$,

$$I_{2^{m-1}-(2^j-1)} \le \frac{(2^m - (2^{j+1}-1))}{(2^{j+1}-1)} J^{\frac12} I^{\frac12}_{2^{m-1}-(2^{j+1}-1)},$$

and iterating $m-2$ times, we finally control $I_{\frac{p}{2}}$ by $J^{1-\eta} I_1^\eta$, which proves that Q_t is bounded on $L^p(\Omega)$.

We now proceed to obtain boundedness of $\mathbf{Q}_t$ on $L^p(\Omega)$ from the Q_t bound; this is worse than using the Gaussian properties of its kernel, as the constants blow up when $p \to 1, +\infty$. It is, however, quite simple. By duality $Q_t^\star$ is bounded on $L^p(\Omega)$, and

$$\mathbf{Q}_t = t\partial_t S(t) = tS(\frac{t}{2})\Delta S(\frac{t}{2}) = 2\sqrt{\frac{t}{2}} S(\frac{t}{2}) \nabla \cdot \sqrt{\frac{t}{2}} \nabla S(\frac{t}{2}) = 2 Q^\star_{\frac{t}{2}} Q_{\frac{t}{2}},$$

and we are done with Lemma 1.4.

From the previous decomposition, we also obtain

$$\|\mathbf{Q}_t f\|_{L^p(\Omega)} \lesssim \|Q_t f\|_{L^p(\Omega)},$$

which implies that any Besov norm defined with $\mathbf{Q}_t$ is bounded by the corresponding norm for Q_t. The reverse bound is true as well, though slightly more involved. We provide the proof for completeness, for homogeneous norm (which restrict the range of p, but the argument can be adapted to the inhomogeneous case as well). Consider $f, h \in C_0^\infty(\Omega)$ and $\langle f, g\rangle = \int_\Omega fg$. Then

$$\begin{aligned}
\langle f, g\rangle &= -\int_0^{+\infty} \langle \partial_t S(t) f, h\rangle\, dt = -2\int_0^{+\infty} \langle \partial_t S(t) f, S(t) h\rangle\, dt \\
&= 2\int_{t<s} \langle \partial_t S(t) f, \partial_s S(s) h\rangle\, dtds = 4\int_0^{+\infty} \langle \nabla S(s) \partial_t S(t) f, \nabla S(s) h\rangle\, dtds \\
&\lesssim \int_s \Big\| \int_0^s \nabla S(t) \partial_s S(s) f\, dt \Big\|_p \|\nabla S(s) h\|_{p'}\, ds \lesssim \int_s \sqrt{s} \|\partial_s S(s) f\|_p \|\nabla S(s) h\|_{p'}\, ds
\end{aligned}$$

where we used our bound on $\sqrt{t}\nabla S(t)$ at fixed t. Then

$$\langle f, h\rangle \lesssim \int_s \|\mathbf{Q}_s f\|_p \|Q_s h\|_{p'} \frac{ds}{s}$$

from which we are done by Hölder.

3.3. **Proof of Proposition 1.5.** As H is separable, we may reduce ourselves to the finite dimensional case; let us consider now $u = (u_l)_{l\in\{1,..,N\}}$ for $N \geq 2$, where each u_l solves (1.4) with Dirichlet condition and initial data f_l. Let H be the Hilbert space with norm $\|u\|_H^2 = \sum_l |u_l|^2$, we aim at obtaining bounds which are uniform in the dimension N.

For the sake of simplicity we consider only real valued u_l, and write

$$|u|^2 = \sum_{l=1}^N u_l^2,\ \ |\nabla u_l|^2 = \sum_{j=1}^n (\partial_j u_l)^2,\ \ |\nabla u|^2 = \sum_{j=1}^n \sum_{l=1}^N (\partial_j u_l)^2$$

Notice that n is the spatial dimension and is fixed through the argument: hence all constants may depend implicitely on n, while N is the dimension of H. For $p = 1, +\infty$, the boundedness of $\mathbf{Q}_t$ follows from the Gaussian character of the time derivative heat kernel, which is diagonal on H.

We proceed with Q_t. Multiplying the equation satisfied by u_l by $u_l|u|^{p-2}$, integrating over Ω and summing up we immediately get (3.4). We now proceed to obtain bounds for $v(x,t) = (v_l(x,t))_l$, where $v_l(x,t) = t^{1/2}\nabla u_l(x,t)$. Multiplying the equation satisfied by v_l by $v_l|v|^{p-2}$ where $|v|^2 = t|\nabla u|^2$, summing up over l and taking the integral over Ω yields, with $T \leq 1$,

$$(3.17)\quad \frac{1}{p}\|v\|^p_{L^p(\Omega)}(T) + \sum_{l=1}^N \sum_{j=1}^n \int_0^T \int_\Omega |\nabla(\partial_j u_l)|^2 |\nabla u|^{p-2}\,dxdt$$
$$+ \frac{(p-2)}{4}\int_0^T \int_\Omega |\nabla|\nabla u|^2|^2 |\nabla u|^{p-4} t^{p/2}\,dxdt \lesssim \int_0^T \|\nabla u\|^p_{L^p(\Omega)} t^{p/2-1}\,dt = \frac{1}{2} I_{\frac{p}{2}},$$

where $|\nabla(\partial_j u_l)|^2 = \sum_{i=1}^n (\partial^2_{i,j} u_l)^2$, $|\nabla u|^2 = \sum_{l=1}^N \sum_{j=1}^n (\partial_j u_l)^2$ and where we discarded the boundary term for the same reasons as in the scalar case (hence the inequality and the restriction $T \leq 1$.)

Denote the last two integrals in the left hand side by J_1, J_2. Like before, we perform integrations by parts in I_k defined in (3.14) to obtain

$$(3.18)\quad \int_\Omega |\nabla u|^{2k}|u|^{p-2k}\,dx = -\sum_{l=1}^N \int_\Omega u_l \Delta u_l |\nabla u|^{2(k-1)}|u|^{p-2k}\,dx$$
$$- (k-1)\sum_{l=1}^N \int_\Omega u_l \nabla u_l \nabla(|\nabla u|^2)|\nabla u|^{2(k-2)}|u|^{p-2k}\,dx$$
$$- (p-2k)\sum_{i=1}^n \int_\Omega (\sum_{l=1}^N \partial_i u_l u_l)^2 |u|^{p-2k-2}\,dx.$$

For $k \geq \frac{p}{4} + 1$ we estimate the first term in the right hand side of (3.18) by

$$\int_\Omega (\sum_{l=1}^N u_l^2)^{1/2} (\sum_{l=1}^N (\Delta u_l)^2)^{1/2} |\nabla u|^{2(k-1)}|u|^{p-2k}\,dx \leq$$
$$(\sum_{l=1}^N \sum_{j=1}^n \int_\Omega \int_\Omega |\nabla(\partial_j u_l)|^2 |\nabla u|^{p-2}\,dx)^{1/2} (\int_\Omega |\nabla u|^{4k-p-2}|u|^{2p-4k+2}\,dx)^{1/2},$$

and the second term in the right hand side of (3.18) by

$$(k-1)\Big(\int_\Omega \sum_{i=1}^{n} (\partial_i(|\nabla u|^2))^2 |\nabla u|^{p-4}\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla u|^{4k-p-2}|u|^{2p-4k+2}\,dx\Big)^{1/2},$$

where we used that

$$\sum_{l=1}^{N} u_l \nabla u_l \nabla(|\nabla u|^2) \le \sum_{i=1}^{n} \Big(\sum_{l=1}^{N} u_l^2\Big)^{1/2}\Big(\sum_{l=1}^{N}(\partial_i u_l)^2\Big)^{1/2}|\partial_i(|\nabla u|^2)| \lesssim |u||\nabla u||\nabla(|\nabla u|^2)|.$$

Since the last term in (3.18) is negative, while the quantity we want to estimate is positive we obtain from the last inequalities

$$\int_0^T \int_\Omega |\nabla u|^{2k}|u|^{p-2k}t^{k-1}\,dxdt \lesssim (J_1^{1/2}+J_2^{1/2})I_{2k-\frac{p}{2}-1} \lesssim (J_1+J_2)^{1/2}I_{2k-\frac{p}{2}-1}. \tag{3.19}$$

From now on we proceed exactly like in the scalar case iterating sufficiently many times to obtain the desired result, since we control $I_{p/2}$ which is the RHS term of (3.18) using (3.19).

3.4. **A simple argument for** (3.1)**.** We now return to the first estimate in Lemma 3.1: while we only deal with $p=2$, there is nothing specific to the L^2 case in what follows. Let χ be a smooth cut-off near the boundary $\partial\Omega$. Then $v=(1-\chi)u$ solves the heat equation in the whole space, with source term $[\chi,\Delta]u$:

$$(1-\chi)u = S_0(t)(1-\chi)u_0 + \int_0^t S_0(t-s)[\chi,\Delta]u(s)\,ds,$$

where S_0 is the free heat semi-group. We have, taking advantage of the localization near the boundary,

$$\|[\chi,\Delta]u\|_{L^2_t(L^{\frac{2n}{n+2}})} \lesssim C(\chi,\chi')\|\nabla u\|_{L^2_t(L^2)} < +\infty,$$

by the energy inequality (3.4). The integral equation on $(1-\chi)u$ features S_0 for which we have trivial Gaussian estimates, and both the homogeneous and inhomogeneous terms are $C_t(L^2)$ and go to zero as time goes to $+\infty$. On the other hand, by Poincaré inequality (or Sobolev),

$$\int_0^t \|\chi u\|_2^2\,ds \lesssim \int_0^t \|\nabla u\|_2^2\,ds,$$

which ensures that $\|\chi u\|_2$ goes to zero as well at $t=+\infty$.

Historical notes and comments

In this section, we collect a few remarks of interest from an historical perspective.

For compact manifolds without boundaries, one may find a direct proof of (1.3) (with Δ_D replaced by the Laplace-Beltrami operator) in [6], which proceeds by reduction to the $\mathbb{R}^n$ case using standard pseudo-differential calculus. Our approach provides an alternative direct proof. However, the true square function bound (1.2) holds on such manifolds, as one has a Mikhlin-Hörmander theorem from [25].

One can also adapt all proofs to the case of Neumann boundary conditions, provided special care is taken of the zero frequency (note that on an exterior domain, a decay condition at infinity solves the issue). The Gaussian bound which is required later holds in the Neumann case, see [9, 8].

Of course, most if not all arguments may be adapted to a variable coefficients elliptic second order operator rather than a Laplacian, and one may even keep tabs on the amount of regularity on its coefficients for the arguments to be carried out.

As mentioned before, Theorem 1.2 is useful, among other things, when dealing with L^p estimates for wave or dispersive evolution equations. For such equations, one naturally considers initial data in Sobolev spaces, and spectral localization conveniently reduces matters to data in L^2, and helps with finite speed of propagation arguments. One however wants to sum eventually over all frequencies in l^2, if possible without loss. Recent examples on domains may be found in [15] or [24], as well as in [21] (see also [17, 16]).

Alternatively, one can derive all results on the heat flow from adapting to the domain case the theory which ultimately led to the proof of the Kato conjecture ([5, 4]). Such a possible development is pointed out by P. Auscher in [3] (chap. 7, p. 66) and was originally our starting point; eventually we were led to the approach we present here, but we provide a sketch of an alternate proof, which was kindly outlined to us by Pascal Auscher: the main drawback from (1.8) is the presence of $\nabla S(t)$ on the right hand-side: one is leaving the functional calculus of Δ_D, and in fact for domains with Lipschitz boundaries the operator $\nabla S(t)$ may not even be bounded. As such, a suitable alternative is to replace $\nabla S(t)$ by $\sqrt{|\partial_t|}S(t)$. Then the square function estimate may be obtained following [3] as follows:

- prove that the associated square function in time is bounded by the L^p norm, for all $1 < p \leq 2$, essentially following step 3 in chapter 6, page 55 in [3]. This requires very little on the semi-group, and Gaussian bounds on $S(t)$ and $\partial_t S(t)$ ([10]) are more than enough to apply the weak $(1, 1)$ criterion from [3] (Theorem 1.1, chapter 1). Moreover, the argument can be extended to domains with Lipschitz boundaries, assuming the Laplacian is defined through the associated Dirichlet form;
- by duality, we get the square function bound for $p > 2$ (step 5, page 56 in [3]);
- from now on one proceeds as in the remaining part of our paper to obtain the bound with spectral localization, and almost orthogonality (3.6) is even easier because we stay in the functional calculus. One has, however, to be careful if one is willing to extend this last step to Lipschitz boundaries, as this would most likely require additional estimates on the resolvent to deal with the Δ_j.

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Appendix: functional calculus

We start by recalling the Dynkin-Helffer-Sjöstrand formula ([13, 14]) and refer to the appendix of [23] for a nice presentation of the use of almost-analytic extensions in the context of functional calculus. Most, if not all, of what follows may be found in Davies'presentation ([11]) and [7]. The last part provides a comprehensive description of the unpublished work [1].

Definition 3.6. (see [23, Lemma A.1]) Let $\Psi \in C_0^\infty(\mathbb{R})$, possibly complex valued. We assume that there exists $\tilde{\Psi} \in C_0^\infty(\mathbb{C})$ such that $|\bar{\partial}\tilde{\Psi}(z)| \leq C|\mathrm{Im} z|$ and $\tilde{\Psi}|_{\mathbb{R}} = \Psi$. Then we have (as a bounded operator in $L^2(\Omega)$)

$$\Psi(-h^2\Delta_D) = \frac{i}{2\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\Psi}(z)(z+h^2\Delta_D)^{-1} d\bar{z}\wedge dz. \tag{3.20}$$

Standard arguments ensure the existence of $\tilde{\Psi}$ in the previous definition (see [23, Lemma A.2] and [28], where it is linked with Hadamard's problem of finding a smooth function with prescribed derivatives at a given point).

Our next lemma lets us deal with Lebesgue spaces: one may find a detailed proof in [7], Proposition 6.2, relying on (complex) heat kernel bounds.

Lemma 3.7. *Let $z \notin \mathbb{R}$ and $|\mathrm{Im} z| \lesssim |\mathrm{Re} z|$, then Δ_D satisfies*

$$\|(z-\Delta_D)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \leq \frac{c}{|\mathrm{Im} z|}\left(\frac{|z|}{|\mathrm{Im} z|}\right)^\alpha, \quad \forall z\notin\mathbb{R} \tag{3.21}$$

for $1 \leq p \leq +\infty$, with a constant $c = c(p) > 0$ and $\alpha = \alpha(n,p) > n|\frac{1}{2}-\frac{1}{p}|$.

Remark that, for all $h \in (0,1]$, the operator $h^2\Delta_D$ satisfies (3.21) with the same constants c and α (this is nothing but scale invariance).

Remark that for $p = 2$ the proof of Lemma 3.7 is trivial by multiplying the resolvent equation $-\Delta_D u + zu = f$ by $\bar{u}$ and we get $\alpha = 0$.

Corollary 3.8. *For $N \geq \alpha+1$ the integral (3.20) is norm convergent and $\forall h \in (0,1]$*

$$\|\Psi(-h^2\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \leq c\sum_{m=0}^{N}\int_{\mathbb{R}} |\partial^m\Psi(x)|(1+|x|^2)^{(m-1)/2}\,dx, \tag{3.22}$$

for some constant c independent of h.

Remark 3.9. Notice how the Mikhlin multiplier condition (2.2) on Ψ does not imply boundedness of $\|\Psi\|_{N+1}$: we need extra decay at infinity.

We refer to [7], Theorem 6.1, for a proof, relying on the previous lemma and definition.

One may then prove that the operator $\Psi(-\Delta_D)$, acting on $L^p(\Omega)$, is independent of $N \geq 1+n/2$ (and of the cut-off function τ in the definition of $\tilde{\Psi}$, see [11].)

We now recall two lemma which were useful when composing operators.

Lemma 3.10 (Lemma 2.2.5,[11])**.** *If $\Psi \in C_0^\infty(\mathbb{R})$ has disjoint support from the spectrum of $-h^2\Delta_D$, then $\Psi(-h^2\Delta_D) = 0$.*

Lemma 3.11 (Lemma 2.2.6, [11])**.** *If Ψ_1, $\Psi_2 \in C_0^\infty(\mathbb{R})$, then $(\Psi_1\Psi_2)(-h^2\Delta_D) = \Psi_1(-h^2\Delta_D)\Psi_2(-h^2\Delta_D)$.*

Last, but not least, we proceed by providing an outline of the proof of the main theorem in [1], relying extensively on the arguments from [2]. The argument is quite general and requires very little on the underlying manifold M (which is the domain Ω in our particular case). The manifold M should be differentiable, have a positive measure that is σ-finite and verify the doubling property: the volume of any ball $B(x, 2r)$ should be controled in a uniform way by that of $B(x, r)$, for any $x \in M$ and $r > 0$. Moreover, we should have a second order operator (the Dirichlet Laplacian in our particular case), denoted by A, that satisfies Gaussian upper bounds, e.g., if $P_t(x, y)$ is the kernel of $\exp(-tA)$, there exists $c > 0$ such that for any $x, y \in M$, $t > 0$,

$$P_t(x,y) \leq \frac{c}{|B(x,\sqrt{t})|}\exp\left(-\frac{d(x,y)^2}{ct}\right). \tag{3.23}$$

In the following, any constant c is allowed to change from line to line as long as it is uniform in the parameters.

In [1], Theorem 2.1 is phrased as follows:

Theorem 3.12. *Let $\phi \in C^\infty(\mathbb{R})$ be a positive function compactly supported on $(1/2, 4)$ such that $\phi(x) = 1$ on $[1, 2]$. Assume that a function m is such that*

$$\sup_{t>0} \|\phi(x)m(tx)\|_{C^{n/2+1}(\mathbb{R})} < +\infty\,, \tag{3.24}$$

then $m(A)$ extends to a bounded operator from L^p to itself, for $1 < p < +\infty$.

Remark 3.13. If we moreover assume that the support of m is a compact $K \subset \mathbb{R}_+$ such that $0 \notin K$, then it may be proved that the Kernel $K(x, y)$ of $m(A)$ is in $L^\infty_y(L^1_x)$. We provide a proof for the reader's convenience in a moment.

Define the annulus $A_j(x) = B(x, 2^{(j+1)/2}) \setminus B(x, 2^{j/2})$ for $j \in \mathbb{Z}$. We start by recalling a few lemma. The firt one is verbatim Lemma 2.1 in [2].

Lemma 3.14. *Assume that $f \in C^n(\mathbb{R})$ and let $M_\alpha = \sup_{t>0,x\in\mathbb{R}} |t^{-\alpha}(f^n(x+t) - f^n(x)|$, for $0 < \alpha \leq 1$. Then, for any $\lambda > 0$, there is a function $\psi_\lambda \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and a constant $c > 0$, uniform with respect to f and λ, such that*

$$\|\hat{\psi}_\lambda\|_\infty \leq c\,, \tag{3.25}$$

$$\text{supp}\hat{\psi}_\lambda \subset [-\lambda, \lambda]\,, \tag{3.26}$$

$$|f(x) - (\psi_\lambda \star f)(x)| \leq cM_\alpha\lambda^{-(n+\alpha)}\,. \tag{3.27}$$

The next one is, up to notations, the first statement from Lemma 3.1 in [2].

Lemma 3.15. *There exist $0 < \eta < 1$ and $c > 0$ such that, for all $p, j \in \mathbb{Z}$ with $j \leq p$, $x \in A_p(y)$ and $|t| \leq \eta 2^{(p-j)/2}$,*

$$|e^{itP_{2^j}}P_{2^j}(x,y)| \leq \frac{c}{|B(y,2^{j/2})|}\exp\left(-\frac{2^{(p-j)/2}}{c}\right). \tag{3.28}$$

Next, we have, assuming that for two kernels K_1 and K_2, we denote by K_1K_2 the composition kernel:

Lemma 3.16. *Let f and α be like in Lemma 3.14. There exists $c > 0$ such that, for $p, j \in \mathbb{Z}$ with $j \leq p$,*

$$\|f(P_{2^j}P_{2^j}(x,y)\|_{L^1(B(y,2^{p/2}))} \leq c2^{n(p-j)/4}\|f\|_\infty \tag{3.29}$$

and

$$\|f(P_{2^j}P_{2^j}(x,y)\|_{L^1(A_p(y))} \leq cM_\alpha 2^{-\alpha(p-j)/4} + \|f\|_1 e^{-c^{-1}2^{(p-j)/2}} \tag{3.30}$$

This lemma is implicit in the proof of Proposition 1.2 in [2]. It follows directly from our hypothesis on α and the two previous lemmas.

From the last lemma, we can immediatly deal with the situation where m has compact support away from 0: let $h(\mu) = m(-\log\mu)\mu^{-1}$, then $m(\lambda) = h(e^{-\lambda})e^{-\lambda}$, moreover, h is compactly supported and $h \in C^{n/2+1}(\mathbb{R})$. Then, if K is the kernel of $m(A)$, we write

$$\begin{aligned}\|K(x,y)\|_{L^1_y} &= \|h(P_1)P_1(x,y)\|_{L^1_y}\\ &= \|h(P_1)P_1(x,y)\|_{L^1(B(y,1))} + \sum_{p\geq 0}\|h(P_1)P_1(x,y)\|_{L^1(A_p(y))}\,.\end{aligned}$$

Picking $n + \alpha = n/2 + 1$ and applying the previous lemma gives the desired result.

We now move to the general case: by interpolation and symmetry of the Kernel, we can reduce to proving that $m(A)$ sends L^1 to the weak Lebesgue space $L^{1,\infty}$. We start with preliminary considerations (following the eponymous subsection in [2]): let $b(\lambda) = \mathbf{1}_{[0,1/t]}(\lambda)e^{t\lambda}$, then

$$\|\int_0^{1/t} dE_\lambda(f)\|_2 = \|b(A)P_t f\|_2 \leq \|b\|_\infty \|P_t f\|_2\,. \tag{3.31}$$

We immediatly conclude that the lefthand side goes to zero when $t \to +\infty$ and that one may then ignore $\lambda = 0$ in the spectral resolution of A. Set $\phi \in C^\infty$ such that supp$\phi \subset (1/2, 2)$ and $\sum_{j\in\mathbb{Z}}\phi(2^j t) = 1$ for any $t > 0$. We then decompose

$$m(\lambda) = \sum_{j\in\mathbb{Z}} m(\lambda)\phi(2^j\lambda) = \sum_{j\in\mathbb{Z}} m_j(\lambda)$$

and accordingly, $K(x,y) = \sum_{j\in\mathbb{Z}} K_j(x,y)$ at the kernel level. Again, we rewrite

$$m_j(\lambda) = m_j(-2^{-j}\log e^{-2^j\lambda})e^{2^j\lambda}e^{-2^j\lambda}\,,$$

and we set $h_j(\mu) = m_j(-2^{-j}\log\mu)\mu^{-1}$, $\zeta_{j,\tau}(\mu) = (1 - \mu^{2^{\tau-j}})h_j(\mu)$, for $\tau \leq j$ and $\xi_{j,\tau}(\mu) = h_j(\mu)s^{2^{\tau-j}}$ for $\tau \geq j$. From our hypothesis on m, one may check that

$$\|\eta_j\|_{C^{n/2+1}(\mathbb{R})} + 2^{j-\tau}\|\zeta_{j,\tau}\|_{C^{n/2+1}(\mathbb{R})} + e^{c^{-1}2^{\tau-j}}\|\xi_{j,\tau}\|_{C^{n/2+1}(\mathbb{R})} \leq c\,. \tag{3.32}$$

Moreover, we may rewrite $m_j(A) = h_j(P_{2^j})P_{2^j}$, $(I - P_{2^\tau})m_j(A) = \zeta_{j,\tau}(P_{2^j})P_{2^j}$, $m_j(A)P_{2^\tau} = \xi_{j,\tau}(P_{2^j})P_{2^j}$.

We now proceed with a Calderon-Zygmund decomposition: consider a positive function $f \in L^1 \cap L^2$ with bounded support, and $a > 0$. There are constants $C, k > 0$ and a sequence of balls $B(x_i, r_i)$ such that

- for almost all $x \in M \setminus \cup_i B(x_i, r_i)$, $f(x) \leq Ca$,
- we have $|B(x_i,r_i)|^{-1}\int_{B(x_i,r_i)} f \leq Ca$,
- we have $\sum_i |B(x_i,r_i)| \leq Ca^{-1}\int_M f$,
- each point $x \in M$ belongs to at most k balls $B(x_i, r_i)$.

Define $\eta_j(x) = 0$ if $x \notin B(x_i, r_i)$ and $\eta_j(x) = (\sum_k \mathbf{1}_{B(x_k,r_k)})^{-1}$ if $x \in B(x_i, r_i)$, and set $w_i = \eta_i f$. If we define $b_i = P_{r_i^2} w_i$, $\theta_i = w_i - b_i$ and $g = f\mathbf{1}_{M\setminus\cup_i B(x_i,r_i)}$, then

$$f = g + \sum_i b_i + \sum_i \theta_i\,,$$

and $|g(x)| \leq ca$ for all $x \in M$.

We now turn to the kernels: from (3.29),(3.30) and (3.32), we deduce

Lemma 3.17. *There exists $c > 0$ such that, for all $p, j, \tau \in \mathbb{Z}$, $p, \tau \geq j$ and $y \in M$,*

$$\|K_j(x,y)\|_{L^1(A_p(y))} \leq c2^{-(p-j)/2}\,, \tag{3.33}$$

$$\|K_j P_{2^\tau}(x,y)\|_{L^1} \leq c \exp -2^{-(\tau-j)}/c\,, \tag{3.34}$$

and for $j \geq \tau$,

$$\|K_j(I - P_{2^\tau})(x,y)\|_{L^1} \leq c2^{(\tau-j)}\,. \tag{3.35}$$

Remark that an immediate consequence of (3.33), (3.34) is the bound, for $j \leq \tau$

$$\|K_j(I - P_{2^\tau})(x,y)\|_{L^1(d(x,y)\geq 2^{\tau/2})} \leq c2^{-(\tau-j)/2}\,. \tag{3.36}$$

Then, one may combine (3.36) and (3.35) to obtain, for all $j, \tau \in \mathbb{Z}$,

$$\|K_j(I - P_{2^\tau})(x,y)\|_{L^1(d(x,y)\geq 2^{\tau/2})} \leq c\,. \tag{3.37}$$

The next lemma is lifted verbatim from [2], e.g. Lemma 5.3.8.

Lemma 3.18. *Let $b \in L^1$ and let assume that* $\operatorname{supp} b \subset B(z,r)$ *for $z \in M$ and $r > 0$. There exists $c > 0$ and independent of b such that for all $u \in L^2$ and all $t \geq r^2$,*

$$\int u P_t b \leq c\|b\|_1 |B(z,r)|^{-1} \int \mathbf{1}_{B(z,r)} \mathcal{M}u\,, \tag{3.38}$$

where $\mathcal{M}u$ is the Hardly-Littlewood maximal function.

As a consequence of this lemma, we prove

$$\|\sum_i b_i\|_2^2 \leq ca\|f\|_1\,. \tag{3.39}$$

We proceed by duality: pick $u \in L^2$, applying the lemma and Cauchy-Schwarz,

$$\begin{aligned}\int u \sum_i b_i &\leq ca\|\mathcal{M}u\|_2 \| \int \sum_i \mathbf{1}_{B(x_i,r_i)}\|_2\,,\\ &\leq ca\|u\|_2\| \int \sum_i \mathbf{1}_{B(x_i,r_i)}\|_2\,.\end{aligned}$$

However, from the properties of the Calderon-Zygmund decomposition, we have

$$\| \int \sum_i \mathbf{1}_{B(x_i,r_i)}\|_2 \leq c\| \int \sum_i \mathbf{1}_{B(x_i,r_i)}\|_1 \leq c\sum_i |B(x_i,r_i)| \leq ca^{-1}\|f\|_1\,,$$

which ends the proof of (3.39).

We are now in a position to proceed with the proof of the main estimate: there exists $c > 0$ such that

$$|\{|m(A)f| \geq a\}| \leq ca^{-1}\|f\|_1\,. \tag{3.40}$$

As usual, we split the measure on the lefthandside according to the Calderon-Zygmund decomposition:

$$|\{|m(A)f| \geq a\}| \leq |\{|m(A)g| \geq a/3\}| + |\{|m(A)\sum_i b_i| \geq a/3\}| \\ + |\{|m(A)\sum_i \theta_i| \geq a/3\}|\,.$$

The first term is easy, as

$$|\{|m(A)g| \geq a/3\}| \leq 9a^{-2}\|m(A)g\|_2^2 \leq ca^{-2}\|g\|_2^2 \\ \leq ca^{-2}a\|ga^{-1}g\|_1 \leq ca^{-1}\|g\|_1 \leq ca^{-1}\|f\|_1\,.$$

The second term is dealt with similarly, using (3.39):

$$|\{|m(A)\sum_i b_i| \geq a/3\}| \leq 9a^{-2}\|m(A)\sum_i b_i\|_2^2 \leq ca^{-2}\|\sum_i b_i\|_2^2 \leq ca^{-1}\|f\|_1\,.$$

For the third and last term, we use (3.37) to get

$$\|m(A)\theta_i(x)\|_{L^1(M\setminus B(x_i,cr_i))} \leq c\|w_i\|_1\,.$$

As such,

$$|\{x \notin \cup_i B(x_i,r_i)\text{s.t.}|m(A)\sum_i \theta_i| \geq a/3\}| \leq ca^{-1}\sum_i \|w_i\|_1 \leq ca^{-1}\|f\|_1\,.$$

But from the Calderon-Zygmund decomposition, we have $|\cup_i B(x_i,cr_i)| \leq ca^{-1}\|f\|_1$, and therefore

$$|\{|m(A)\sum_i \theta_i| \geq a/3\}| \leq ca^{-1}\sum_i \|w_i\|_1 \leq ca^{-1}\|f\|_1\,,$$

which concludes the proof of the L^1 to $L^{1,\infty}$ estimate.

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