Dispersive estimates for the wave equation in strictly convex domains with boundary

Oana Ivanovici
(joint work with Fabrice Planchon)

Consider the wave equation inside a domain Ω of dimension $d \geq 2$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, & x \in \Omega \\ u(0,x) = \delta_a, \ \partial_t u(0,x) = 0, \end{cases} \tag{1}$$

where $a \in \Omega$, δ_a is the Dirac function and Δ_g denotes the Laplace-Beltrami operator on Ω. In the case of a non empty boundary we consider the Diriclet condition $u|_{\partial\Omega} = 0$ on the boundary.

If Ω is the free space $\mathbb{R}^d$ with the Euclidian metric $g_{i,j} = \delta_{i,j}$ and if $u_{\mathbb{R}^d}(t,x)$ is the Green function (i.e. the solution to (1) in $\mathbb{R}^d$) then it is given by

$$u_{\mathbb{R}^d}(t,x) = \frac{1}{(2\pi)^d} \int \cos(t|\xi|) e^{i(x-a)\xi} d\xi$$

and it satisfies the classical dispersive estimates:

$$\|\psi(hD_t)u_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d} \min\{1, (h/t)^{\frac{d-1}{2}}\}. \tag{2}$$

Here $\psi \in C_0^\infty$ is a smooth function supported outside a neighborhood of 0.

In this note we are interested in domains with boundary: the difficulties arise from the behavior of the singularities of the solutions to (1) near the points of $\partial\Omega$. In the case of a concave boundary, sharp dispersive estimates should follow using the Melrose and Taylor parametrix and the approach in [7]. In the opposite situation of a strictly convex domain, the presence of the gliding rays prevent the construction of such a parametrix.

Gilles Lebeau was the first who described in [6] the dispersive estimates on small time intervals for the solutions of (1) inside a strictly convex domain (Ω, g) of dimension $d \geq 2$. The result he had announced reads as follows:

Theorem 1. *If $a > 0$ is sufficiently small, then there exists $T > 0$, $C > 0$ so that for every $h \in (0,1]$ and $t \in (0,T]$ the solution u to (1) satisfies*

$$|\psi(hD_t)u(t,x)| \leq C(d)h^{-d} \min\{1, (h/t)^{\frac{d-2}{2}+\frac{1}{4}}\}. \tag{3}$$

Remark 2. *The estimate (3) means that, compared to the dispersive estimate in the free space (2), there is a loss of a power of $\frac{1}{4}$ of $\frac{h}{t}$ inside a strictly convex domain, and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. This loss is optimal.*

Remark 3. *In [6] Gilles Lebeau sketched the main steps of the proof and gave a full description of the geometry behind. However, many details are missing and therefore, our forthcoming work [5] in collaboration with Fabrice Planchon is intended to complete the analytical part of Gilles Lebeau's result.*

Remark 4. *The loss of* $\frac{1}{4}$ *comes only from the dispersion in the normal variable, therefore it will be enough to prove the result in dimension* $d = 2$ *only.*

Remark 5. *Theorem 1 allows to prove sharp results in dimension* $d \geq 2$ *for the spectral projector estimates generalizing the work* [8] *in dimensions* $d \geq 3$ *in the case of convex domains. It also gives the sharp range of indices for which optimal Strichartz estimates hold (this is a work in progress, in collaboration with Fabrice Planchon); moreover, using* (3) *we can prove that the counterexamples constructed in* [3, 4] *are optimal.*

Proof. Before starting the proof, the first thing to understand is the type of concentration phenomena such as *caustics* that may occur near the boundary.

What are caustics? Caustics are envelopes of light rays that appear in a given problem. At the caustic point the intensity of light is singularly large, causing different physical phenomena. Mathematically, caustics could be characterized as points were usual bounds on oscillatory integrals are no longer valid. It is well known that the asymptotic behavior of an oscillatory integral is governed by the number and the order of their critical points which are real. Let

$$(4) \qquad u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int_\zeta e^{\frac{i}{h}\Phi(z,\zeta)} g(z,\zeta,h) d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0,1].$$

If there are degenerate critical points, known as caustics, then $\|u_h(z)\|_{L^\infty}$ is no longer uniformly bounded. The order of a caustic κ is defined as the infimum of κ' so that $\|u_h(z)\|_{L^\infty} = O(h^{-\kappa'})$. For example, recall that in [3] we considered phase functions of the form $\Phi_F(z,\zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$ and obtained a loss in the Strichartz estimates of $\frac{1}{6}$ derivatives. This phase corresponds to a fold and has order precisely $\kappa = \frac{1}{6}$. In the proof of Theorem 1 a crucial role will be played by the Pearcey type integrals, with phase function of the form $\Phi_C(z,\zeta) = \frac{\zeta^4}{4} + z_1\frac{\zeta^2}{2} + z_2\zeta + z_3$ and order $\kappa = \frac{1}{4}$. They correspond to a cusp type singularity; the swallowtail canonical form involves the phase $\Phi_S(z,\zeta) = \frac{\zeta^5}{5} + z_1\frac{\zeta^3}{3} + z_2\frac{\zeta^2}{2} + z_3\zeta + z_4$, with order $k = \frac{3}{10}$.

Let $\Omega = \{(x,y) \in \mathbb{R}^2, x > 0\}$ and $\Delta_g = \partial_x^2 + (1+x)\partial_y^2$ define a strictly convex domain in $\mathbb{R}^2$. A first step in the proof of Theorem 1 consists in a detailed description of the set of points of Ω which can be reached following all the optical rays starting from a of length t. We split the data in packets in such a way that each packet corresponds to a number of reflections on the boundary for a fixed time T. At high frequency $\frac{1}{h}$, the "worst" packets will be those for which $a \simeq h^{1/2}$ and which propagate along directions parallel to $\partial\Omega$. These localized data will involve "swallowtail" type singularities in the wave front set of the solution. Hence it will be sufficient to prove the estimates (3) for the following initial data:

$$u_0(x,y) = \frac{1}{(2\pi h)^2} \int e^{\frac{i}{h}((x-a)\xi + y\eta)} \psi(\eta) \rho\Big(\frac{\xi}{h^{1/4}\eta}\Big) d\xi d\eta,$$

where ψ, ρ are smooth functions compactly supported in a neighborhood of 1 and 0, respectively, $\psi \in C_0^\infty(\frac{1}{2}, 2)$, $\rho \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$. If the initial distance a to the boundary is small, namely $a \leq h^{\frac{1}{2}}$, we use the fact that the essential support of the Fourier transform of u remains small, together with the elementary estimate [6][(2.24)]. For $a > h^{1/2}$ we construct a parametrix u for small time t between 0 and the moment the wave reaches the boundary the first time; we then solve the Airy equation with this data on the boundary. We repeat this construction a number of times $N \simeq \frac{1}{\sqrt{a}}$. We obtain a parametrix of the form

$$U_h(t,x,y) = \sum_{n=0}^{N} u_n(t,x,y),$$

$$u_n(t,x,y) = \int e^{\frac{i}{h}\eta\phi_n(t,x,y,\xi)} g_h^n(x,y,t,\xi)\psi(\eta)\rho(h^{-1/4}(\frac{a}{\xi} - \frac{\xi}{4}))d\xi d\eta.$$

The symbols g_h^n are chosen so that u_n to have almost orthogonal supports in time and so that the Dirichlet condition to be satisfied. We study the asymptotic behavior of the parametrices u_n. We otain the the equivalent of [6][Lemma 3.7]:

Theorem 6. *For every $n \in \{1,..,N\}$, the phase ϕ_n has saddle points of order at most 3; for each $n \in \{1,..,N\}$ there exists a unique time $t = t_{S,n}$ for which $\phi_n(t)$ has a critical point ξ_S of order 3.*

From the above Lemma it follows, using Arnold's classification, that ϕ_n is a Pearcey type integral with order $\frac{1}{4}$. Writing the asymptotic expansion of $u_n(t)$ near $t_{S,n}$, we deduce that a loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ is unavoidable for $\|u_n\|_{L^\infty}$.

Theorem 7. *The loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ in the dispersive estimates (3) is optimal in any dimension $d \geq 2$.*

The optimality follows from the fact that there is a swallowtail type singularity in the wavefront set $WF_h(u_n)$ for each $n \in \{1,..,N\}$. Then use Remark 4. □

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