Strichartz inequalities for 3D waves in a strictly convex domain

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(joint work with Gilles Lebeau and Fabrice Planchon)

Consider the wave equation in a domain Ω of dimension $d \geq 2$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, & x \in \Omega \\ u(0,x) = \delta_a, \ \partial_t u(0,x) = 0, \end{cases} \tag{1}$$

where $a \in \Omega$, δ_a is the Dirac function and Δ_g denotes the Laplace-Beltrami operator on Ω. If $\partial\Omega \neq \emptyset$ we consider the Dirichlet condition $u|_{\partial\Omega} = 0$.

Let Ω be the free space $\mathbb{R}^d$ with the Euclidian metric $g_{i,j} = \delta_{i,j}$ and $\chi \in C_0^\infty$ be a smooth function supported near 1. If $u_{\mathbb{R}^d}(t,x)$ is the solution to (1) in $\mathbb{R}^d$ then it is given by

$$u_{\mathbb{R}^d}(t,x) = \frac{1}{(2\pi)^d} \int \cos(t|\xi|) e^{i(x-a)\xi} d\xi$$

and it satisfies the classical dispersive estimates:

$$\|\chi(hD_t)u_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C(d) h^{-d} \min\{1, (h/t)^{\alpha_d}\}. \tag{2}$$

Interpolating between (2) and the energy estimate and using the so called TT^* argument, yields the following Strichartz estimates:

$$h^\beta \|\chi(hD_t)u\|_{L^q([0,T],L^r(\mathbb{R}^d))} \leq C\Big(\|u(0,x)\|_{L^2} + \|hD_t u\|_{L^2}\Big). \tag{3}$$

Here $q \in (2,\infty]$, $r \in [2,\infty]$ satisfy $(q,r,d) \neq (2,\infty,3)$, $\frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}$ and

$$\frac{1}{q} = \alpha_d\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad \beta = (d - \alpha_d)\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

Our aim in the present note, based on [3], is to obtain Strichartz estimates inside domains: in this situation, the difficulties arise from the behaviour of the wave flow near the points of the boundary. Before stating our main result, we briefly introduce the Friedlander's model domain of the half space $\Omega_d = \{(x,y) | x > 0, y \in \mathbb{R}^{d-1}\}$ with Laplace operator given by $\Delta_F = \partial_x^2 + (1+x)\Delta_{\mathbb{R}_y^{d-1}}$. Clearly, the manifold Ω with the metric g_F inherited from Δ_F is a strictly convex domain; moreover, (Ω_2, g_F) may be seen as a simplified model for the disk $D(0,1)$ with polar coordinates (r,θ), where $r = 1 - x/2$, $\theta = y$. Our main result is the following:

Theorem 1. [3] *Strichartz inequality holds true for the solution to* (1) *inside* (Ω_d, g_F) *with* $\alpha_d = \frac{d-1}{2} - \frac{1}{6}$.

Remark 2. *This was proved by M.Blair, H.Smith and C.Sogge in the case $d = 2$ for arbitrary boundary (i.e. without convexity assumption). The above theorem improves all the known results for $d \geq 3$. The case of a general strictly convex boundary is a work in progress with G.Lebeau, F.Planchon and R.Lascar.*

In [2], we proved the following dispersive estimate for (Ω_d, g_F), $d \geq 2$:

Theorem 3. *There exists $T > 0$, $C(d) > 0$ such that for every $a \in (0,1]$, $h \in (0,1]$ and $t \in (0,T]$ the solution $u_a(t,x,y) = \cos(t\sqrt{|\Delta_F|})(\delta_{x=a,y=0})$ to (1) satisfies*

$$|\chi(hD_t)u(t,x)| \leq C(d)h^{-d}\min\{1,(h/t)^{\frac{d-2}{2}}\gamma(t,h,a)\}, \tag{4}$$

where

$$\gamma(t,h,a) = \begin{cases} (\frac{h}{t})^{1/2} + a^{1/8}h^{1/4}, \text{ for } a \geq h^{4/7-\epsilon} \\ (\frac{h}{t})^{1/3} + h^{1/4}, \text{ for } a \leq h^{1/2}. \end{cases}$$

Moreover, there is a sequence of moments of times $t_n = 4n\sqrt{a}\sqrt{1+a}$ for which equality holds in (4); for $t \notin (t_n - \epsilon\frac{a^{1/2}}{n}, t_n + \epsilon\frac{a^{1/2}}{n}) := I_n$, $\gamma(t,h,a)$ can be bounded by $(\frac{h}{t})^{1/3}$ independently of a.

Remark 4. *The estimate (4) means that, compared to the dispersive estimate in the free space (2), there is a loss of a power of $\frac{1}{4}$ of h inside a strictly convex domain, and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. Such caustics occur because optical rays sent from a source point under different directions are no longer diverging from each other.*

Remark 5. *As a corollary to Theorem 3, we immediately obtain Strichartz estimates in any dimension $d \geq 2$ with $\alpha_d = \frac{d-2}{2} + \frac{1}{4}$.*

Proof. (of Theorem 1) We distinguish two different regimes:

- In the range $t > h$ and $a < h^{1/2+\epsilon}$, we prove the stronger estimate

$$|\chi(hD_t)u_a(t,x,y)| \leq Ch^{-d}(\frac{h}{t})^{\frac{d-2}{2}}(\frac{h}{t})^{1/3}.$$

  This means that in the range $a < h^{1/2+\epsilon}$, one can kill the bad factor $h^{1/4}$ of Theorem 3. The geometry is irrelevant when a is very small, since there are too many singularities in the wave front set and the new estimates are obtained using a finer analysis on the sum of gallery modes (inspired by exponential sum methods). Using the spectral decomposition, we obtain an explicit representation of the Green function as a sum of gallery modes, valid for any a. Taking its Poisson transformation yields a superposition of waves similar to the parametrix we obtained in [2] for $a > h^{4/7-\epsilon}$.
- In the range $a > h^{4/7-\epsilon}$, we observe that the "bad" factor $h^{1/4}$ occurs only near the discrete set of times t_n, with an estimation of $\gamma(t,h,a)$ for t near t_n ($t \in I_n$) by

$$\gamma(t,h,a) \leq (\frac{h}{t})^{1/2} + h^{1/3} + \frac{a^{1/8}h^{1/4}}{n^{1/4} + h^{-1/12}a^{-1/24}|t^2 - t_n^2|^{1/6}}. \tag{5}$$

  Notice also that for $t \notin I_n$, the last factor is $\leq h^{1/3}$. The refinement on $\gamma(t,h,a)$ follows from inspection of the (degenerate) stationary phase argument in [2]

End of the proof: For simplicity restrict to Strichartz with $\alpha_d = \frac{d-1}{2} - \frac{1}{6}$ for $d = 3$. We consider the Green function $G(t,x,y,a) = \chi(hD_t)e^{it\sqrt{|\Delta_F|}}(\delta_{x=a,y=0})$ and for f compactly supported in $(s, a \geq 0, b)$ we set

$$A(f)(t,x,y) = \int G(t-s,x,y-b,a)f(s,a,b)dsdadb.$$

The dispersive exponent is in this case $\alpha_3 = \frac{5}{6}$. We have to prove the end point estimate for $r = \infty$ and $q = 12/5$:

$$h^{2\beta}\|A(f)\|_{L^{12/5}_{t\in[0,1]}L^\infty_{x,y}} \leq C\|f\|_{L^{12/7}_s L^1_{a,b}}, \quad 2\beta = (d-\alpha_d) = 3 - 5/6 = 13/6.$$

We summarise: the swallowtail singularities occur only at $t_n = 4n\sqrt{a}$, $x = a$, $y_n := t_n + O(a^{3/2}n)$; they have an effect on $I_n := (t_n - \frac{\epsilon\sqrt{a}}{n}, t_n + \frac{\epsilon\sqrt{a}}{n})$. outside I_n there are only cusps with $(\frac{h}{t})^{-1/6}$ loss. The estimate of $\gamma(t,h,a)$ in (5) allows to track precisely where the usual TT^* argument fails.

We write $G(t,x,y,a) = G_0(t,x,y,a) + G_s(t,x,y,a)$ where G_s is the singular part, associated to a cutoff of G in balls centred at the swallowtail singularities

$$|x-a| \leq \frac{a}{n^2}, \quad |t - 4n\sqrt{a}\sqrt{1+a}| \leq \frac{\sqrt{a}}{n}.$$

Going back to [2], we obtain the following:

Proposition 1.

$$h^{2\beta}\sup_{x,y}|G_0(t,x,y,a)| \leq C|t|^{-5/6};$$

$$h^{2\beta}\sup_{x,y}|G_s(t,x,y,a)| \leq D(t,a,h), \quad \sup_{a,h}\int_{-1}^{1}|D(t,a,h)|^p dt < \infty, \quad \forall p < 2.$$

Let $A = A_0 + A_s$ corresponding to the previous decomposition. The estimate for A_0 follows easily, since the convolution by $|t|^{-5/6}$ maps $L^{12/7}$ in $L^{12/5}$. By the preceding proposition, $h^{2\beta}A_s$ is bounded from $L^1_s L^1_{a,b}$ into $L^{2-\epsilon}_t L^\infty_{x,y}$. Since the cutoff in balls near the swallowtails singularities is symmetric in (x,a), by duality, $h^{2\beta}A_s$ is bounded from $L^{2+\epsilon}_s L^1_{a,b}$ into $L^\infty_t L^\infty_{x,y}$ and, by interpolation, we get

$$h^{2\beta}A_s \quad \text{is bounded from} \quad L^{12/7}_s \quad \text{into} \quad L^{12-\epsilon}_t L^\infty_{x,y},$$

which is more than enough.

□

References

[1] M. Blair, H. Smith, C. Sogge, *Strichartz estimates for the wave equation on manifolds with boundary*, Ann.Inst.H.Poincaré, to appear

[2] O. Ivanovici, G. Lebeau, F. Planchon *Dispersive estimates for the wave equation in strictly convex domains* , to appear in Annals of Mathematics

[3] O. Ivanovici, G. Lebeau, F. Planchon *Stricharz inequalities for the wave equation in strictly convex domains* , to be submitted