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# Équations dispersives et problèmes aux limites

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**Résumé**

Nous nous intéressons ici à l'étude des propriétés des solutions des équations aux dérivées partielles hyperboliques. Plus précisément, les travaux portent sur les inégalités de type Strichartz pour les solutions des équations des ondes et de Schrödinger sur des variétés à bord.

De nombreux auteurs se sont intéressés à l'étude du problème de Cauchy pour de telles équations. Pendant longtemps, des résultats ont été obtenus dans le cas où le domaine est $\mathbb{R}^d$ : en effet, des formules explicites sont alors disponibles pour l'étude du semi-groupe d'évolution linéaire et elles permettent d'établir une large classe d'estimations à priori : *inégalités de Strichartz* ou *effet régularisant* (pour l'équation de Schrödinger dans ce dernier cas). Plus récemment, ces techniques ont pu être développées dans le cas des géométries non triviales. Cependant, la situation reste loin d'être aussi bien comprise que dans le cas de $\mathbb{R}^d$...

Dans cette thèse nous avons montré que les estimations de Strichartz usuelles sont fausses pour l'équation des ondes sur un convexe strict, en raison de phénomènes microlocaux liés aux caustiques générées en temps arbitrairement petit près du bord. Pour l'équation de Schrödinger sur des domaines extérieurs nous avons obtenu des estimations d'effet régularisant précisé au voisinage du bord, à l'extérieur d'une boule de $\mathbb{R}^3$ qui nous a permis d'obtenir un résultat d'existence globale pour l'équation de Schrödinger cubique à l'extérieur d'un nombre fini de boules de $\mathbb{R}^3$ satisfaisant l'hypothèse d'Ikawa. Nous avons ensuite obtenu des inégalités de Strichartz optimales pour l'équation semi-classique sur le billard de Sinaï et nous avons déduit de cela les mêmes estimations de Strichartz que dans le cas plat pour l'équation de Schrödinger classique à l'extérieur d'un convexe strict de $\mathbb{R}^d$. Enfin, dans le cas de l'extérieur d'un obstacle non-captant, nous avons montré que l'équation de Schrödinger quintique ($H^1$ - critique) est bien posée, ainsi qu'un phénomène de type scattering pour l'équation de Schrödinger sous-critique, en développant des estimations adéquates à partir de résultats récents sur les projecteurs spectraux, qui se substituent efficacement dans ce cadre aux estimations de Strichartz.

**Mots-clefs** : Inégalités de Strichartz, dispersion, équations des ondes et de Schrödinger, effet régularisant.

Dispersive equations and boundary value problems


**Abstract**

This thesis work is in the area of hyperbolic PDE and the specific research topics concern Strichartz type estimates for the wave and the Schrödinger equations on manifolds with boundary, which represent fundamental tools in the study of nonlinear problems.

In $\mathbb{R}^d$ the Cauchy problem in the context of the wave and Schrödinger equations have a long history, explicit formulas being available for the study of the linear evolution group which allow to establish a large class of estimates: *Strichartz estimates* or *smoothing effect* (for the Schrödinger equation). Recently, these techniques had been developed for nontrivial geometries; however, for general manifolds much less is known and the situation if far from being completely understood..

In this work we show that the Strichartz estimates for the wave equation inside a strictly convex domain suffer losses when compared to the usual Euclidian case, at least for a subset of the usual set of indices and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. Dealing with the Schrödinger equation outside a ball in $\mathbb{R}^3$, we obtain a precise smoothing effect near the boundary, which yields global well-posedness results for the cubic Schrödinger equation outside a finite union of balls in $\mathbb{R}^3$ satisfying Ikawa's assumptions. Another result of this work consists of establishing sharp Strichartz estimates for the semi-classical Schrödinger equation on the Sinaï billiard; combining this with the smoothing effect yields sharp Strichartz estimates for the classical Schrödinger equation outside a strictly convex domain of $\mathbb{R}^d$. Finally, outside a non-trapping obstacle (but not convex) in $\mathbb{R}^3$ we prove a different set of (scale-invariant) estimates, combining spectral projector estimates and smoothing effect, which yield local well-posedness for the energy-critical (quintic) Schrödinger equation and scattering for the sub-critical equation.



**Keywords** : Strichartz estimates, wave and Schrödinger equations, smoothing effect.

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# Première partie

# Introduction

## Sommaire

# 1 Equation des ondes

## 1.1 L'équation des ondes dans $\mathbb{R}^d$. Inégalités de Strichartz

On se place dans le cadre $\Omega = \mathbb{R}^d$ avec $d \geq 2$ et on note $\Delta$ l'opérateur de Laplace dans $\mathbb{R}^d$. Soient $u_0 \in L^2(\mathbb{R}^d)$ et $\Psi \in C_0^\infty(\mathbb{R}^*)$. On considère le problème (semi-classique) suivant

$$ih\partial_t u - h\sqrt{-\Delta}u = 0, \quad u|_{t=0} = \Psi(h\sqrt{-\Delta})u_0. \tag{1.1}$$

Si on note $u(x,t) = e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0(x)$ le flot linéaire, la solution de (1.1) s'écrit

$$e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0(x) = \frac{1}{(2\pi h)^d}\int e^{\frac{i}{h}(<x,\xi>-t|\xi|)}\Psi(|\xi|)\hat{u_0}(\frac{\xi}{h})d\xi. \tag{1.2}$$

On définit une paire admissible en dimension $d$ pour l'équation des ondes un couple $(q,r)$ tel que $(q,r,d) \neq (2,\infty,3)$, $q,r \geq 2$ et

$$\frac{2}{q} + \frac{d-1}{r} = \frac{d-1}{2}. \tag{1.3}$$

Les inégalités de Strichartz pour l'équation des ondes (1.1) s'écrivent comme suit

**Proposition 1.1.** *Soit $(q,r)$ une paire $d$-admissible pour l'équation des ondes en dimension $d \geq 2$ et $h \in (0,1]$. Si $u$ est solution de (1.1) avec donnée initiale $u_0 \in L^2(\mathbb{R}^d)$, alors il existe une constante $C > 0$ telle que*

$$h^{\frac{(d+1)}{2}(\frac{1}{2}-\frac{1}{r})}\|e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|\Psi(h\sqrt{-\Delta})u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

Cette inégalité a été démontrée par R.H.Strichartz [100] dans le cas particulier $q = r$ et ensuite établie par J.Ginibre et G.Velo [49, 51] pour des paires admissibles $(q,r)$ avec $q > 2$ et simultanément par H.Lindblad et C.Sogge [75]. Le point limite $q = 2$ a été montré par M.Keel et T.Tao [70]. En dimension $d = 3$, les paires extrémales ne sont pas admissibles et l'estimation (1.4) est fausse : on obtient une perte logarithmique de dérivées dans les estimations de Strichartz.

L'inégalité de Strichartz s'obtient en partant de l'inégalité de dispersion classique,

$$\|e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0\|_{L^\infty(\mathbb{R}^d)} \leq C_0 h^{-d}\gamma_d(\frac{t}{h})\|\Psi(h\sqrt{-\Delta})u_0\|_{L^1(\mathbb{R}^d)} \tag{1.5}$$

avec $C_0$ indépendant de $(t,x)$ et $h \in (0,1]$ et où on a posé

$$\gamma_d(\lambda) = \sup_{z\in\mathbb{R}^d}|\int e^{i\lambda(z\xi-|\xi|)}\Psi(|\xi|)d\xi| \simeq \lambda^{-\frac{(d-1)}{2}}. \tag{1.6}$$

Pour $q \in (2,\infty]$ et $r \in [2,\infty]$ on introduit

$$\frac{1}{q} = \alpha(\frac{1}{2}-\frac{1}{r}), \quad \beta = (d-\alpha)(\frac{1}{2}-\frac{1}{r}). \tag{1.7}$$

On note que la paire $(q,r)$ est $d$-admissible si $\alpha = \frac{d-1}{2}$. On a le résultat suivant :

**Lemme 1.1.** *Soit $\alpha \geq 0$ et soit $(q,r)$ une paire $\alpha$-admissible en dimension $d$ avec $q > 2$. Soit $\beta$ définit dans (1.7). Si la solution $e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0$ de (1.1) satisfait l'équation de dispersion (1.5) pour une fonction $\gamma_{d,h} : \mathbb{R} \to \mathbb{R}_+$, alors il existe une constante $C > 0$ indépendante de $h$ telle que pour $T > 0$ l'inégalité suivante soit vérifiée*

$$h^{\beta}\|e^{-it\sqrt{-\Delta}}\Psi(h\sqrt{-\Delta})u_0\|_{L^q((0,T),L^r(\mathbb{R}^d))} \leq C\Big(\sup_{s\in(0,\frac{T}{h})} s^{\alpha}\gamma_{d,h}(s))\Big)^{\frac{1}{2}-\frac{1}{r}}\|\Psi(h\sqrt{-\Delta})u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

## 1.2 Influence de la géométrie de l'espace

Dans le cas d'un domaine $\Omega$ à bord, pour les problèmes qui nous intéressent la difficulté principale vient du comportement des singularités au voisinage d'un point du bord. À l'intérieur de l'ouvert, ces singularités se propagent, d'après un théorème du à L.Hörmander [56], le long des rayons optiques (que l'on peut voir, en adjoignant la direction de propagation, comme des courbes du fibré cotangent). L'étude de la propagation au bord a été faite notamment par R.Melrose et J.Sjöstrand [77], [78] qui ont introduit la notion de *rayon bicaractéristique généralisé*, généralisant celle de rayon optique. Le cas le plus simple, correspondant aux lois, classiques, de l'optique géométrique, est celui des points pour lesquels le flot est transverse au bord (appelés *points hyperboliques*). Des difficultés apparaissent près des points où les rayons sont tangents au bord. Les points dits *diffractifs* sont ceux par lesquels passe un rayon optique qui frôle le bord sans être dévié. Pour décrire correctement la propagation, il faut également considérer des rayons qui restent dans le bord de $\Omega$, appelés rayons *glissants*, et les limites des rayons se rapprochant du bord et se réfléchissant un grand nombre de fois. La projection spatiale d'un tel rayon est une géodésique du bord.

On va rappeler quelques résultats antérieurs dans le cas des géométries à bord :

### 1.2.1 Variétés à bord strictement concave

Lorsque $\Omega$ est l'extérieur d'un obstacle strictement convexe, H.Smith et C.Sogge ont prouvé dans [95] la validité des estimations de Strichartz de l'espace libre. Ils ont utilisé pour cela la paramétrice pour la diffraction construite par R.Melrose et M.Taylor près des rayons tangents au bord de l'obstacle.

Si l'hypothèse de stricte concavité du bord est enlevée, la présence des rayons géodésiques multi-réfléchis et de leurs limites, les rayons glissants, ne permet pas d'avoir une telle paramétrice ! Remarquer qu'à l'extérieur d'un convexe, un point source ne génère jamais de caustiques, et que ce sont les caustiques générées en temps petit par des points sources près du bord qui posent problème à l'intérieur d'un convexe strict.

Nous rappelons brièvement la paramétrice construite par R.Melrose et M.Taylor à l'extérieur d'un convexe strict, qui va s'avérer un outil essentiel aussi pour montrer la validité des estimations de Strichartz pour l'équation de Schrödinger semi-classique à l'intérieur d'un compact à bord strictement concave, ainsi que des inégalités de Strichartz à l'extérieur d'un obstacle strictement convexe. Le résultat concernant l'équation de Schrödinger, qui va faire l'objet d'un chapitre prochain, est un résultat original de cette thèse.

**Proposition 1.2.** *(voir H.Smith et C.Sogge [95, 94], M.Zworski [111], R.Melrose et M.Taylor [81]) Soit* $(\Omega, g)$ *une variété Riemannienne compacte de dimension d à bord régulier strictement concave où l'extérieur d'un domaine strictement convexe. On considère l'équation des ondes dans* $\Omega$ *avec condition de Dirichlet sur le bord*

$$\begin{cases} \partial_t^2 w - \Delta_g w = 0, \quad \Omega \times \mathbb{R}_+, \\ w(x,0) = f(x) \in C^\infty(\Omega), \quad \partial_t w(x,0) = g(x) \in C^\infty(\Omega), \\ w|_{\partial\Omega \times \mathbb{R}_+} = 0, \end{cases} \tag{1.9}$$

*où* $\Delta_g$ *désigne l'opérateur de Laplace associé à la métrique* $g$ *de* $\Omega$*. Alors près d'un point diffractif, la solution sortante de* (1.9) *s'écrit, modulo des termes réguliers*

$$w(x,t) = \int_{\mathbb{R}^d} e^{i(\theta(x,\xi)+t\xi_1)}(a(x,\xi)A_+(\zeta(x,\xi)) + b(x,\xi)A'_+(\zeta(x,\xi))) \\ \times \frac{Ai(\zeta_0(\xi))}{A_+(\zeta_0(\xi))}\hat{K}(f,g)(\xi)d\xi, \tag{1.10}$$

*où les symboles a et b sont de type* $(1,0)$ *et d'ordre* $1/6$ *et* $-1/6$*, respectivement, supportés dans un voisinage conique de l'axe* $\xi_1$ *et* $K$ *est un opérateur intégral de Fourier classique d'ordre* $0$ *en* $f$ *et d'ordre* $-1$ *en* $g$*, à support compact. Les phase* $\theta$ *et* $\zeta$ *sont réelles, régulières et homogènes de degrés* $1$ *et* $2/3$*, respectivement. Ici* $Ai(\zeta)$ *est la fonction d'Airy et* $A_+(\zeta) = Ai(e^{-2\pi i/3}\zeta)$*.*

Si on choisit des coordonnées locales près d'un point diffractif dans lesquelles $\Omega$ est donné par $x_d > 0$, alors les phases $\theta$, $\zeta$ vérifient les équations eikonales

$$\begin{cases} \xi_1^2 - < d\theta, d\theta >_g + \zeta < d\zeta, d\zeta >_g = 0, \\ < d\theta, d\zeta >_g = 0, \\ \zeta(x', 0, \xi) = \zeta_0(\xi) = -\xi_1^{-1/3}\xi_d, \end{cases} \tag{1.11}$$

dans la région $\zeta \leq 0$. Ici $< .,. >_g$ désigne le produit scalaire induit par la métrique $g$.

### 1.2.2 Dispersion avec perte dans un convexe strict

G.Lebeau a été le premier à décrire les estimations de dispersion en temps petit pour les solutions de (1.1) dans un domaine strictement convexe de $\Omega \subset \mathbb{R}^d$. Plus précisément, il a montré dans [74, Thm.1.1] qu'une perte de dérivées par rapport à l'estimation de dispersion libre (1.5) est inévitable :

$$\|\chi(hD_t)u(t,.)\|_{L^\infty(\Omega)} \leq C_0 h^{-d} \min\{1, (h/t)^{(d-2)/2+1/4}\}\|u_0\|_{L^1(\Omega)}, \tag{1.12}$$

où $\chi(hD_t)u$ est une localisation de $u$ aux fréquences $hD_t \simeq 1$, $D_t = \frac{1}{i}\partial_t$, et où $t \in (0, T]$ pour un $T > 0$. L'estimation (1.12) signifie qu'on perd une puissance de $1/4$ de $h/t$ à l'intérieur d'un domaine strictement convexe par rapport à l'estimation de dispersion libre (1.5). On choisit d'écrire l'exposant de dispersion de la formule (1.12) sous la forme $\frac{(d-2)}{2} + \frac{1}{4}$ au lieu de $\frac{(d-1)}{2} - \frac{1}{4}$, car $d-1$ est la dimension du bord $\partial\Omega$, de sorte que (1.12) montre que l'effet de dispersion reste meilleur que la seule dispersion dans les variables tangentielles. En plus, pour $T > 0$, la perte d'une puissance $1/4$ de $h/t$ est optimale pour l'estimation $L^\infty(\Omega)$ uniforme, en raison de la présence de caustiques de type queue d'aronde dans le support singulier de $u$ dès que le point source est suffisamment proche de bord. La présence de caustiques en temps arbitrairement petit explique la perte par rapport à la dispersion libre.

Pourtant, ce résultat, optimal pour la dispersion, implique quand même des estimations de type Strichartz **sans pertes !** (en terme d'échelle) mais avec des indices modifiés. En utilisant l'argument $TT^*$ usuel, on déduit de (1.12) des estimations de Strichartz pour l'équation des ondes

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 \quad \text{dans} \quad \Omega \times \mathbb{R}, \\ u = 0 \quad \text{sur} \quad \partial\Omega \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad h\partial_t u(0,x) = u_1(x) \end{cases} \tag{1.13}$$

comme suit :

**Proposition 1.3.** *(G.Lebeau [74, Prop.1.2]) Soit $d \geq 2$. Pour tout $T > 0$ il existe $C > 0$ tel que pour tout $h \in (0,1]$ et toute solution $u$ de (1.13) localisée à fréquence spatiale $1/h$ on a*

$$h^\beta \|u\|_{L^q((0,T],L^r(\Omega))} \leq C(\|u_0\|_{L^2(\Omega)} + h\|u_1\|_{L^2(\Omega)}), \tag{1.14}$$

*où les indices vérifient*

$$\frac{1}{q} = (\frac{d-2}{2} + \frac{1}{4})(\frac{1}{2} - \frac{1}{r}), \quad \beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}.$$

Notons bien que $\alpha_d := \frac{d-2}{2} + \frac{1}{4}$ soit optimal pour l'estimation de dispersion (1.12), on ne sait pas quel est le $\alpha_d$ optimal pour l'estimation de Strichartz.

**Proposition 1.4.** *([74, Prop.1.4]) Soit $d \geq 3$ et $q \in (2, \infty]$, $r \in [2, \infty)$ des exposants tels que*

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1, \quad q \geq \frac{2d+3}{2d-3}.$$

*Pour tout $T > 0$ il existe $C > 0$ tel que pour tout $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, la solution $u$ de (1.13) vérifie l'inégalité de Strichartz*

$$\|u\|_{L^q((0,T],L^r(\Omega))} \leq C(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}). \tag{1.15}$$

Remarquer que l'estimation (1.15) est la même que celle de l'espace libre, sauf que les valeurs admissibles de $q$ sont restreintes à $q \geq \frac{2d+3}{2d-3}$ au lieu de $q > 2$ en dimension $d \geq 3$. Bien sûr, pour l'estimation (1.15) cela ne fait une différence que pour $d \in \{3, 4\}$.

### 1.2.3 Variétés compactes à bord

Les estimations des projecteurs spectraux de H.Smith et C.Sogge [96] ont été le point clé d'un travail récent de N.Burq, G.Lebeau et F.Planchon [28] (voir aussi [30]) permettant d'établir des inégalités de Strichartz pour un certain ensemble d'indices $(q, r, \beta)$. L'ensemble d'indices qui peuvent être obtenus de cette façon est pourtant soumis à des restrictions sur $r$, imposées par l'ensemble des valeurs admissibles dans l'estimation de la fonction carrée : en dimension $d = 3$ par exemple cela restreint les indices à $q, r \geq 5$.

Dans [15], M.Blair, H.Smith et C.Sogge généralisent les résultats de [28] et [74] et démontrent les estimations **sans pertes** (1.14) avec des indices qui vérifient

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \begin{cases} \frac{1}{q} \leq \frac{(d-1)}{3}(\frac{1}{2} - \frac{1}{r}), & d \leq 4, \\ \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, & d \geq 4. \end{cases} \tag{1.16}$$

La remarque clé de [96] est que la solution linéaire $u$ satisfait de meilleures estimations si elle n'est pas microlocalisée dans des directions tangentielles à $\partial\Omega$. Cela est du au fait qu'on peut construire des paramétrices sur de grands intervalles de temps lorsqu'on suit des trajectoires transversales au bord. Plus précisément, la paramétrice pour des directions d'angle $\simeq \theta$ qui ne sont pas tangentes au bord est valable pour des intervalles de temps de taille $\theta$, ce qui devrait impliquer des Strichartz avec une perte dépendant de $\theta$. Pourtant, cette perte peut être "neutralisée" par le fait que de telles directions vivent dans un petit cône dans l'espace des fréquences. Pour des estimations avec indices $(q, r)$ sous-critiques, c'est à dire pour lesquelles on a une inégalité stricte dans (1.3), cette localisation en fréquence apporte un gain pour de petits angles $\theta$. La restriction des indices $(q, r)$ dans (1.16) vient du fait qu'on impose que ce gain neutralise la perte venant de la sommation sur $\theta^{-1}$ intervalles de temps sur lesquels on a de bonnes estimations. Cette restriction est donc naturellement imposée par la nature locale de la construction de la paramétrice dans [96].

## 1.3 Stratégie possible. Modes de galerie

Une méthode pour caractériser des éventuelles pertes de dérivées serait d'utiliser les modes de galerie, qui accumulent leur énergie près du bord et contribuent à des normes $L^r(\Omega)$ élevées. On considère ici le cas du demi-espace

$$\Omega = \{(x, y) \in \mathbb{R}^d | x > 0, y \in \mathbb{R}^{d-1}\}$$

avec Laplacien de Dirichlet donné par

$$\Delta_D = \partial_x^2 + (1 + x) \sum_{j=1}^{d-1} \partial_{y_j}^2. \tag{1.17}$$

L'opérateur obtenu par la transformée de Fourier dans la variable tangentielle $y$, noté

$$-\Delta_{D,\eta} = -\partial_x^2 + (1 + x)|\eta|^2, \tag{1.18}$$

est autoadjoint, positif sur $L^2(\mathbb{R}_+)$ et à résolvante compacte à condition que $\eta \neq 0$. Il existe une base de fonctions propres $v_k(., \eta)$ de $-\Delta_{D,\eta}$ associées aux valeurs propres $\lambda_k(\eta) \to \infty$,

$$v_k(x, \eta) = Ai(|\eta|^{\frac{2}{3}} x - \omega_k), \tag{1.19}$$

où $(-\omega_k)_k$ dénotent les zéros de la fonction d'Airy en ordre décroissante.

*Définition* 1.1. Pour $x > 0$, soit $E_k(\Omega)$ la fermeture dans $L^2(\Omega)$ de l'ensemble

$$\{u(x, y) = \frac{1}{(2\pi)^{d-1}} \int e^{iy\eta} Ai(|\eta|^{\frac{2}{3}} x - \omega_k)\hat{\varphi}(\eta) d\eta, \varphi \in \mathcal{S}(\mathbb{R}^{d-1})\}, \tag{1.20}$$

où $\mathcal{S}(\mathbb{R}^{d-1})$ est l'espace de Schwartz des fonctions à décroissance rapide,

$$\mathcal{S}(\mathbb{R}^{d-1}) = \{f \in C^\infty(\mathbb{R}^{d-1}) | \|z^\alpha D^\beta f\|_{L^\infty(\mathbb{R}^{d-1})} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^{d-1}\}.$$

Pour $k$ fixé, une fonction $u \in E_k(\Omega)$ est appelée mode de galerie (ou onde de surface).

On peut décomposer $L^2(\Omega)$ en une somme directe de modes de galerie,

$$L^2(\Omega) = \bigoplus_{\perp k} E_k(\Omega).$$

Un résultat obtenu dans cette thèse ([60, Thm.2]) est le suivant :

**Proposition 1.5.** *Soit* $d \geq 2$, $A \in C_0^\infty(\mathbb{R}^{d-1} \setminus \{0\})$, $k \geq 1$ *est* $u_0 \in E_k(\Omega)$. *Soit* $(q, r)$ *une paire d-admissible avec* $q > 2$. *La solution* $u$ *de l'équation* (1.13) *avec le Laplacien de Dirichlet* $\Delta$ *introduit dans* (1.17) *et avec données initiales* $u_0 = A(hD_y)u_0$, $u_1 = 0$ *vérifie*

$$\|u\|_{L^q([0,T], L^r(\Omega))} \leq C h^{-d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \|A(hD_y)u_1\|_{L^2(\Omega)}.$$

*Ce résultat montre que la solution de l'équation des ondes* (1.13) *avec donnée initiale localisée près du bord et qui se propage le long du bord (ce type de donnée est bien illustré par les modes de galerie) vérifie les mêmes estimations de Strichartz (locales en temps) que celles de l'espace libre* (1.4). *D'autre part, lorsqu'on considère* (1.13) *avec données initiales supportées loin d'un voisinage fixe du bord, par vitesse finie de propagation on en déduit les même estimations* (1.4) *(localement en temps) ; on a le même résultat même si la donnée est près du bord mais l'onde est transverse au bord, de façon à ce qu'elle ne se réflechisse qu'une seule fois.*

Ce sont ces cas "extrêmes" qui ont laissé penser, pendant plusieurs années, qu'à l'intérieur d'un compact l'équation des ondes pourrait vérifier les mêmes estimations de Strichartz que celles de l'espace libre.

Un des résultats de cette thèse montre que, contrairement au fait que les cas "extrêmes" indiqueraient des Strichartz comme dans l'espace libre, une perte de dérivées est inévitable à l'intérieur d'un domaine strictement microlocalement convexe (où, plus généralement, un domaine avec au moins un point sur le bord et une bicaractéristique tangente au bord en ce point et qui a un contact exactement d'ordre 2 avec le bord) et cela apparaît à cause des caustiques générées en temps petit près du bord.

## 1.4 Résultats concernant l'équation des ondes à l'intérieur d'un domaine

On se place dans un domaine satisfaisant l'hypothèse 1.1 et on construit des suites de données initiales pour le problème (1.1), localisées dans un voisinage de taille dépendant de la fréquence, pour lesquelles les solutions correspondantes montrent qu'une perte de dérivées est inévitable dans les inégalités de Strichartz, au moins pour un sous ensemble d'indices admissibles.

### 1.4.1 Énoncé du résultat principal

*Hypothèse* 1.1. Soit $(\Omega, g)$ une variété Riemannienne de dimension $d \geq 2$ compacte, à bord régulier telle qu'en un point du bord il existe au moins une bicaractéristique qui passe par ce point et a un contact exactement d'ordre deux avec le bord $\partial\Omega$ en ce point. On note $\Delta_g$ l'opérateur de Laplace associé à la métrique $g$ de $\Omega$.

Le résultat principal concernant l'équation des ondes est le suivant :

**Théorème 1.1.** *(O.I. [61, 62]) Sous l'hypothèse 1.1, si* $(q, r)$ *est une paire admissible pour l'équation des ondes en dimension* $d$ *avec* $r > 4$, *alors pour tout* $\epsilon > 0$ *suffisamment petit il existe des suites de données initiales* $V_{h,j,\epsilon} \in C^\infty(\bar{\Omega})$, $j = \overline{0,1}$ *telles que la solution* $V_{h,\epsilon}$

*de l'équation des ondes avec condition de Dirichlet*

$$\begin{cases} (\partial_t^2 - \Delta_g) V_{h,\epsilon} = 0, \\ V_{h,\epsilon}|_{t=0} = V_{h,0,\epsilon}, \quad \partial_t V_{h,\epsilon}|_{t=0} = V_{h,1,\epsilon}, \\ V_{h,\epsilon}|_{\partial\Omega \times [0,1]} = 0, \end{cases} \tag{1.21}$$

*vérifie pour* $j \in \{0, 1\}$

$$\sup_{\epsilon > 0, h} \|V_{h,j,\epsilon}\|_{\dot{H}^{\frac{(d+1)}{2}(\frac{1}{2}-\frac{1}{r})+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-2\epsilon-j}(\Omega)} \leq 1, \quad \lim_{h\to 0} \|V_{h,\epsilon}\|_{L_t^q([0,1],L^r(\Omega))} = \infty. \tag{1.22}$$

*Remarque* 1.1. Le résultat est pertinent en dimensions $d \in \{2, 3, 4\}$ seulement, car pour $d \geq 5$ il n'existe pas de paires $(q, r)$ admissibles avec $r > 4$.

On présente ici la démarche suivie dans le cas du demi-espace

$$\Omega = \{(x, y) \in \mathbb{R}^2, x > 0, y \in \mathbb{R}\} \subset \mathbb{R}^2$$

avec Laplacien donné par $\Delta_D = \partial_x^2 + (1 + x)\partial_y^2$. Notons d'abord que $\Omega$ avec la métrique héritée de $\Delta_D$ devient un domaine strictement convexe et, en particulier, satisfait l'hypothèse 1.1.

Dans ce cadre l'analyse est simplifiée par la présence des translations en $(y, t)$, grâce auxquelles tous les calculs sont explicites. Après l'esquisse de la preuve dans ce cas modèle de Friedlander qui contient les idées principales de la construction on va donner l'ingrédient qui permet, dans le cas d'un domaine satisfaisant l'hypothèse 1.1, de se réduire à appliquer les arguments de la dimension 2.

### 1.4.2 Esquisse de preuve pour le modèle de Friedlander

L'hypothèse de stricte convexité du bord implique la présence, en tout point de l'espace cotangent $T^*(\bar{\Omega} \times \mathbb{R})$ d'une bicaractéristique tangente au bord avec contact exactement d'ordre deux (et qui reste dans le complémentaire de $\Omega$) : si on note

$$p(x, y, t, \xi, \eta, \tau) = \xi^2 + (1 + x)\eta^2 - \tau^2$$

le symbole de $\Delta_D$ et si $(\rho_0, \vartheta_0) = (0, y, t, \xi, \eta, \tau)$ est un tel point *diffractif*, cela se traduit par

$$p|_{(\rho_0,\vartheta_0)} = 0, \quad \{p, x\}|_{(\rho_0,\vartheta_0)} = 2\xi = 0, \quad \{\{p, x\}, p\}|_{(\rho_0,\vartheta_0)} = 2\frac{\partial p}{\partial x} > 0, \tag{1.23}$$

où $\{.,.\}$ désigne le crochet de Poisson. On peut supposer que $(\rho_0, \vartheta_0) = (0, 0, 0, 0, 1, -1)$. Dans tout voisinage d'un point diffractif il y a des points hyperboliques : on se place dans un voisinage conique, hyperbolique de $(\rho_0, \vartheta_0)$ et on regarde des points $(\rho, \vartheta)$ tels que

$$p(\rho, \vartheta)|_{\partial\Omega} = \xi^2 + (1 + x)\eta^2 - \tau^2|_{x=0} = a\eta^2, \quad (\rho, \vartheta) = (x, y, t, \xi, \eta, \tau),$$

où $a = h^\delta$, $0 < \delta < 2/3$ à déterminer. Dans un certain sens $a$ mesure la "distance" au point diffractif $(\rho_0, \vartheta_0)$.

**Paramétrice :** On commence par chercher une solution approchée du problème avec condition de Dirichlet

$$\begin{cases} \partial_t^2 V - \partial_x^2 V - (1+x)\partial_y^2 V = 0, & \text{dans} \quad \Omega \times \mathbb{R}, \\ V|_{\partial\Omega\times[0,1]} = 0, \end{cases} \tag{1.24}$$

sous la forme

$$u_h(x,y,t) = \int_{\xi,\eta,\tau} e^{\frac{i}{h}(y\eta+t\tau+(x+1-\frac{\tau^2}{\eta^2})\eta\xi+\eta\frac{\xi^3}{3})} g(t,\xi,\eta,\tau,h)\Psi(\eta)\delta(\tau=-\eta(1+a)^{1/2})d\xi d\eta d\tau, \tag{1.25}$$

où le symbole $g$ est à déterminer, $\delta$ est l'application de Dirac en 0 et où $\Psi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ est une fonction régulière, à support dans un petit voisinage de 1, $0 \leq \Psi(\eta) \leq 1$. Ce choix est motivé par le fait que la transformée de Fourier $\hat{u}(x,\eta,\tau)$ s'obtient comme solution d'une équation d'Airy. La phase est homogène d'ordre 1 en $(\eta,\tau)$ et s'écrit

$$\Phi(x,y,t,\xi,\eta,\tau) = t\tau + y\eta + (x+1-\frac{\tau^2}{\eta^2})\eta\xi + \eta\frac{\xi^3}{3}. \tag{1.26}$$

La lagrangienne associée est donnée par

$$\Lambda_\Phi = \{(x,y,t,\xi,\eta,\tau)|\xi^2 + (x+1-\frac{\tau^2}{\eta^2}) = 0, \partial_\eta\Phi = 0\} \subset T^*(\bar{\Omega}\times\mathbb{R})\setminus o. \tag{1.27}$$

On note que hors de tout voisinage de $\Lambda_\Phi$ la contribution de $u_h$ est $O_{L^2}(h^\infty)$, car le front d'onde de $u_h$ satisfait $WF_h(u_h) \subset \Lambda_\Phi$. Soit $\pi : \Lambda_\Phi \to \bar{\Omega}\times\mathbb{R}$ la projection canonique (dans l'espace physique). On introduit $\Sigma$ comme étant l'ensemble des points singuliers de $\pi$, c'est à dire pour lesquels le Jacobien de la matrice $d\pi$ s'annule, et on trouve $\Sigma = \{\xi = 0\}$. La *caustique* $\mathcal{C}$ est définie comme l'image par la projection $\pi$ de l'ensemble singulier, $\mathcal{C} = \pi(\{\xi = 0\}) = \{x = a\}$. Si on regarde la projection dans l'espace $(x, y-(1+a)^{1/2}t)$ on remarque une singularité de type *cusp* en $x = a$, car sur la lagrangienne on a

$$y - (1+a)^{1/2}t = \pm\frac{2}{3}(a-x)^{3/2}. \tag{1.28}$$

Lorsqu'on regarde la trace sur le bord, la distribution de Dirac $\delta(\tau = -\eta(1+a)^{1/2})$ a le rôle important de localiser loin de la caustique, et la lagrangienne $\Lambda_\Phi|_{\partial\Omega}$ devient ainsi le graphe d'une paire de transformations canoniques ; il s'agit des *applications de billard* $\delta_\pm : T^*(\partial\Omega\times\mathbb{R}) \to T^*(\partial\Omega\times\mathbb{R})$, définies dans la région hyperbolique, continues jusqu'au bord, régulières à l'intérieur et définies en un point hyperbolique en suivant les bicaractéristiques issues de ce point jusqu'à ce qu'elles croisent de nouveau le bord. Dans le modèle de Friedlander, grâce aux translations en temps et en variable tangentielle, les applications de billard sont données par des formules explicites,

$$\delta^\pm(y,t,\eta,\tau) = (y \pm 4(\frac{\tau^2}{\eta^2}-1)^{1/2} \pm \frac{8}{3}(\frac{\tau^2}{\eta^2}-1)^{3/2}, t \mp 4(\frac{\tau^2}{\eta^2}-1)^{1/2}\frac{\tau}{\eta}, \eta, \tau). \tag{1.29}$$

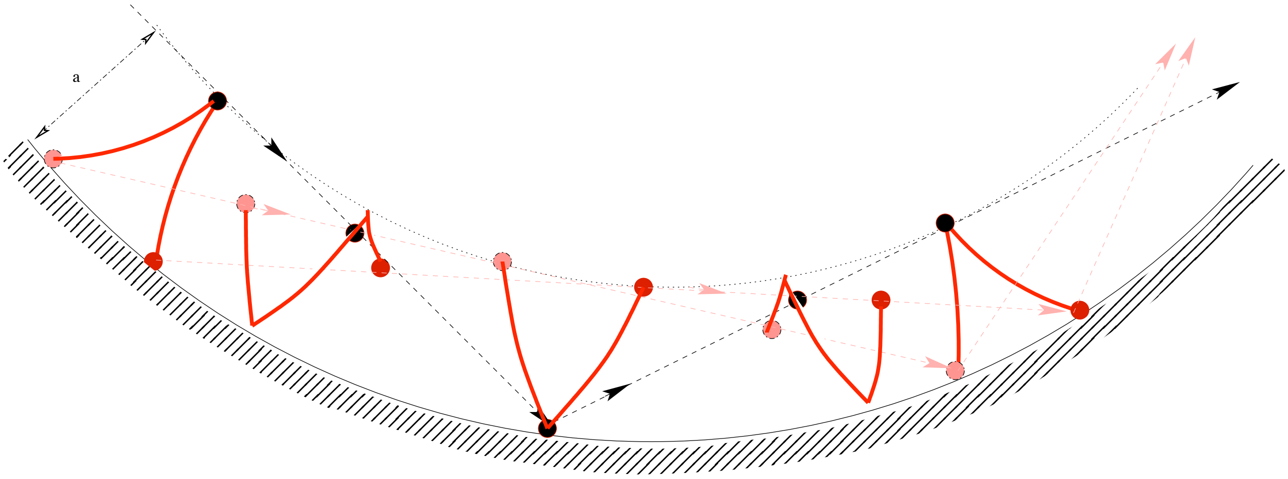



FIGURE 1 – Propagation du cusp

La relation obtenue en composant $n$ fois

$$\Lambda_{\Phi|_{x=0}} \circ ... \circ \Lambda_{\Phi|_{x=0}}$$

possède toujours loin de la caustique $n+1$ composantes, obtenues en utilisant les graphes des itérées

$$(\delta^{\pm})^n(y,t,\eta,\tau) = (y \pm 4n(\frac{\tau^2}{\eta^2}-1)^{1/2} \pm \frac{8}{3}n(\frac{\tau^2}{\eta^2}-1)^{3/2}, t \mp 4n(\frac{\tau^2}{\eta^2}-1)^{1/2}\frac{\tau}{\eta}, \eta, \tau). \quad (1.30)$$

Comme $\Lambda_{\Phi|_{x=0}}$ admet comme paramétrisation $y\eta + t\tau + \frac{4}{3}\eta(\frac{\tau^2}{\eta^2}-1)^{3/2}$, les lagrangiennes obtenues après $n$ itérations $(\Lambda_{\Phi|_{x=0}})^{\circ n}$ seront paramétrées par

$$y\eta + t\tau + \frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2}$$

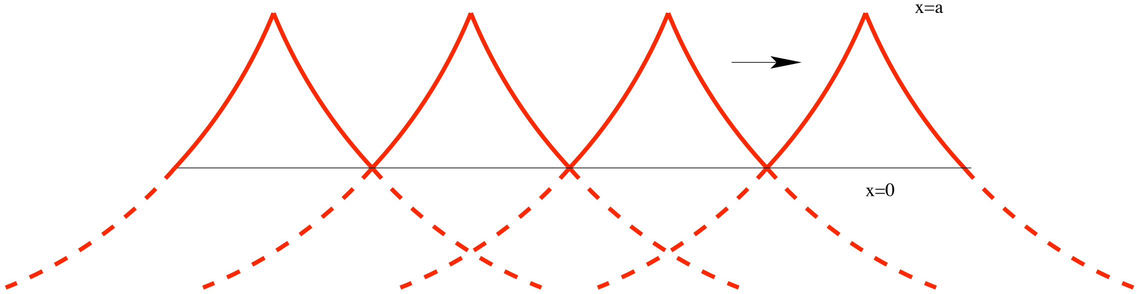
et les phases $\Phi_n$ associées aux $(\Lambda_{\Phi})^{\circ n}$ correspondantes seront données par

$$\Phi_n(x,y,t,\xi,\eta,\tau) := t\tau + y\eta + (x+1-\frac{\tau^2}{\eta^2})\eta\xi + \eta\frac{\xi^3}{3} + \frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2}. \quad (1.31)$$

**Choix du symbole :** On va déterminer le symbole $g$ tel qu'il vérifie deux conditions :

1. Il doit résoudre la première équation de transport dans

$$\partial_t^2 u_h - \Delta_D u_h = \int e^{\frac{i}{h}\Phi}\Big(\partial_t^2 g + \frac{i}{h}\eta(\partial_\xi g - 2(1+a)^{1/2}\partial_t g)\Big) d\xi d\eta.$$




2. Au temps $t = 0$ il doit être supporté dans un petit voisinage $V$ de la caustique : en fait si en $t = 0$ on se localise près de $\mathcal{C}$, le petit bout de cusp défini dans (1.28) où $u_h$ va "vivre" (c'est à dire hors duquel la contribution de $u_h$ sera $O_{L^2}(h^\infty)$) va se propager jusqu'à atteindre le bord, mais très peu de temps après il va disparaître. Cette deuxième condition va donc assurer le fait que $u_h$ vit sur un intervalle de temps de taille $a^{1/2}$.

*Définition* 1.2. Soit $\lambda \geq 1$. Pour un compact $K \subset \mathbb{R}$ on introduit l'espace $\mathcal{S}_K(\lambda)$ des fonctions $\varrho(z,\lambda) \in C^\infty(\mathbb{R})$ qui vérifient

1. $\sup_{z\in\mathbb{R},\lambda\geq 1} |\partial_z^\alpha \varrho(z,\lambda)| \leq C_\alpha$, où $C_\alpha$ ne dépendent pas de $\lambda$,
2. Si $\psi(z) \in C_0^\infty$ est égale à 1 dans un voisinage de $K$, $0 \leq \psi \leq 1$ alors $(1-\psi)\varrho \in O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$.

On prend $0 < c_0 \ll 1$ suffisamment petit, fixé, $\lambda = a^{3/2}/h$ et $\varrho \in \mathcal{S}_{[-c_0,c_0]}(\lambda)$ et on pose

$$g(t,\eta,\xi,h) := \varrho(\frac{t + 2(1+a)^{1/2}\xi}{2(1+a)^{1/2}a^{1/2}}, \lambda).$$

On vérifie que ce symbole satisfait les deux conditions précédentes avec $V = [(1-c_0^2)a, a]$. Pour le moment on a construit une solution approchée de (1.24) localisée à fréquence $1/h$ en variable tangentielle $y$, supportée essentiellement en variable normale $x$ dans un voisinage de taille $A$ du bord et en temps dans un intervalle

$$[-2(1+c_0)(1+a)^{1/2}a^{1/2}, 2(1+c_0)(1+a)^{1/2}a^{1/2}].$$

En effet, si $x > a$ ou bien si $t$ n'appartient pas à un tel intervalle (ce qui implique, grâce à la deuxième condition sur le support du symbole $g$, que $|\xi| > a^{1/2}$) on remarque qu'on quitte un voisinage de la lagrangienne $\Lambda_\Phi$ qui contient le front d'onde de $u_h$.

Dans ce qui suit on utilise les itérées de l'application de billard $\delta_+$ pour construire des solutions approchées $u_h^n$ de la forme

$$u_h^n(x,y,t) = \int_{\xi,\eta} e^{\frac{i}{h}\Phi_n(x,y,t,\xi,\eta,-\eta(1+a)^{1/2})} g_n(t,\xi,\eta,h)\Psi(\eta)d\xi d\eta, \quad u_h^0(x,y,t) = u_h(x,y,t), \tag{1.32}$$

où les phases $\Phi_n$ sont données par (1.31) et paramètrent les graphes des itérées $(\delta_+)^{\circ n}$ et où $\tau = -(1+a)^{1/2}\eta$. Il reste à déterminer les symboles $g_n$ solutions des (premières) équations de transport

$$\partial_\xi g_n - 2(1+a)^{1/2}\partial_t g_n = 0,$$

tels que chaque $u_h^n$ vit essentiellement sur un intervalle de temps

$$I_n := [-2(1+c_0)(1+a)^{1/2}a^{1/2} + 4n(1+a)^{1/2}a^{1/2}, 2(1+c_0)(1+a)^{1/2}a^{1/2} + 4n(1+a)^{1/2}a^{1/2}]$$

et soit localisé en $t = 4n(1+a)^{1/2}a^{1/2}$ dans un petit voisinage de la caustique $\{x = a\}$ (de la même taille que le voisinage $V$ correspondant à $u_h$). En fait dans ce cas, si on définit

$$U_h(x, y, t) := \sum_{n=0}^{N} u_h^n(x, y, t), \tag{1.33}$$

où $N$ est un entier tel que $a^{1/2}N \simeq 1$, alors $U_h$ est une solution approchée de (1.24) supportée dans un intervalle de temps fixe (indépendant de $h$) et (quitte à restreindre encore le voisinage $V$) on va pouvoir estimer

$$\|U_h\|^q_{L^q([0,1],L^r(\Omega))} \geq \sum_{k\leq N/5} \int_{J_k} \|\sum_{n=0}^{N} u_h^n(.,t)\|^q_{L^r(\Omega)} dt + O(h^\infty), \tag{1.34}$$

où $J_k = [-c_0(1+a)^{1/2}a^{1/2}+4k(1+a)^{1/2}a^{1/2}, c_0(1+a)^{1/2}a^{1/2}+4k(1+a)^{1/2}a^{1/2}]$ et où, grâce aux propriétés des supports des symboles $g_n$, pour $t \in J_k$ il existe un seul $u_h^n(.,t)$ (correspondant à $n = k$) dont la contribution n'est pas $O_{L^2}(h^\infty)$ dans la somme $\sum_{n=0}^{N} u_h^n(.,t)$. Un calcul explicite montre que les normes $L^r(\Omega)$ des $u_h^n$ sont équivalentes et en particulier pour tout $n$ et $t \in I_n$ on a

$$\|u_h^n(.,t)\|_{L^r(\Omega)} \simeq \begin{cases} h^{\frac{1}{r}+\frac{1}{2}}a^{\frac{1}{r}-\frac{1}{4}}, & 2 \leq r < 4, \\ h^{\frac{1}{3}+\frac{5}{3r}}, & r > 4, \end{cases} \tag{1.35}$$

d'où on peut facilement minorer $\|U_h\|_{L^q([0,1])L^r(\Omega))}$ par $Ch^{-\frac{2}{3}+\frac{5}{3r}-\frac{\delta}{4}}\|U_h|_{t=0}\|_{L^2(\Omega)}$, où $C$ est une constante indépendante de $h$.

Il reste à déterminer $g_n$ telle qu'en plus la condition de Dirichlet sur le bord soit satisfaite. Chaque $u_h^n(0,.)$ s'écrit sur le bord comme une somme de deux opérateurs de trace $Tr_\pm(g^n)(y,t)$ localisés pour $y - (1+a)^{1/2}t + \frac{4}{3}na^{3/2}$ près de $\pm\frac{2}{3}na^{3/2}$. Pour obtenir une contribution $O_{L^2}(h^\infty)$ de $U_h|_{t\in[0,1]}$ restreinte au bord on choisit les symboles tels que pour tout $0 \leq n \leq N-1$ on ait

$$Tr_-(g^n) + Tr_+(g^{n+1}) = O_{L^2}(h^\infty). \tag{1.36}$$

**Choix des symboles $g_n$ :** Pour chaque $n$ on veut que le symbole $g^n$ satisfasse les mêmes propriétés que $g_0$ : résoudre la première équation de transport et être supporté en $t = 4n(1+a)^{1/2}a^{1/2}$ dans un petit voisinage de la caustique $V_n$, *indépendent de* $n$ : en fait, si la taille de $V_n$ devient de plus en plus grande avec $n$, les bouts de cusps $u_h^n$ commencent à "vivre" de plus en plus longtemps et après un certain nombre d'itérations leurs supports en temps commencent à interférer ce qui empêche de déterminer une borne inférieure de la norme $L^q$ en temps de la somme $U_h$ à partir seulement d'une estimation des normes des $u_h^n$, qui est la seule information de laquelle on dispose (par calcul explicite). Par conséquence, le seul choix possible de $g^n$ serait de la forme

$$g_n(t,\eta,\xi,h) := \varrho^n(\frac{t+2(1+a)^{1/2}\xi}{2(1+a)^{1/2}a^{1/2}} - 2n, \eta, \lambda),$$

avec $\varrho^n(.,\eta,\lambda)$ supportés (pour tout $n$ !) dans un voisinage *fixe* de 0, $[-c_0, c_0]$ pour $c_0$ petit. Si ce choix est possible, tout en satisfaisant la condition (1.36) au bord, il assure que chaque $u_h^n$ est essentiellement supporté dans la variable $t$ dans l'intervalle de temps $I_n$ et on conclut par (1.34) en utilisant (1.35).

**Condition au bord :** Il reste à voir si on peut déterminer $\varrho^n$ qui satisfassent les conditions énoncées plus haut : on calcule explicitement, pour $\lambda = a^{3/2}/h$

$$Tr_\pm(\varrho^n)(y,t) = 2\pi\sqrt{\frac{a}{\lambda}}\int e^{i\frac{\eta}{h}(y-t(1+a)^{1/2}+\frac{4}{3}na^{3/2}\mp\frac{2}{3}a^{3/2})}\eta^{-1/2}\Psi(\eta) \\ \times I_\pm(\varrho^n)_\eta(\frac{t}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda)d\eta, \quad (1.37)$$

où, pour $\eta \in \text{supp}(\Psi)$, $I_{\pm,\eta}$ sont donnés par

$$I_\pm(\varrho^n(.,\eta,\lambda))_\eta(z,\lambda) = e^{\pm i\pi/2 - i\pi/4}\frac{\eta\lambda}{2\pi}\int_{w,z'} e^{i\eta\lambda(w(z-z')\mp\frac{2}{3}((1-w)^{3/2}-1))} \\ \times \kappa(w)a_\pm(w,\eta\lambda)\varrho^n(z',\eta,\lambda)dz'dw, \quad (1.38)$$

où $a_\pm$ sont des symboles elliptiques et où $\kappa$ est supportée près de 0. Les opérateurs $I_{\pm,\eta}$ sont elliptiques et $I_{\pm,\eta} : \mathcal{S}_{[-c_0,c_0]}(\lambda) \to \mathcal{S}_{[-c_0\mp 1, c_0\mp 1]}(\lambda)$. La condition (1.36) se traduit par

$$e^{\frac{i\pi}{2}}I_-(T_1(\varrho^n(.,\eta,\lambda)))_\eta + e^{-\frac{i\pi}{2}}I_+(T_{-1}(\varrho^{n+1}(.,\eta,\lambda)))_\eta = O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}),$$

où $T_{\pm 1}$ sont les opérateurs de translation qui à $\varrho(z)$ associent $\varrho(z \pm 1)$. On a en plus

$$((I_{+,\eta})^{-1} \circ I_{-,\eta})^{\circ n} : \mathcal{S}_{[-c_0,c_0]}(\lambda) \to \mathcal{S}_{[-c_0+2n, c_0+2n]}(\lambda).$$

Comme les opérateurs $I_{\pm,\eta}$ ont des noyaux de convolution ils commutent par conséquent avec les translations et si on définit pour $0 \le n \le N$

$$\varrho^n(z-2n,\eta,\lambda) := ((I_{+,\eta})^{-1} \circ I_{-,\eta})^{\circ n}(\varrho(.,\lambda))(z),$$

on voit facilement que la condition (1.36) est vérifiée et que $\varrho^n \in \mathcal{S}_{[-c_0,c_0]}(\lambda/(n+1))$. Pour qu'en dehors de l'intervalle $[-c_0, c_0]$ la contribution de $\varrho^n(z-2n,\eta,\lambda)$ soit $O_{\mathcal{S}(\mathbb{R})}(h^\infty)$ (comme celle de $\varrho(z,\lambda)$) il faut imposer aussi $\lambda/N \le h^\epsilon$ pour un $\epsilon > 0$ si petit qu'on veut. Cette contrainte, ainsi la condition $a^{1/2}N \simeq 1$ qui assure une construction sur un temps $[0,1]$, impliquent

$$a \simeq h^{(1-\epsilon)/2} \quad \text{et aussi} \quad N \simeq h^{-(1-\epsilon)/4}.$$

## 1.5 Généralisation du résultat pour un domaine satisfaisant l'hypothèse 1.1

On va juste énoncer les étapes principales de la construction d'un contre-exemple en dimension $d \geq 2$ pour une variété $(\Omega, g)$ satisfaisant l'hypothèse 1.1.

### 1.5.1 Réduction à la dimension 2

D'abord, on montre qu'il suffit de prouver le Théorème 1.1 pour $d = 2$ : en fait on se donne un point du bord par lequel passe une bicaractéristique tangente au bord ayant un contact d'ordre 2 et on introduit des coordonnées géodésques sur le bord $y = (y_1, y') \in \mathbb{R}^{d-1}$ telles que $y_1$ indique la direction de la bicaracthéristique considérée et on considère l'équation

$$\begin{cases} (\partial_t^2 - \partial_x^2 - (1 + xb(y_1))\partial_{y_1}^2)V = 0, \\ V|_{\partial\Omega\times[0,1]} = 0, \end{cases} \tag{1.39}$$

où on a supposé (ce qui est toujours possible) que la métrique $g$ vérifie

$$g(0, y, \xi, \eta) = \xi^2 + \sum_{j,k=1}^{d-1} B_{j,k}(0, y)\eta_j\eta_k,$$

avec $B_{1,1}(0, y) = 1$, $B_{1,k}(0, y) = 0$ pour $j \in \{2, .., d-1\}$ et où on a définit $b(y_1) := \partial_x B_{1,1}(0, y_1, 0) > 0$ ce qui revient à dire que le point $(x = 0, y = 0, \xi = 0, \eta_1 = 1, \eta' = 0)$ vérifie l'hypothèse 1.1.

Supposons qu'on a construit une solution approchée $u_h(x, y_1, t)$ de l'équation (1.39) pour laquelle on obtient une perte d'au moins $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ dérivées dans l'inégalité de Strichartz. Il suffit ensuite de définir

$$V(x, y, t) := h^{-(d-2)/4} u_h(x, y_1, t) e^{-\frac{|y'|^2}{2h}} \chi(y'),$$

où $\chi$ localise dans une carte de coordonnées ; comme le facteur exponentiel de $V$ localise dans un voisinage tubulaire de taille $h^{1/2}$ de $y' = 0$ et en plus il sature les inégalités de Stricharz en dimension $d-2$, $V$ va être aussi une solution approchée de (1.21) et le terme de reste dans l'équation vérifiée par $V$ sera suffisamment petit en norme $L^q([0, 1], L^r(\Omega))$ pour ne pas compter dans l'inégalité de Strichartz.

### 1.5.2 Construction de solutions approchées en dimension 2

On cherche des solutions approchées de (1.39) sous la forme

$$u_h^n(x, y_1, t) = \int_{\xi,\eta} e^{\frac{i}{h}\Phi^n(x,y_1,t,\xi,\eta,-\eta(1+a)^{1/2})}\Psi(\eta) \times g_h^n\Big(\frac{\partial_\tau\theta + \xi\partial_\tau\zeta}{\partial_\tau\zeta_0(-\zeta_0)^{1/2}} - 2n, y, \eta\Big)\Big|_{(x,y_1,t,\eta,-\eta(1+a)^{1/2})} d\xi d\eta, \quad (1.40)$$

oú les phases $\Phi^n$ sont de la forme

$$\Phi^n(x, y_1, t, \xi, \eta, \tau) = \theta + \xi\zeta + \frac{\xi^3}{3} + \frac{4}{3}n(-\zeta_0)^{3/2}|_{(x,y_1,t,\eta,\tau)}, \quad (1.41)$$

avec $\theta$ et $\zeta$ homogènes d'ordres 1 et 2/3, respectivement, en $(\eta, \tau)$ et où la restriction au bord $\zeta_0$ coïncide avec celle du cas modèle, $\zeta_0 := \zeta|_{x=0} = -(\frac{\tau^2}{\eta^2} - 1)\eta^{2/3}$ (et, en particulier, cette restriction est indépendante de la variable tangentielle $y_1$).

La construction de telles fonctions de phases $\theta$ et $\zeta$ est possible grâce à un résultat de R.Melrose ([5]) et constitue un outil essentiel de la preuve car pour garder les mêmes propriétés (sur la localisation des supports et l'estimation des normes $L^r$) pour chaque $u_h^n$ après un très grand nombre de réflections $n$ on ne peut procéder qu'avec une forme normale de la phase (près d'un point diffractif) qui est donnée justement par (1.41).

Les symboles vérifient la première équation de transport (car $\partial_\tau\theta + \xi\partial_\tau\zeta$ est une courbe intégrale du champs de vecteurs qui y apparaît) et $\Psi(\eta)$ est supporté, comme dans le cas modèle, pour $\eta$ près de 1.

### 1.5.3 Condition au bord

Chaque symbole $g_h^n$ est déterminé par la condition au bord : plus précisément, si $J$ est un opérateur intégral de Fourier elliptique dont la relation canonique conjugue l'application de billard $\delta^+$ définie dans (1.29) à celle du cas général, on choisit $g_h^n|_{x=0}$ tel que

$$u_h^n|_{x=0} = \sum_{\pm} J \circ Tr_\pm(\varrho^n),$$

où on rappelle que $Tr_\pm$ sont les opérateurs de trace qui apparaissent dans la restriction au bord de la solution du problème du cas modèle (de Friedlander) et sont définis dans (1.37). Ainsi, après avoir appliqué plusieurs fois la formule de la phase stationnaire, on déduit une formule explicite de $g_h^n$ comme une somme asymptotique en $a^{1/2}/h$ des dérivées de $\varrho^n$.

Le reste de la preuve découle en suivant essentiellement les pas du cas modèle.

# 2 Équation de Schrödinger

## 2.1 L'équation de Schrödinger classique dans $\mathbb{R}^d$. Inégalités de Strichartz

On se place dans le cadre $\Omega = \mathbb{R}^d$ avec $d \geq 2$ et on note $\Delta$ l'opérateur de Laplace sur $\mathbb{R}^d$. On définit une paire $d$-admissible pour l'équation de Schrödinger un couple $(q,r)$ tel que $(q,r,d) \neq (2,\infty,2)$, $q,r \geq 2$ et

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{2.1}$$

Soit $u_0 \in L^2(\mathbb{R}^d)$. On considère le problème de Schrödinger

$$i\partial_t u + \Delta u = 0, \quad u|_{t=0} = u_0. \tag{2.2}$$

Si on note $u(x,t) = e^{it\Delta}u_0(x)$ le flot linéaire, grâce à la transformée de Fourier on obtient la forme exacte de la solution de (2.2)

$$u(x,t) = \frac{1}{4\pi i|t|^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy. \tag{2.3}$$

L'inégalité de Strichartz pour l'équation de Schrödinger (2.2) s'écrit comme suit

**Proposition 2.1.** *Soit $(q,r)$ une paire $d$-admissible pour l'équation de Schrödinger en dimension $d \geq 2$. Si $u$ est solution de (2.2) avec donnée initiale $u_0 \in L^2(\mathbb{R}^d)$, alors il existe une constante $C > 0$ telle que*

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

Grâce à l'invariance de l'équation linéaire par le changement d'échelle $(t,x) \to (\lambda^2 t, \lambda x)$, on déduit de l'inégalité de Strichartz (2.4) que la condition (2.1) sur le couple $(q,r)$ est nécessaire.

Les inégalités de Strichartz pour l'équation de Schrödinger dans le contexte classique ont été établies par R.H.Strichartz et généralisées par J.Ginibre et G.Velo [49, 50, 51], L.Kapitanski [65], T.Cazenave et F.Weissler [33], T.Kato [68], etc, qui les ont ensuite utilisées pour montrer des résultats d'existence globale pour l'équation semi-linéaire avec non-linéarité polynomiale. Elles se démontrent à partir de l'inégalité de dispersion

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}, \tag{2.5}$$

qui se déduit facilement dans le cas euclidien de la forme explicite de $u$, en utilisant la conservation de l'énergie, le théorème d'interpolation de Riesz-Thorin et l'argument $TT^*$.

### 2.1.1 Équation de Schrödinger semi-classique

Dans ce qui suit, on va aborder l'équation de Schrödinger par des méthodes semi-classiques, permettant le travail dans des coordonnées de carte avec des données localisées spectralement. Soit $\Psi \in C_0^\infty(\mathbb{R}^*)$ et $h \in (0,1]$. Dans l'équation (2.2) on introduit le temps semi-classique $t = hs$ et on considère une donnée initiale localisée à fréquence $1/h$, c'est à dire telle que $u_0 = \Psi(-h^2\Delta)u_0 \in L^2(\mathbb{R}^d)$. Si on pose $v(x,s) = u(x,hs)$, on obtient le problème semi-classique suivant

$$ih\partial_s v + h^2\Delta v = 0, \quad v|_{s=0} = \Psi(-h^2\Delta)u_0. \tag{2.6}$$

Si on note $v(x,s) = e^{ish\Delta}\Psi(-h^2\Delta)u_0(x)$ le flot semi-classique linéaire, la solution de (2.6) s'écrit

$$e^{ihs\Delta}\Psi(-h^2\Delta)u_0(x) = \frac{1}{(2\pi h)^d}\int e^{\frac{i}{h}(<x,\xi>-s|\xi|^2)}\Psi(|\xi|^2)\hat{u_0}(\frac{\xi}{h})d\xi. \tag{2.7}$$

Les inégalités de Strichartz pour l'équation de Schrödinger semi-classique (2.6) s'écrivent

$$h^{\frac{d}{2}(\frac{1}{2}-\frac{1}{r})}\|e^{ihs\Delta}\Psi(-h^2\Delta)u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|\Psi(-h^2\Delta)u_0\|_{L^2(\mathbb{R}^d)}, \tag{2.8}$$

où $(q,r)$ une paire $d$-admissible pour l'équation de Schrödinger en dimension $d \geq 2$. L'inégalité de dispersion pour le problème semi-classique devient

$$\|e^{ihs\Delta}\Psi(-h^2\Delta)u_0\|_{L^\infty(\mathbb{R}^d)} \leq C(h|s|)^{-d/2}\|\Psi(-h^2\Delta)u_0\|_{L^1(\mathbb{R}^d)}. \tag{2.9}$$

**Vitesse finie de propagation pour l'équation de Schrödinger semi-classique :** On rappelle la méthode de G.Lebeau [73] pour montrer la vitesse finie de propagation de (2.6). Soit $\Omega$ un ouvert borné de $\mathbb{R}^d$ à frontière régulière $C^\infty$ et soit $\Delta_D$ l'opérateur de Laplace dans $\Omega$. On note $(e_\nu, \lambda_\nu)_{\nu\geq 1}$ un système orthonormal dans $L^2(\Omega)$ de fonctions propres de $-\Delta_D$ avec condition de Dirichlet sur $\partial\Omega$. Soit $\alpha \in (0,1/2)$ et $\beta > 2$ ; on note

$$J_k := \{\nu | 2^k\alpha \leq \sqrt{\lambda_\nu} \leq 2^k\beta\}, \quad F_k := \text{span}\{e_\nu | \nu \in J_k\} \subset L^2(\Omega).$$

On identifie les éléments $v_{k,0} \in F_k$ à la solution du problème d'évolution, où $h_k = 2^{-k}$

$$\begin{cases} ih_k\partial_s v_k + h_k^2\Delta_D v_k = 0, & \text{dans} \quad \Omega\times\mathbb{R}, \\ v_k(x,0) = v_{k,0}(x), & v_k|_{\partial\Omega\times\mathbb{R}} = 0. \end{cases} \tag{2.10}$$

1. *Micro-localisation du problème* (2.10) *dans un domaine compact de l'espace des phases :* On a un recouvrement fini par des ouverts $U$ de coordonnées locales $(x', x_d)$ tels que
$$U \cap \Omega = \{(x',x_d) | |x'| \leq 1, x_d > 0\} \quad \text{si} \quad U \cap \partial\Omega \neq \emptyset.$$
On écrit $v_k = \sum_\pm Op_{h_k}(q_\pm)v_k$, où $Op_h(q_\pm)$ sont des opérateurs $h$-pseudodifférentiels avec $\text{supp}(q_-) \subset \{|\xi| \leq 2D\}$, $\text{supp}(q_+) \subset \{|\xi| \geq D\}$. Pour $D$ assez grand, la contribution $Op_{h_k}(q_+)v_k$ sera $O_{L^2(\Omega\times\mathbb{R})}(h_k^\infty)$, c'est à dire
$$\forall M \geq 1 \quad \exists C_M > 0 \quad \|Op_{h_k}(q_+)v_k\|_{L^2(\Omega\times\mathbb{R})} \leq C_M h_k^M. \tag{2.11}$$

D'autre part, si $\chi(\tau) \in C_0^\infty(\mathbb{R}_+)$ est égale à 1 au voisinage de $[\alpha^2, \beta^2]$, $\psi \in C_0^\infty$, on montre aussi que

$$\lim_{k\to\infty} \|(1-\chi)(h_k D_t)\psi(t) Op_{h_k}(q_-) v_k\|_{L^2(\Omega\times\mathbb{R})} = 0. \tag{2.12}$$

À ce stade $\chi(hD_t)Op_h(q_-)$ est un opérateur $h$-pseudodifférentiel en $(x,t)$, à support compact dans l'espace des phases $\tau \in [\alpha^2, \beta^2]$, $|\xi| \leq D$ pour un $D > 1$.

2. *Propagation des singularités (jusqu'au bord) [77], [78] :*
*Définition* 2.1. On dit qu'un point $\rho_0 = (x_0, \xi_0) \in T^*(\partial\Omega \times \mathbb{R}) \cup T^*(\Omega \times \mathbb{R})$ n'est pas dans le front d'onde au bord de $v = (v_k)_k$, $v_k \in F_k$, noté $WF_b(v)$, s'il existe un opérateur $h$-pseudodifférentiel de symbole $q(x,\xi,h)$ à support compact en $(x,\xi)$, elliptique en $\rho_0$, et $\psi \in C_0^\infty$ égale à 1 près de $x_0$ tel que

$$\forall M \geq 1 \quad \exists C_M \quad \|Op_{h_k}(q)\psi v_k\|_{L^2(\Omega\times\mathbb{R})} \leq C_M h_k^M.$$

D'après les estimations précédentes (2.11), (2.12), on a

$$WF_b(v) \subset \{\tau \in [\alpha^2, \beta^2], |\xi| \leq D\}.$$

Pour obtenir des informations plus précises et en particulier la propagation sur le flot géodésique, on utilise une transformation qui ajoute une variable et supprime le paramètre $h_k$ pour se ramener aux théorèmes d'analyse microlocale standard pour les problèmes aux limites d'ordre deux. Soit $\Theta(v)(x,t,s) := \sum_k e^{-i2^k s} v_k(x,t)$ qui vérifie

$$\partial_{s,t}^2 \Theta(v) = \Delta_D \Theta(v), \quad \text{dans} \quad \Omega \times \mathbb{R}_t \times \mathbb{R}_s, \quad \Theta(v)|_{\partial\Omega\times\mathbb{R}_t\times\mathbb{R}_s} = 0.$$

On définit $WF_b(v)$ en utilisant des opérateurs pseudodifférentiels standard (tangentiels près du bord) comme sous ensemble fermé de $T^*(\Omega\times\mathbb{R}_t\times\mathbb{R}_s) \cup T^*(\partial\Omega\times\mathbb{R}_t\times\mathbb{R}_s)$. Par le Théorème de R.Melrose et J.Sjöstrand [77], [78], le front d'onde au bord $WF_b(\Theta(v))$ est invariant par le flot bicaractéristique généralisé de $\sigma\tau - |\xi|^2$. On démontre dans [73, Lemme 1] le résultat suivant :
**Proposition 2.2.** *(Vitesse finie de propagation) Pour* $\rho \in T^*(\Omega\times\mathbb{R}_t) \cup T^*(\partial\Omega\times\mathbb{R}_t)$ *soit* $\theta(\rho, s)$ *le point de* $T^*(\Omega \times \mathbb{R}_t \times \mathbb{R}_s) \cup T^*(\partial\Omega \times \mathbb{R}_t \times \mathbb{R}_s)$

$$\theta(\rho, s) = (\rho; s, \sigma = 1), \quad (s,\sigma) \in T^*\mathbb{R}_s.$$

*Alors pour tout* $s_0$, $v = (v_k)_k$, $v_k \in F_k$, $\rho_0 \in T^*(\Omega \times \mathbb{R}_t) \cup T^*(\partial\Omega \times \mathbb{R}_t)$

$$\rho_0 \in WF_b(v) \quad \textit{si et seulement si} \quad \theta(\rho_0, s_0) \in WF_b(\Theta(v)).$$

## 2.2 Influence de la géométrie de l'espace

Dans le cas des géométries non-triviales, la situation reste quand même loin d'être aussi bien comprise que dans l'espace libre. Une question naturelle est de se demander dans quelle mesure les estimations de Strichartz (fondamentales pour la résolution des équations de Schrödinger non-linéaires) restent stables quand on perturbe la métrique. Un début de réponse à cette question a été apporté par le travail de G.Staffilani et D.Tataru [98] qui démontrent que pour une perturbation $C^2$ à support compact en espace et non-captive l'estimation (2.4) reste vraie (au moins localement en temps).

L'obtention des inégalités de Strichartz en *millieu borné* est un fait remarquable, car les estimations de dispersion pour l'équation (2.2) ne sont plus vérifiées ni sur des petits intervalles de temps. Voici une jolie preuve de cela : soit $M$ une variété compacte et supposons que l'estimation de dispersion (2.5) est vrai pour $|t| \leq T$, pour un $T > 0$. Cela implique que le noyau de l'opérateur qui à $u_0$ associe $u(.,t)$ devrait être dans $L^\infty(M \times M) \subset L^2(M \times M)$, la dernière inclusion étant une conséquence de la compacité de $M$. Un tel opérateur est de Hilbert-Schmidt et, en particulier, compact ; d'autre part, l'application

$$L^2(M) \ni u_0 \to u(.,t) \in L^2(M)$$

est une isométrie de $L^2(M)$ qui ne peut être compacte que si la dimension de $L^2(M)$ est finie, d'òu une contradiction.

### 2.2.1 Variétés compactes sans bord

En se plaçant sur une variété Riemannienne compacte sans bord $(\Omega, g)$, N.Burq, P.Gérard et N.Tzvetkov ont montré dans [27] une inégalité de Strichartz avec perte de dérivées pour l'équation classique

$$\|e^{it\Delta_g} u_0\|_{L^q([0,T],L^r(\Omega))} \leq C\|u_0\|_{H^{1/q}(\Omega)}, \tag{2.13}$$

où $\Delta_g$ désigne le Laplacien sur $\Omega$ associé à la métrique $g$. Cette inégalité est obtenue en travaillant dans le contexte semi-classique, avec des données initiales spectralement localisées. Ceci permet de travailler dans une carte et de construire une solution approchée par la méthode BKW sur des petits intervalles de temps de taille l'inverse de la fréquence. Précisément, on considère $v(x,s) = e^{ihs\Delta_g}\Psi(-h^2\Delta_g)u_0 = u(x,hs)$ solution de l'équation de Schrödinger semi-classique

$$ih\partial_s v + h^2\Delta_g v = 0, \quad v|_{s=0} = \Psi(-h^2\Delta_g)u_0. \tag{2.14}$$

On peut alors, en utilisant des techniques semi-classiques standard et la vitesse finie de propagation, écrire une paramétrice pour $v$ en temps $s$ petit, ce qui permet, en appliquant une formule de phase stationnaire, de démontrer l'estimation de dispersion (2.9) pour $|s| \lesssim 1$, où $\Delta$ est remplacé par $\Delta_g$.

Même si on ne sait pas si elle est optimale, la perte de $1/q$ dérivées dans le cas d'un compact $C^\infty$ sans bord est un résultat naturel du à l'argument heuristique suivant : pour une variété compacte quelconque il n'existe pas de points conjugués pour le flot géodésique à distance plus petite que le rayon d'injectivité de la variété. Étant donnée une solution de l'équation de Schrödinger localisée à fréquence $1/h$, l'énergie se propage à vitesse $1/h$. Par conséquent, la solution spectralement localisée devrait garder de bonnes propriétés de dispersion au moins pendant un temps égal à l'inverse de la fréquence. On s'attend alors à pourvoir montrer une estimation de Strichartz sans perte de dérivées sur un intervalle de temps de taille $h$. En prenant la somme on obtient, grâce à la théorie de Littlewood-Paley, une inégalité de Strichartz en temps 1 pour l'équation classique avec constante $C(h) \simeq h^{-1/q}$.

### 2.2.2 Intérieur d'un domaine strictement convexe

Soit $\Omega$ un domaine strictement convexe et $\Delta_g$ le Laplacien sur $\Omega$ associé à la métrique $g$ dans $\Omega$. On montre dans un chapitre prochain (voir [60]) le résultat suivant :

**Proposition 2.3.** *(O.I. [60]) Soit $(q, r)$ une paire Schrödinger admissible en dimension $d \geq 2$ et $u_0 \in E_k(\Omega)$ pour un $k \geq 1$ (voir Définition 1.1). Alors le flot de Schrödinger semi-classique $e^{ihs\Delta_g}u_0$ vérifie*

$$\|e^{ihs\Delta_g}u_0\|_{L^q(\mathbb{R},L^r(\Omega))} \lesssim h^{-(\frac{d}{2}+\frac{1}{6})(\frac{1}{2}-\frac{1}{r})}\|u_0\|_{L^2(\Omega}. \tag{2.15}$$

*En plus la perte $1/6$ est optimale (pour des données initiales dans un espace $E_k(\Omega)$ de modes de galeries).*

En effet les modes de galerie accumulent leur énergie près du bord et contribuent de cette façon à des normes $L^r(\Omega)$ élevées. Appliquant le flot d'évolution linéaire semi-classique de Schrödinger à ces fonctions propres du Laplacien on s'aperçoit qu'une perte d'au moins $1/6$ dérivées est inévitable pour les estimations de Strichartz. Cela laisse penser que de telles données initiales pourraient caractériser des pertes de dérivées pour l'équation des ondes aussi, mais la Proposition 1.5 montre que ce n'est pas le cas.

### 2.2.3 Variétés compactes avec bord

En utilisant les méthodes semi-classiques de N.Burq, P.Gérard, N.Tzvetkov, R.Anton [7] a abordé le cas d'une variété $(\Omega, g)$ avec bord (régulier, compact). La présence du bord rend plus difficile la mise en place de cette méthode. En faisant une symétrie par rapport au bord avec des coordonnées transverses à celui ci (ce qui est toujours possible en utilisant par exemple, des voisinages tubulaires) on se ramène au cas d'une variété $C^\infty$ sans bord, mais avec une métrique $g$ à coefficients lipschitziens (à cause du prolongement de $g$ de façon symétrique par rapport au bord). En régularisant la métrique à une fréquence dépendante de la fréquence de la donnée initiale et en utilisant la méthode BKW et les travaux de H.Bahouri et J.Y.Chemin [11], D.Tataru [103], elle démontre les inégalités de Strichartz suivantes :

$$\|e^{it\Delta_g}u_0\|_{L^q([0,T],L^r(\Omega))} \leq C(s,T)\|u_0\|_{H^s(\Omega)}, \quad \forall s > 3/2q, \tag{2.16}$$

où $(q, r)$ est un couple Schrödinger admissible en dimension $d \in \{2, 3\}$ et $0 < T < \infty$.

Récemment, M.Blair, H.Smith et C.Sogge [16] ont amélioré ce résultat sur un compact avec bord ou un compact $(\Omega, g)$ sans bord mais avec une métrique $g$ Lipschitz :

$$\|e^{it\Delta_g}u_0\|_{L^q([0,T],L^r(\Omega))} \leq C\|u_0\|_{H^{4/3q}(\Omega)}, \tag{2.17}$$

où $(q, r)$ est une paire admissible en dimension $d \geq 2$. L'argument de réflexion de la métrique symétriquement par rapport au bord (dans des coordonnées géodésiques) est de nouveau utilisé pour se réduire à une variété $C^\infty$ sans bord et métrique Lipschitz qui sera ensuite remplacée par une métrique régularisée avec points conjugués à distance $h^{1/3}$ les uns des autres. Par conséquence, la solution localisée à fréquence $1/h$ possède de bonnes propriétés de dispersion sur un temps $h^{4/3}$, ce qui implique la perte $4/3q$ dans (2.17). Ainsi, l'inégalité (2.17) serait l'analogue de l'inégalité (2.13) sur un domaine sans bord.

### 2.2.4 Extérieur d'un obstacle satisfaisant une condition de non-capture

Afin d'assurer la dispersion, plusieurs auteurs ont imposé la condition de non-capture pour la métrique. Une métrique est dite non captante si ses géodésiques ne restent pas dans un compact pour un temps infini (c'est à dire si toute courbe géodésique $\gamma(s)$, $s \in \mathbb{R}$ vérifie $\lim_{s\to\pm\infty} \gamma(s) = \infty$). Pour une métrique perturbation compacte à coefficients $C^2$ de la métrique euclidienne, G.Staffilani et D.Tataru [98] ont montré une estimation de Strichartz identique à celle obtenue pour la métrique euclidienne.

**Effet régularisant :** Dans $\mathbb{R}^d$ avec la métrique euclidienne, l'équation (2.2) induit un gain de régularité spatiale pour presque tout $t$ par rapport à la donnée initiale que l'on peut écrire, si $\chi \in C_0^\infty$ est une fonction régulière à support compact,

$$\|\chi e^{it\Delta}u_0\|_{L^2(-T,T)H^{1/2}(\mathbb{R}^d)} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{2.18}$$

On appelle cette propriété *effet régularisant* (local).

Cet effet a été remarqué par P.Constantin et J.C.Saut [38, 37], P.Sjölin [91], L.Vega [110], W.Craig, T.Kappeler et W.Strauss [39], M.Ben Artzi et S.Klainerman [13], M.Ben Artzi et A.Devinatz [12], S.I.Doi [45], T.Kato et K.Yajima [69] et généralisé par plusieurs auteurs : L.Robbiano et C.Zuily [88, 89] sur des perturbations de courte portée, A.Hassel, T.Tao et J.Wunch [54] pour des variétés asymptotiquement coniques, J.M.Bouclet et N.Tzvetkov [17] pour le cas des perturbations à longue portée de la métrique euclidienne, c'est à dire pour une métrique $g$ qui satisfait :

$$\exists \epsilon > 0, \forall \alpha \in \mathbb{N}^d : \quad |\partial_x^\alpha (g_{i,j} - \delta_{i,j})| \lesssim (1 + |x|)^{-(|\alpha|+\epsilon)}.$$

Lorsqu'on rajoute un obstacle (et que l'on considère éventuellement un Laplacien associé à une métrique variable), on peut démontrer l'effet régularisant avec gain (de régularité) d'une demi-dérivée moyennant l'hypothèse de non-capture suivante :

*Hypothèse* 2.1. Soit $\Theta$ un obstacle à bord régulier et $\Omega = \mathbb{R}^d \backslash \Theta$. Alors tout rayon lumineux qui se propage dans $\Omega$ selon les lois de l'optique géométrique ne reste pas dans un compact de $\bar{\Omega}$ un temps infiniment long. On dit alors que $\Theta$ est *non-captant*.

Sous cette hypothèse de non-capture N.Burq, P.Gérard et N.Tzvetkov ont montré dans [26, Prop.2.7] l'effet régularisant

$$\|\chi e^{it\Delta_g} u_0\|_{L^2(-T,T)H^{s+1/2}(\Omega)} \leq C\|u_0\|_{H^s(\Omega)}, \quad s \in [0,1], \tag{2.19}$$

où $\chi$ est une troncature qui localise près du bord. En effet, ce théorème est démontré dans un cadre plus général par S.I.Doi [60] (localement en temps). Dans le cas particulier d'une perturbation à support compact ($T = \infty$), on obtient ce résultat comme conséquence des estimations sur la résolvante sortante $(-\Delta_g - (1/h - i0)^2)^{-1}$ suivantes ([26, Prop.2.1])

$$\begin{gathered} \forall \chi \in C_0^\infty(\mathbb{R}^d), \quad \exists C > 0, \quad \forall h \in (0,1], \quad \forall 0 < \epsilon \ll 1 \\ \|\chi(-\Delta_g - (1/h - i0)^2)^{-1}\chi\|_{L^2(\Omega)\to L^2(\Omega)} \leq Ch. \end{gathered} \tag{2.20}$$

Ce résultat a été obtenu avec une généralité de plus en plus grande pour $0 < h \ll 1$ par P.D.Lax et R.S.Phillips [72], R.Melrose et J.Sjöstrand [78], S.H.Tang et M.Zworski [102], N.Burq [24]. La preuve pour les basses fréquences peut être trouvée dans [22, Annexe B.2].

Loin d'un voisinage du bord on a les estimations suivantes ([27, Prop.2.10])

$$\|(1-\chi)e^{it\Delta_g} u_0\|_{L^2(-T,T)H^{s+1/2}(\Omega)} \leq C\|u_0\|_{H^s(\Omega)}, \quad s \in [0,1], \tag{2.21}$$

où $\chi$ est une troncature égale à 1 près de $\partial\Omega$. En effet, comme $\chi$ est identiquement égale à 1 dans un voisinage de $\partial\Omega$, on peut regarder $(1-\chi)e^{-it\Delta_g}u_0$ comme étant la solution d'un problème dans l'espace libre $\mathbb{R}^d$ et par conséquence satisfait les inégalités de Strichartz usuelles. A l'aide de ces estimations N.Burq, P.Gérard et N.Tzvetkov [27] ont obtenu des inégalités de Strichartz avec perte $\gamma = 1/q$ à l'extérieur d'un obstacle non-captant :

$$\|e^{it\Delta_g} u_0\|_{L^q([0,T],L^r(\Omega))} \leq C\|u_0\|_{H^{1/q}(\Omega)}, \tag{2.22}$$

où $(q,r)$ est une paire admissible en dimension $d \geq 2$. On remarque ici que la perte est égale à celle (2.13) sur une variété compacte sans bord : pourtant, les raisons sont complètement différentes. Ainsi, cette estimation (2.22) est basée sur des estimations de résolvante du Laplacien et traduit l'effet régularisant du flot linéaire.

**Effet régularisant pour l'exemple d'Ikawa :** On considère un obstacle compact, régulier $\Theta \subset \mathbb{R}^d$ de complémentaire connexe et on définit $\Omega := \mathbb{R}^d \setminus \Theta$.

*Hypothèse* 2.2. (M.Ikawa [57, 58]) On suppose que $\Theta = \cup_{i=1}^N \Theta_i$ est une réunion finie d'obstacles strictement convexes vérifiant les conditions suivantes

1. si on note $\overline{\mathrm{Conv}(\Theta_i \cup \Theta_j)}$ l'enveloppe convexe fermée de l'union des obstacles $\Theta_i$ et $\Theta_j$, alors pour tous $1 \leq i,j,k \leq N$, $i \neq j$, $j \neq k$, $k \neq i$ on a
$$\overline{\mathrm{Conv}(\Theta_i \cup \Theta_j)} \cap \Theta_k = \emptyset.$$

2. Soit $\kappa$ la valeur minimale des courbures principales des bords des obstacles $\Theta_i$ et soit $L$ la valeur minimale des distances entre deux obstacles. Alors, si $N > 2$, on suppose $kL > N$ (aucune autre hypothèse si $N = 2$).

Notons que si $N = 2$ les deux conditions sont automatiquement satisfaites. La première condition de l'hypothèse 2.2 est purement technique, alors que la deuxième est une condition concernant l'hyperbolicité forte du système dynamique donné par l'application de billard.

**Théorème 2.1.** *(Effet régularisant O.I.[25, Thm.1.7]) Sous l'hypothèse 2.2, pour tout $\epsilon > 0$ et $\chi \in C_0^\infty(\mathbb{R}^d)$, il existe une constante $C > 0$ telle que pour toute donnée initiale $u_0 \in L^2(\Omega)$ le flot linéaire de Schrödinger sur $\Omega$ vérifie*

$$\|\chi e^{it\Delta_D} u_0\|_{L^2(\mathbb{R})H^{1/2-\epsilon}(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}. \tag{2.23}$$

Notons que dans (2.23) on a une perte de $\epsilon$ dérivées par rapport à l'estimation d'effet régularisant (2.19) dans le cas d'un domaine non-captant : l'hypothèse 2.1 n'est clairement pas satisfaite par l'exemple d'Ikawa.

## 2.3 Problème de Cauchy pour l'équation de Schrödinger : existence globale et comportement de type scattering

Soit $d \geq 1$, $p \in \mathbb{R}$ et $\epsilon \in \{-1, 1\}$. On considère le problème

$$\begin{cases} i\partial_t u + \Delta u = \epsilon |u|^{p-1} u \quad \text{dans} \quad \mathbb{R}^{d+1}, \\ u(x, 0) = u_0(x). \end{cases} \tag{2.24}$$

Lorsqu'on dispose de solutions globales de (2.24), il est naturel d'étudier leur comportement à grand temps. Pour les équations des ondes et de Schrödinger dans le cas défocalisant ($\epsilon = 1$) on rencontre souvent un phénomène de type scattering. On pourrait noter que dans le cas des données petites il s'agit en général d'une simple conséquence de la construction des solutions, mais que le cas de grandes données est sensiblement plus difficile.

*Définition* 2.2. Supposons que l'équation (2.24) admet une solution globale pour une donnée $u_0 \in H^1(\mathbb{R}^d)$. On dit que cette solution admet un comportement de type *scattering* dans $H^1(\mathbb{R}^d)$ en $\pm\infty$ s'il existe des fonctions $v_\pm(x)$ (appelées états asymptotiques) telles que

$$\lim_{t \to \pm\infty} \|u(x, t) - e^{it\Delta} v_\pm\|_{H^1(\mathbb{R}^d)} = 0. \tag{2.25}$$

Autrement dit, en norme $H^1(\mathbb{R}^d)$, la solution se rapproche d'une solution linéaire (qui **n'est pas** la solution $e^{it\Delta_D} u_0$).

On rappelle un résultat de Nakanishi (voir aussi J.Ginibre et G.Velo [52] pour $d \geq 3$) concernant le comportement de type scattering pour l'équation de Schrödinger défocalisante $H^1$ - sous-critique (et $L^2$- sur-critique) :

**Théorème 2.2.** *Soit* $u_0 \in H^1(\mathbb{R}^d)$*,* $d \geq 1$ *et soit* $u$ *la solution du problème* (2.24) *avec* $\epsilon = 1$*,* $1 + 4/d < p < 1 + 4/(d-2)$*. Alors* $u$ *admet un comportement de type scattering.*

Un résultat plus fort, mais concernant seulement l'équation cubique ($p = 3$) défocalisante ($\epsilon = 1$) est démontré dans [36], où on obtient scattering dans $H^\sigma$ pour tout $\sigma > 4/5$. On pourra consulter [52] pour une perspective moderne sur ces questions.

### 2.3.1 Existence globale et scattering dans un domaine extérieur

Soit $\Omega \subset \mathbb{R}^d$ un domaine non-captant à bord régulier $d \geq 1$, $p \in \mathbb{R}$, $\epsilon \in \{-1, 1\}$. On considère le problème

$$\begin{cases} i\partial_t u + \Delta_D u = \epsilon |u|^{p-1} u & \text{dans } \Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \quad u|_{\partial\Omega} = 0. \end{cases} \tag{2.26}$$

Comme dans un domaine non-captant quelconque (qui ne soit pas à bord strictement concave) on n'a pas des estimations de Strichartz invariantes d'échelle, prouver le caractère bien posé pour l'équation (2.26) s'avère être une tache bien plus difficile que dans le cas euclidien où (2.4) a lieu. En effet, on sait que l'équation (2.26) est bien posée pour $p < 3$ (N.Burq, P.Gérard et N.Tzvetkov [26]) et pour $p = 3$ (R.Anton [6]) ; dans ces travaux on établit des estimations de Strichartz pas optimales (qui ne sont pas invariantes d'échelle) à l'aide desquelles on en déduit des résultats d'existence locale.

Dans un travail récent [87], F.Planchon et L.Vega ont montré l'existence locale pour (2.26) avec $1 < p < 5$ en dimension 3, ainsi que l'existence globale et le comportement de type scattering pour l'équation cubique en dimension trois de l'espace, à condition que le domaine $\Omega$ soit étoilé. Plus précisément, ils ont montré :

**Théorème 2.3.** *Soient* $1 < p < 5$ *et* $d = 3$*. Soit* $\Theta$ *un obstacle non-captant,* $\Omega = \mathbb{R}^d \setminus \Theta$ *et soit* $u_0 \in H_0^1(\Omega)$*. Alors il existe* $T > 0$ *et une solution* $u$ *de* (2.26) *qui est dans* $C([0,T), H_0^1(\Omega))$*. L'unicité a lieu dans l'espace* $C([0,T), H_0^1(\Omega)) \cap L^4([0,T), W^{3/4,4}(\Omega)) \cap L^2([0,T), H^{3/2}(\Omega)) \cap L^4([0,T), L^\infty(\Omega))$*.*

**Théorème 2.4.** *Soient* $p = 3$*,* $d = 3$ *et* $\epsilon = 1$*. Soient* $\Omega$ *l'extérieur d'un obstacle étoilé et* $u \in H_0^1(\Omega)$*. Alors la solution globale* $u$ *de l'équation cubique défocalisante* (2.26) *admet un comportement de type scattering dans* $H_0^1(\Omega)$*.*

Pour montrer l'existence globale ils ont montré la validité des inégalités de Strichartz usuelles à l'extérieur d'un obstacle étoilé pour $q = r = 4$

$$\|e^{it\Delta_D} u_0\|_{L^4_{t,x}} \lesssim \|u_0\|_{\dot{H}^{1/4}},$$

mais comme cela n'implique pas des estimations de la norme $L^4_t L^\infty_x$ il faut ensuite utiliser l'effet régularisant (2.19) près du bord et les estimations de Strichartz classiques (2.4) loin d'un voisinage du bord.

## 2.4 Résultats concernant l'effet régularisant à l'extérieur d'un nombre fini de boules dans $\mathbb{R}^3$

Dans un tout premier travail de cette thèse [59, Thm.1.3] on démontre un effet régularisant "amélioré" à l'extérieur d'une ou de plusieurs boules de $\mathbb{R}^3$ qui permet d'établir des inégalités de Strichartz avec pertes et existence globale pour l'équation de Schrödinger cubique en dimension 3 de l'espace pas seulement à l'extérieur d'une boule mais aussi pour un domaine qui est le complémentaire dans $\mathbb{R}^3$ d'un nombre fini de boules - et donc forcément captant ! (l'exemple d'Ikawa).

Les travaux de S.H.Doi [46] et N.Burq [25, 24] montrent que l'effet régularisant est intimement lié à la vitesse infinie de propagation des solutions de Schrödinger. Plus précisément, si on considère une solution localisée à fréquence $\lambda$, alors elle se propage à vitesse $\lambda$ et va rester dans un domaine borné seulement un temps $1/\lambda$. Par conséquent, lorsqu'on calcule la norme $L^2$ on obtient un gain de $1/\lambda^{1/2}$ par rapport à la norme $L^\infty$, ce qui induit un gain de 1/2 dérivées. Cet argument heuristique laisse penser qu'on pourrait "raffiner" et améliorer l'effet régularisant en considérant des domaines d'espace de taille dépendant de l'inverse de la fréquence : un contexte naturel est l'extérieur d'un ou plusieurs obstacles convexes, dans quel cas des candidats naturels pour des domaines $\lambda$-dépendants sont les $\lambda^{-\alpha}$-voisinages du bord.

**Théorème 2.5.** *(O.I. [59, Thm.1.1]) Soit* $\Omega = \mathbb{R}^3 \setminus B(0,1)$*, où* $B(0,1)$ *désigne la boule unité de* $\mathbb{R}^3$ *et* $T > 0$*. Soient* $\psi$ *et* $\chi \in C_0^\infty(\mathbb{R}^*)$ *des fonctions régulières à support compact,* $\psi = 1$ *près de* 1*,* $\chi = 1$ *près de* 0*. On définit* $\chi_\lambda(|x|) := \chi(\lambda^\alpha(|x|-1))$*, où* $x$ *dénote la variable sur* $\Omega$ *et où* $0 \leq \alpha < \frac{2}{3}$*,* $\lambda \geq 1$*. Alors on a*

1. *Pour* $s \in [-1,1]$ *et* $v(t) = \int_0^t e^{i(t-\tau)\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda g d\tau$

$$\|\chi_\lambda \psi(\frac{-\Delta_D}{\lambda^2})v\|_{L^2_T H^{s+1}_D(\Omega)} \leq C\lambda^{-\frac{\alpha}{2}}\|\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda g\|_{L^2_T H^s_D(\Omega)}, \tag{2.27}$$

2. *Pour* $s \in [0,1]$

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{s+\frac{1}{2}}(\Omega)} \leq C\lambda^{s-\frac{\alpha}{4}}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{2.28}$$

*Ici la constante* $C$ *est indépendante de* $T$*, c'est à dire les estimations sont globales en temps.*

**Idée de la preuve :** On souhaite estimer les normes $H^s(\Omega)$ du flot de Schrödinger dans une couronne de taille $\lambda^{-\alpha}$ du bord : pour cela on commence par étudier les solutions sortantes de l'équation $(\Delta_D + \lambda^2)w = 0$, $w|_{\partial\Omega} = \chi_\lambda f$. Rappelons d'abord la définition de telles solutions : on se place dans le système de coordonnées polaires où on note $\omega$ la variable sur la sphère $\mathbb{S}^2$ et on décompose toute solution $w$ à l'extérieur d'une boule de rayon 1 en harmoniques sphériques, c'est à dire sous la forme

$$w = \sum_\mu w_\mu(r)e_\mu(\omega), \quad r \geq 1,$$

où $(e_\mu, \mu)$ désigne la base orthonormale de $L^2(\mathbb{S}^2)$ formée des fonctions propres pour l'opérateur de Laplace -Beltrami sur la sphère associées aux valeurs propres $\mu^2$. Les fonctions $w_\mu$ sont alors solutions d'une équation de Bessel et seront données par

$$w_\mu = C_\mu r^{-1/2} H_\nu(\lambda r) + C'_\mu r^{-1/2} \bar{H}_\nu(\lambda r), \quad \text{avec} \quad \nu = \sqrt{\mu^2 + 1/4},$$

où $H_\nu(\lambda r)$ est la fonction de Hankel d'ordre $\nu$.

*Définition* 2.3. On dit que la fonction $w$ solution de $(\Delta_D + \lambda^2)w = 0$ à l'extérieur d'une boule est sortante si et seulement si dans la décomposition précédente on a $C'_\mu = 0$ pour tout $\mu$.

Par la méthode de variation de la constante on calcule explicitement la solution sortante de l'équation de Helmholz en fonction de $H_\nu(\lambda r)$ et grâce au comportement asymptotique des fonctions de Hankel on en déduit des estimations de la norme $L^2$ de la transformée de Fourier en temps de l'équation de Schrödinger.

### 2.4.1 Applications

*Hypothèse* 2.3. On considère un obstacle compact, régulier $\Theta \subset \mathbb{R}^3$ de complémentaire connexe satisfaisant les conditions de l'hypothèse 2.2 avec $N \geq 1$ et $\Theta_i = B(o_i, r_i)$, où $B(o_i, r_i) \subset \mathbb{R}^3$ sont des boules de centre $o_i$ et de rayon $r_i > 0$. Si $N > 2$ la deuxième condition de l'hypothèse 2.2 se traduit par $kL > 2$, où $L$ est le minimum des distances entre toutes les deux boules et $k = \min\{1/\sqrt{r_i}, i = \overline{1, N}\}$. Si $N = 1$ alors $\Theta$ est une boule de $\mathbb{R}^3$. On définit $\Omega := \mathbb{R}^3 \setminus \Theta$ et note $\Delta_{D,N}$ l'opérateur de Laplace sur $\Omega$ avec condition de Dirichlet (ou de Neumann, respectivement) sur le bord.

**Théorème 2.6.** *(O.I. [59, Thm.1.3]) Soit $\Theta \subset \mathbb{R}^3$ un obstacle satisfaisant les conditions de l'hypothèse 2.3 et $\Omega = \mathbb{R}^3 \setminus \Theta$. Soient $T > 0$ et $(p, q)$ une paire admissible en dimension 3. Alors pour tout $\epsilon > 0$ suffisamment petit il existe une constante $C > 0$ telle que pour toute donnée initiale $u_0 \in H_{D,N}^{\frac{4}{5p}+\epsilon}(\Omega)$ on a*

$$\|e^{it\Delta_{D,N}} u_0\|_{L^p([-T,T], L^q(\Omega))} \leq C \|u_0\|_{H_{D,N}^{\frac{4}{5p}+\epsilon}(\Omega)}. \tag{2.29}$$

**Théorème 2.7.** *(O.I. [59, Thm.1.4]) Soit $\Omega \subset \mathbb{R}^3$ satisfaisant l'hypothèse 2.2. Pour toute donnée initiale $u_0 \in H_0^1(\Omega)$ le problème*

$$(i\partial_t + \Delta_D)u = |u|^2 u \quad \textit{dans} \quad \mathbb{R} \times \Omega, \quad u(0, x) = u_0(x), \quad \textit{dans} \quad \Omega, \quad u|_{\partial\Omega} = 0 \tag{2.30}$$

*admet une unique solution $u \in C(\mathbb{R}, H_0^1(\Omega))$ globale en temps qui satisfait les lois de conservations*

$$\frac{d}{dt} \int_\Omega |u(t, x)|^2 dx = 0; \quad \frac{d}{dt}\Big( \int_\Omega |\nabla u(t, x)|^2 dx + \int_\Omega V(u(t, x)) dx \Big) = 0. \tag{2.31}$$

*En plus, pour tout $T > 0$ le flot $u_0 \to u$ est Lipschitz continu de tout sous-ensemble borné de $H_0^1(\Omega)$ dans $C(\mathbb{R}, H_0^1(\Omega))$.*

Le Théorème 2.7 améliore un résultat de N.Burq, P.Gérard et N.Tzvetkov [26] dans le cas de l'extérieur d'une boule, où on démontre l'existence globale pour l'équation de Schrödinger cubique à l'extérieur d'un obstacle non-captant de $\mathbb{R}^3$ mais seulement pour des données initiales de norme $H^1$ petite.

Dans le cas d'une réunion de plusieurs obstacles strictement convexes, N.Burq, C.Guillarmou et A.Hassel ont démontré très récemment des inégalités de Strichartz optimales. Ils ont utilisé pour cela de façon essentielle les inégalités de Strichartz (2.41) démontrées dans [61] (voir la section suivante).

## 2.5 Résultats concernant l'équation de Schrödinger sur une variété compacte à bord strictement concave où à l'extérieur d'un obstacle strictement convexe

Dans [61] nous considérons une variété Riemannienne compacte $M$ à bord strictement concave (le billard de Sinaï par exemple qui est défini comme étant le complémentaire d'un obstacle strictement convexe sur le tore) et nous démontrons que les inégalités de Strichartz semi-classiques (locales en temps) de l'espace libre restent vraies pour la solution de l'équation de Schrödinger (semi-classique) dans $M$. Une conséquence directe est la validité des estimations de Strichartz pour l'équation de Schrödinger (classique) à l'extérieur d'un obstacle strictement convexe de $\mathbb{R}^d$.

La stratégie consiste à utiliser la paramétrice de R.Melrose et M.Taylor pour l'équation des ondes près d'un rayon tangent au bord (introduite dans la Proposition 1.2) et la vitesse finie de propagation pour l'équation semi-classique (démontrée par G.Lebeau [73]) pour déduire, en suivant les travaux de H.Smith et C.Sogge [95], M.Zworski [111], des estimations de Strichartz semi-classiques sans pertes de dérivées.

En utilisant l'effet régularisant (2.19) à l'extérieur d'un obstacle strictement convexe (en particulier non-captant !), ainsi que des estimations de la forme (2.21) loin d'un voisinage fixé du bord, on en déduit des inégalités de Strichartz globales sans pertes de dérivées dans le complémentaire $\Omega$ d'un obstacle strictement convexe de $\mathbb{R}^d$.

On donne deux applications directes de ce dernier résultat :

1. L'existence locale des solutions à données $H^1(\Omega)$ pour l'équation de Schrödinger critique (avec une non-linéarité quintique en dimension 3 de l'espace) ;
2. Scattering pour l'équation de Schrödinger sous-critique en dimension 3.

*Remarque* 2.1. Remarquons quand même que cette dernière application ne constitue pas un résultat entièrement nouveau : rappelons ici un travail récent de F.Planchon et L.Vega [87] où on montre des Strichartz sans pertes à l'extérieur d'un obstacle non-captant mais seulement pour des indices satisfaisant $q = r = 4$ en dimension $d = 3$. À partir de cela ils obtiennent le scattering à l'extérieur d'un obstacle étoilé pour une non linéarité cubique (voir les théorèmes 2.3, 2.4). Pourtant, le fait de connaître les estimations de Strichartz pour tous les indices admissibles (sauf les cas extrêmes) permet de simplifier considérablement la preuve de [87] et de l'étendre à tous les $7/3 < p < 5$.

Dans ce qui suit on va énoncer les résultats obtenus dans [61] et présenter l'essentiel des méthodes utilisées.

### 2.5.1 Billard de Sinaï à bord strictement concave

*Hypothèse* 2.4. Soit $(M, g)$ une variété Riemannienne compacte de dimension $d \geq 2$ à bord $C^\infty$. On suppose que $\partial M$ est géodésiquement strictement concave. On note $\Delta_g$ l'opérateur de Laplace associé à la métrique $g$ de domaine $H^2(M) \cap H_0^1(M)$. On suppose

$$\Delta_g = \sum_{j,k=2}^{d} g^{j,k}(x)\partial_j\partial_k + \sum_{j=1}^{d} a_j(x)\partial_j, \tag{2.32}$$

où les coefficients appartiennent à un sous-ensemble borné de $C^\infty$ et la partie principale de $\Delta_g$ est uniformément elliptique.

Soit $\Psi \in C_0^\infty((1/2, 2))$ à support compact telle que $\sum_{k\in\mathbb{Z}} \Psi(2^{-2k}\lambda) = 1$, $\forall \lambda \in \mathbb{R}$.

Le premier résultat principal de [61] est le suivant :

**Théorème 2.8.** *(O.I. [61, Thm.1.3])(Inégalités de Strichartz semi-classiques) Sous l'hypothèse 2.4, étant donnée une paire $(q, r)$ admissible pour l'équation de Schrödinger avec $q > 2$ et $T > 0$ suffisamment petit, il existe une constante $C = C(T) > 0$ telle que la solution $v(x, s)$ du problème de Schrödinger semi-classique sur $M \times \mathbb{R}$ avec condition de Dirichlet au bord*

$$\begin{cases} ih\partial_s v + h^2\Delta_g v = 0, \quad dans \quad M \times \mathbb{R} \\ v(x, 0) = \Psi(-h^2\Delta_g)v_0(x), \quad v|_{\partial M} = 0, \end{cases} \tag{2.33}$$

*vérifie*

$$\|v\|_{L^q((-T,T),L^r(M))} \leq Ch^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})}\|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}. \tag{2.34}$$

**Corollaire 2.1.** *(O.I. [61, Cor.1.5]) Sous l'hypothèse 2.4, étant donnée une paire $(q, r)$ admissible pour l'équation de Schrödinger avec $q > 2$ et $I$ un intervalle de temps fini, il existe une constante $C(I) > 0$ telle que la solution $v(x, t)$ du problème de Schrödinger classique sur $M \times \mathbb{R}$ avec condition de Dirichlet au bord*

$$\begin{cases} i\partial_t v + \Delta_g u = 0, \quad dans \quad M \times \mathbb{R}, \\ v(x, 0) = v_0(x), \quad u|_{\partial M} = 0 \end{cases} \tag{2.35}$$

*vérifie*

$$\|v\|_{L^q((I,L^r(M))} \leq C(I)\|v_0\|_{H^{\frac{1}{q}}(M)}. \tag{2.36}$$

Le corollaire découle en suivant l'approche proposée par N.Burq, P.Gérard et N.Tzvetkov dans [27] pour une variété sans bord : en effet, en utilisant les inégalités de Strichartz (6.6) pour des intervalles de temps de taille $h$ et en faisant la somme en temps, on obtient une estimation en temps 1 pour l'équation classique (2.35) avec constante $C(h) \simeq h^{-1/q}$.

**Idée de la preuve du Théorème 2.8 :** En utilisant la paramétrice (1.10) pour les ondes et la vitesse finie de propagation, on écrit la solution $v$ de l'équation semi-classique (2.33) près d'un point diffractif

$$v(x,t) = \frac{1}{h^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)} 2\xi_1(a(x,\xi/h)A_+(\zeta(x,\xi/h))+b(x,\xi/h)A'_+(\zeta(x,\xi/h)))\times \quad (2.37)$$

$$\times\frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\hat{K}(f_h,g_h)(\frac{\xi}{h})d\xi,$$

où les symboles et les phases ont été introduits dans la Proposition 1.2 et où $f_h$, $g_h$ vérifient $\|f_h\|_{L^2(M)} \simeq h\|g_h\|_{L^2(M)} \simeq \|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}$ et sont supportés loin d'un voisinage de bord. Grâce à la continuité $L^2$ de $K$, on est ramené à montrer que l'opérateur $A_h$ définit par

$$A_h(f)(x,t) := \frac{1}{h^n}\int_{\mathbb{R}^n} 2\xi_1(a(x,\xi/h)A_+(\zeta(x,\xi/h)) + b(x,\xi/h)A'_+(\zeta(x,\xi/h)))\times \quad (2.38)$$

$$\times e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)}\frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\hat{f}(\frac{\xi}{h})d\xi$$

vérifie

$$\|A_hf\|_{L^q((0,T],L^r(\mathbb{R}^d))} \leq Ch^{-d/2(1/2-1/r)}\|f\|_{L^2(\mathbb{R}^d)}, \quad (2.39)$$

On suit l'approche de H.Smith et C.Sogge [95] qui consiste à introduire une fonction $\chi \in C^\infty$ telle que $\mathrm{supp}(\chi) \subset (-\infty,-1]$, $\mathrm{supp}(1-\chi) \subset [-2,\infty)$. Alors $A_h$ va s'écrire comme une somme de deux opérateurs $A_{h,j}$ de la même forme qui correspondent à la décomposition $1 = \chi + (1-\chi)$.

1. Le premier opérateur $A_{h,1}$ est supporté pour $\zeta \leq -1$ et dans ce régime la phase s'écrit
$$-t\xi_1^2 + \theta(x,\xi) - \frac{2}{3}(-\zeta)^{3/2}(x,\xi).$$
On peut appliquer le lemme de phase stationnaire dans l'intégrale qui définit $A_{h,1}A^*_{h,1}$ car l'application $x \to \nabla_\xi(\theta(x,\xi) - \frac{2}{3}(-\zeta)^{3/2}(x,\xi))$ est un difféomorphisme loin de $\{\zeta = 0\}$, pour déduire des estimations de dispersion (2.5) pour des temps $h^{1/3} \lesssim |t-s|$. Pour $|t-s| \ll h^{1/3}$ on va "geler" les coefficients du noyau de $A_{h,1}$ et, en restreignant le support dans des cubes de taille $h^{1/3}$ (en espace), on va pouvoir les regarder comme des multiplicateurs de Fourier.

2. Le terme "diffractif" dont le symbole est supporté pour $\zeta \geq -2$, se traite un peu différemment : en utilisant le comportement asymptotique de $Ai$ et en remarquant qu'une dérivée en $x_d$ produit un effet similaire que la multiplication par un symbole d'ordre 2/3 on arrive, en suivant [95], à se réduire à devoir montrer des estimations de type (2.39) mais pour un opérateur dont le symbole $c$ vérifie $x_d^j \partial_{x_d}^k c(t,x,\xi/h)) \in S^{2(k-j)/3}_{2/3,1/3}(\mathbb{R}^d_{t,x'} \times \mathbb{R}^d_\xi)$. A ce point on va pouvoir appliquer les mêmes arguments que ceux du premier cas (de l'opérateur $A_{h,1}$) pour en déduire la dispersion et, par conséquent, (2.39).

### 2.5.2 Extérieur d'un obstacle strictement convexe

*Hypothèse* 2.5. Soit $\Omega = \mathbb{R}^d \setminus \Theta$, $d \geq 2$ où $\Theta$ est un obstacle compact à bord régulier. On suppose que $\partial\Omega$ est strictement géodésiquement concave. On note $\Delta_D = \sum_{j=1}^d \partial_j^2$ l'opérateur de Laplace sur $\Omega$ à coefficients constants avec la condition de Dirichlet sur le bord.

**Théorème 2.9.** *(O.I.[61, Thm1.7])(Inégalités de Strichartz classiques) Sous l'hypothèse 2.5, étant donnée une paire $(q,r)$ admissible pour l'équation de Schrödinger avec $q > 2$ et $u_0 \in L^2(\Omega)$, il existe une constante $C > 0$ telle que la solution $u(x,t)$ de l'équation de Schrödinger dans $\Omega \times \mathbb{R}$ avec condition de Dirichlet sur le bord*

$$\begin{cases} i\partial_t u + \Delta_D u = 0, \quad \text{dans} \quad \Omega \times \mathbb{R} \\ u(x,0) = u_0(x), \quad u|_{\partial\Omega} = 0, \end{cases} \tag{2.40}$$

*vérifie*

$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{2.41}$$

**Idée de la preuve :** D'après le Théorème 1.1 et la vitesse finie de propagation de l'équation semi-classique [73] on déduit des inégalités de Strichartz sans pertes mais sur des intervalles de temps de taille égale à l'inverse de la fréquence. On suit l'approche suggérée par N.Burq [23] qui consiste à décomposer $u = \chi u + (1-\chi)u$, où $\chi \in C_0^\infty(\mathbb{R}^d)$ est égale à 1 près de l'obstacle $\Theta$.

1. Si $\varphi \in C_0^\infty((-1,2))$ est égale à 1 sur $[0,1]$ alors $\varphi(t/h-l)\Psi(-h^2\Delta_D)\chi u$ est supportée sur un intervalle de temps de taille $h$ et elle est solution d'une équation de Schrödinger avec une non-linéarité notée $V_{h,l}$. La formule de Duhamel, l'inégalité de Minkovski et l'estimation de Strichartz en temps $h$ impliquent

$$\|\varphi(t/h-l)\Psi(-h^2\Delta_D)\chi u\|_{L^q_t L^r(\Omega)} \leq Ch^{1/2}\|V_{h,l}\|_{L^2_t L^2(\Omega)}.$$

   On utilise ensuite l'effet régularisant (2.19).

2. Loin de l'obstacle on regarde $(1-\chi)u$ comme solution d'un problème dans l'espace libre et on conclut en utilisant un lemme de Christ et Kisselev.

Notons que, étant quand même que sur une variété à bord, on ne peut pas appliquer directement la méthode de [27] : pour conclure, on a essentiellement besoin de l'estimation de fonction carrée sur un domaine à bord qu'on a montrée dans un travail récent [63] en collaboration avec F.Planchon (voir la section suivante).

### 2.5.3 Applications

Une fois le Théorème 2.9 établi, on obtient (par les méthodes classiques de point fixe et l'approche de [87]) les résultats suivants :

**Théorème 2.10.** *(O.I. [61, Thm.1.8])(Existence locale pour l'équation de Schrödinger critique) Soit* $\Omega$ *une variété Riemannienne de dimension* 3 *satisfaisant l'hypothèse 2.5. Soit* $T > 0$ *et* $u_0 \in H_0^1(\Omega)$. *Alors il existe une unique solution* $u \in C([0,T], H_0^1(\Omega)) \cap L^5((0,T], W^{1,30/11}(\Omega))$ *de l'équation non-linéare quintique*

$$i\partial_t u + \Delta_D u = \pm |u|^4 u \quad dans \quad \Omega \times \mathbb{R}, \quad u|_{t=0} = u_0 \quad sur \quad \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.42}$$

*En plus pour tout* $T > 0$, *le flot* $u_0 \to u$ *est lipschitzien continu de tout sous-ensemble* $B$ *borné de* $H_0^1(\Omega)$ *à valeurs dans* $C([-T,T), H_0^1(\Omega))$. *Si la norme* $H^1(\Omega)$ *de la donnée initiale* $u_0$ *est suffisamment petite, alors la solution est globale en temps.*

**Théorème 2.11.** *(O.I. [61, Thm.1.9])(Scattering pour l'équation de Schrödinger sous-critique) Soit* $\Omega$ *une variété Riemannienne de dimension* 3 *satisfaisant l'hypothèse 2.5. Soit* $3 \leq p < 5$ *et* $u_0 \in H_0^1(\Omega)$. *Alors la solution (globale en temps) de l'équation de Schrödinger défocalisante*

$$i\partial_t u + \Delta_D u = |u|^{p-1} u, \quad u|_{t=0} = u_0 \quad dans \quad \Omega, \quad u|_{\partial\Omega} = 0 \tag{2.43}$$

*admet un comportement de type scattering dans* $H_0^1(\Omega)$. *Si* $p = 5$ *et la norme* $H^1(\Omega)$ *de la donnée initiale* $u_0$ *est suffisamment petite, alors la solution globale de l'équation de Schrödinger critique admet aussi un comportement de type scattering dans* $H_0^1(\Omega)$.

## 2.6 Résultats concernant une estimation de fonction carrée

Pour démontrer la validité des inégalités de Strichartz (2.41) à l'extérieur d'un obstacle strictement convexe on a utilisé dans [61] le théorème suivant, dont une preuve élémentaire et auto-contenue est donnée dans [63], en collaboration avec Fabrice Planchon.

*Hypothèse* 2.6. Soit $\Omega$ un domaine de $\mathbb{R}^d$, $d \geq 2$, avec bord régulier $\partial\Omega$. Soit $\Delta_D$ l'opérateur de Laplace dans $\Omega$ avec conditions de Dirichlet sur le bord, de domaine $H^2(\Omega) \cap H_0^1(\Omega)$.

On rappelle la formule de Dynkin-Helffer-Sjöstrand (voir E.M.Dynkin [47], B.Hellfer et J.Sjöstrand [55]) :

*Définition* 2.4. Étant donné $\Psi \in C_0^\infty(\mathbb{R})$ on définit

$$\Psi(-h^2\Delta_D) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\Psi}(z)(z+h^2\Delta_D)^{-1}dL(z), \tag{2.44}$$

où $dL(z)$ dénote la mesure de Lebesgue sur $\mathbb{C}$ et $\tilde{\Psi}$ est une extension presque analytique de $\Psi$, c'est à dire, pour $< z >= (1+|z|^2)^{1/2}, \quad N \geq 0$,

$$\tilde{\Psi}(z) = \Big(\sum_{m=0}^{N} \partial^m \Psi(\mathrm{Re}z)(i\mathrm{Im}z)^m/m!\Big)\tau(\mathrm{Im}z/ < \mathrm{Re}z >),$$

où $\tau$ est une fonction $C^\infty$ à valeurs positives telle que $\tau(s) = 1$ si $|s| \leq 1$ et $\tau(s) = 0$ si $|s| \geq 2$.

On a le résultat suivant :

**Théorème 2.12.** *(O.I. et F.Planchon [63, Thm.1.1]) Sous l'hypothèse 2.6, soit $f \in C^\infty(\Omega)$ et soit $\Psi \in C_0^\infty(\mathbb{R}^*)$ telle que*

$$\sum_{j\in\mathbb{Z}} \Psi(2^{-2j}\lambda) = 1, \quad \lambda \in \mathbb{R}. \tag{2.45}$$

*Alors pour tout $p \in [2,\infty)$ on a*

$$\|f\|_{L^p(\Omega)} \leq C_p\Big(\sum_{j\in\mathbb{Z}} \|\Psi(-2^{-2j}\Delta_D)f\|^2_{L^p(\Omega)}\Big)^{1/2}, \tag{2.46}$$

*où l'opérateur $\Psi(-2^{-2j}\Delta_D)$ est défini par la formule* (2.44).

Le lecteur familier avec la théorie des espaces fonctionnels va reconnaître l'inclusion $\dot{B}^{0,2}_p(\Omega) \hookrightarrow L^p(\Omega)$, où on a défini les espace de Besov en utilisant comme norme le terme de droite dans (2.46). On s'attendrait, par analogie avec le cas de $\mathbb{R}^d$, d'obtenir, pour tout $1 < p < +\infty$, la plus forte équivalence

$$\|f\|_{L^p(\Omega)} \approx \|(\sum_j |\Psi(-2^{-2j}\Delta_D)f|^2)^{\frac{1}{2}}\|_{L^p(\Omega)};$$

autrement dit $L^p(\Omega)$ et l'espace de Triebel-Lizorkin $\dot{F}^{0,2}_p(\Omega)$ coïncident. Une telle équivalence est montrée par H.Triebel dans [107, 108, 109], pourtant il faudrait la reconstruire en utilisant des résultats de différentes sections (les espaces fonctionnels sont introduits différemment, seuls les cas non-homogènes sont traités, etc). Par contre, la note [63] présente une preuve auto-contenue et quasiment élémentaire de ce théorème, basée essentiellement sur des intégrations par parties. La stratégie utilisée dans cette note est de se réduire à des estimations sur le flot de la chaleur, en démontrant une propriété de presque orthogonalité entre les projecteurs spectraux et la localisation du flot de la chaleur.

## 2.7 Résultats concernant l'équation de Schrödinger non-linéaire en dimension 3 dans un domaine non-captant

Soit $\Theta \subset \mathbb{R}^3$ un obstacle non-captant, c'est à dire tel que tout rayon lumineux qui se propage dans $\Omega := \mathbb{R}^d \setminus \Theta$ selon les lois de l'optique géométrique ne reste pas dans un compact de $\bar{\Omega}$ un temps infiniment long. Les contre-exemples construits dans le Théorème 1.1 [61, 62] pour l'équation des ondes et la perte induite par les données initiales dans un espace de modes de galerie pour l'équation de Schrödinger (voir la Proposition 4.2) laissent penser que des pertes de dérivées pourraient apparaître dans les inégalités de Strichartz pour le flot de Schrödinger dans $\Omega$ dès que le bord $\partial\Omega$ n'est pas strictement convexe.

En collaboration avec Fabrice Planchon nous avons montré dans [64] une classe d'estimations (invariantes d'échelle) dans ce contexte - en combinant des estimations de projecteurs spectraux (obtenues récemment par H.Smith et C.Sogge [96] pour des domaines bornés) et l'effet régularisant (voir N.Burq, Gérard, N.Tzvetkov [26] ou G.Staffilani et D.Tataru [98]) - qui permettent d'établir un résultat d'existence locale pour l'équation de Schrödinger quintique (critique en dimension 3) ainsi que le scattering pour l'équation de Schrödinger sous-critique défocalisante.

**Théorème 2.13.** *(O.I. et F.Planchon [64], Existence locale pour l'équation quintique) Soit $u_0 \in H^1_D(\Omega)$. Il existe $T(u_0)$ tel que l'équation quintique*

$$i\partial_t u - \Delta_D u = \pm|u|^4 u \text{ dans } \Omega \times \mathbb{R}, \quad u|_{t=0} = u_0 \text{ dans } \Omega, \tag{2.47}$$

*(où $\Delta_D = \sum_{j=1}^3 \partial_j^2$ est le laplacien sur $\Omega$) admet une unique solution $u \in C([-T,T], H^1_0(\Omega)) \cap \dot{B}^{1,2}_5(L^{20/11}_T)$. En plus, la solution est globale en temps et admet un comportement de type scattering dans $H^1_0(\Omega)$ si la donnée initiale est suffisamment petite (en norme $H^1_0(\Omega)$).*

**Idée de la preuve :** Dans un premier temps on établit une estimation de la norme $L^5_x L^2_t$ pour le flot linéaire de Schrödinger $e^{-it\Delta_D}u_0(x)$, d'où on déduit, pour $q \geq 2$

$$\|\Psi(-h^2\Delta_D)e^{-it\Delta_D}u_0\|_{L^5(\Omega)L^q(\mathbb{R})} \lesssim h^{\frac{2}{q}-\frac{9}{10}}\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}, \tag{2.48}$$

où $\Psi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ est une localisation spectrale à fréquence $1/h$. Pour cela on introduit une troncature $\chi$ qui localise près du bord et on montre que $\chi e^{-it\Delta_D}u_0$ est solution d'une équation de Schrödinger non-linéaire sur le billard de Sinaï de bord $\partial\Omega$ (qui est défini comme étant l'extérieur de $\Theta$ dans un cube de $\mathbb{R}^d$ avec des conditions périodiques au bord). On réduit de cette façon la preuve à montrer (2.48) où le laplacien $\Delta_D$ (sur le domaine extérieur $\Omega$) est maintenant remplacé par l'opérateur de Laplace $\Delta_S = \sum_{j=1}^3 \partial_j^2$ sur une variété compacte $S$, pour lequel on peut utiliser les estimations des projecteurs spectraux de H.Smith et C.Sogge [96]. En fait, le point clé est le résultat suivant (voir [96, Thm.7.1])

$$\|\Pi_\lambda\|_{L^2(S)\to L^5(S)} \leq \lambda^{2/5},$$

où $\Pi_\lambda = 1_{\sqrt{-\Delta_S}\in[\lambda,\lambda+1)}$ désigne le projecteur spectral associé à l'opérateur $\Delta_S$ sur $S$.

L'estimation (2.48) avec $q = 20/9$ implique $e^{-it\Delta_D}u_0 \in W^{1,5}(\Omega)L_t^{20/9}$ et d'autre part, en prenant $q = 40$, on obtient $e^{-it\Delta_D}u_0 \in W^{3/20,5}(\Omega)L_t^{40} \subset L^{20/3}(\Omega)L_t^{40}$. On conclut par un argument de point fixe dans l'espace $X_T := W^{1,5}(\Omega)L_T^{20/9} \cap L^{20/3}(\Omega)L_T^{40}$ pour démontrer l'existence d'une solution $u$ de l'équation non-linéaire (2.46) sur un temps $[0, T)$ pour $T$ suffisamment petit.

À partir de (2.48) on obtient aussi le comportement de type scattering pour l'équation sous-critique défocalisante

$$i\partial_t u + \Delta_D u = \pm|u|^{p-1}u \text{ dans } \Omega \times \mathbb{R}, \quad u|_{t=0} = u_0 \text{ dans } \Omega, \tag{2.49}$$

sur un domaine non-captant en dimension trois de l'espace pour tout $3 < p < 5$ (rappelons que le cas d'un domaine étoilé avec $p = 3$ a été traité dans [87]). Notons ici que le cas $3 + 2/5 < p < 5$ est obtenu comme conséquence du caractère bien posé dans $H^{s_p}$, $s_p = \frac{3}{2} - \frac{2}{p-1}$ et des méthodes de [87] ; le cas $3 < p \leq 3 + 2/5$ est traité en étendant l'argument de [87].

**Théorème 2.14.** *(O.I. et F.Planchon [64]) Soit* $3+\frac{2}{5} < p < 5$, $s_p = \frac{3}{2}-\frac{2}{p-1}$ *et* $u_0 \in H_0^{s_p}(\Omega)$. *Il existe* $T(u_0)$ *tel que le problème non linéaire* (2.49) *admet une unique solution* $u \in C([-T, T], H_0^{s_p}(\Omega)) \cap \dot{B}_5^{s_p,2}(\Omega; L^{\frac{20}{11}}(-T, T))$. *En plus la solution est globale en temps et admet un comportement de type scattering dans* $H^{s_p}(\Omega)$ *si la donnée initiale est suffisamment petite.*

Finalement, on considère l'asymptotique à temps grand de (2.49) dans le cas défocalisant, c.a.d. avec signe $+$ dans le terme de droite.

**Théorème 2.15.** *(O.I. et F.Planchon [64]) On suppose que le domaine* $\Omega$ *est l'extérieur d'un obstacle étoilé, compact, de* $\mathbb{R}^3$ *(ce qui implique que* $\Omega$ *est non captant). Soient* $3 \leq p < 5$ *et* $u_0 \in H_0^1(\Omega)$. *Alors il existe une unique solution u globale en temps, qui appartient à l'espace* $C(R, H_0^1(\Omega))$, *du problème non linéaire* (2.49). *En plus, il existe* $u^\pm \in H_0^1(\Omega)$ *tels que*

$$\lim_{t\to\pm\infty} \|u(x,t) - e^{it\Delta_D}u^\pm\|_{H_0^1(\Omega)} = 0.$$

# 3 Conclusions et perspectives

Dans cette thèse nous avons mis en évidence des phénomènes d'un type nouveau, liés au bord, dans l'étude de la dispersion pour les ondes. Les contre-exemples montrent que l'on doit renoncer à obtenir les estimations des variétés sans bord. Il reste maintenant à comprendre quelles estimations obtenir, dans le cas des ondes mais également pour Schrödinger où la situation semble encore moins claire. Dans le cadre des applications aux équations non-linéaires, nous avons obtenu des résultats semblables à ceux du cas plat (équation de Schrödinger non-linéaire critique sur un domaine extérieur non-captant), et ce en l'absence des estimations de Strichartz usuelles. Ces estimations sont cruciales pour étudier l'asymptotique à grand temps, et comprendre contourner leur absence semble important pour traiter ces problèmes critiques. Voici quelques problèmes ouverts :

— Soient $d \geq 2$, $(q, r)$ une paire $d$- admissible pour l'équation des ondes et $\beta$ tel que
$$\frac{1}{q} = \alpha(\frac{1}{2} - \frac{1}{r}), \quad \beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}.$$
Un problème intéressant est de déterminer l'ensemble d'indices optimaux $(q, r, \beta)$ tels que les inégalités de Strichartz pour l'équation des ondes dans un domaine compact (1.13) (ou au moins strictement convexe) soient vérifiées en toute dimension $d \geq 2$.

— Le résultat de G.Lebeau [74] laisse penser que les estimations de type Strichartz (1.14) pourraient être vraies avec $\alpha = \frac{d-1}{2} - \frac{1}{6}$ en toute dimension $d \geq 2$ à l'intérieur d'un domaine strictement convexe. M.Blair, H.Smith et C.Sogge [15] démontrent ce résultat sur toute variété compacte mais seulement en dimension 2 (voir (1.16)). Ce serait intéressant, en suivant l'approche de G.Lebeau [74], d'améliorer ce résultat en toute dimension $d \geq 3$ (et, éventuellement, en combinant les deux méthodes, d'obtenir un résultat similaire pour toute variété compacte à bord).

— Les contre-exemples construits dans le Théorème 1.1 (voir [61, 62]) pour l'équation des ondes dans un domaine strictement microlocalement convexe $\Omega$ à bord laissent penser que la même méthode pourrait s'appliquer pour en déduire des pertes de dérivées pour les inégalités de Strichartz pour l'équation de Schrödinger semi-classique. D'autre part, dans [61] on montre aussi que pour des données initiales dans un espace des modes de galeries des pertes sont inévitables pour le flot de Schrödinger semi-classique et, plus précisément, la perte de $\frac{1}{6}(\frac{1}{2} - \frac{1}{r})$ par rapport aux inégalités de l'espace libre (2.4) est optimale dans les inégalités de Strichartz dans $\Omega$ si $u_0$ est un mode de galeries. Une question naturelle est de se demander à quoi on pourrait s'attendre si on adaptait la construction précédente dans le cas de l'équation de Schrödinger, et en particulier est-ce qu'une telle approche pourrait mettre en évidence une perte plus importante de dérivées que dans le cas des modes de galeries ?

— Une conséquence immédiate de Théorème 2.8 est la validité des inégalités de Strichartz (classiques) locales en temps avec une perte $1/q$ de dérivées sur le billard de Sinaï. En effet, cela découle par la même méthode que celle proposée par N.Burq, P.Gérard et N.Tzvetkov sur un compact sans bord dans [27]. Une question naturelle est de se demander si on pourrait encore améliorer ces estimations sur une variété compacte à bord strictement concave.

— Dans un travail récent, N.Burq, C.Guillarmou et A.Hassel démontrent que dans une géométrie captive hyperbolique $\Omega$ satisfaisant l'hypothèse 2.2, la présence des trajectoires captives n'empêche pas la validité des estimations de Strichartz sans pertes pour les solutions d'équations de Schrödinger et des ondes. La preuve est basée sur les estimations de Strichartz usuelles (2.41) à l'extérieur d'un obstacle strictement convexe. Un problème intéressant serait de comprendre la dynamique en grand temps lorsque le domaine $\Omega$ est le complémentaire de plusieurs obstacles étoilés (mais pas strictement convexes) dans $\mathbb{R}^d$ mais tel qu'il n'y a que des rayons captés hyperboliques.

# Deuxième partie

# Équation des ondes

## Sommaire

# 4 Counterexamples to Strichartz estimates for the wave equation in domains

## 4.1 Introduction

Let $\Omega$ be the upper half plane $\{(x,y) \in \mathbb{R}^2, x > 0, y \in \mathbb{R}\}$. Define the Laplacian on $\Omega$ to be $\Delta_D = \partial_x^2 + (1+x)\partial_y^2$, together with Dirichlet boundary conditions on $\partial\Omega$: one may easily see that $\Omega$, with the metric inherited from $\Delta_D$, is a strictly convex domain. We shall prove that, in such a domain $\Omega$, Strichartz estimates for the wave equation suffer losses when compared to the usual case $\Omega = \mathbb{R}^2$, at least for a subset of the usual range of indices. Our construction is microlocal in nature; in [60] we prove that the same result holds true for any regular domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3, 4$, provided there exists a point in $T^*\partial\Omega$ where the boundary is microlocally strictly convex.

*Definition* 4.1. Let $q, r \geq 2$, $(q, r, \alpha) \neq (2, \infty, 1)$. A pair $(q, r)$ is called $\alpha$-admissible if

$$\frac{1}{q} + \frac{\alpha}{r} \leq \frac{\alpha}{2}, \tag{4.1}$$

and sharp $\alpha$-admissible whenever equality holds in (4.1). For a given dimension $d$, a pair $(q, r)$ will be *wave-admissible* if $d \geq 2$ and $(q, r)$ is $\frac{d-1}{2}$-admissible; it will be *Schrödinger-admissible* if $d \geq 1$ and $(q, r)$ is sharp $\frac{d}{2}$-admissible. Finally, notice that the endpoint $(2, \frac{2\alpha}{\alpha-1})$ is sharp $\alpha$-admissible when $\alpha > 1$.

When $\alpha = 1$ the endpoint pairs are inadmissible and the endpoint estimates for wave equation ($d = 3$) and Schrödinger equation ($d = 2$) are known to fail: one obtains a logarithmic loss of derivatives which gives Strichartz estimates with $\epsilon$ losses.

Our main result reads as follows:

**Theorem 4.1.** *Let $(q, r)$ be a sharp wave-admissible pair in dimension $d = 2$ with $4 < r < \infty$. There exist $\psi_j \in C_0^\infty(\mathbb{R})$ and for every small $\epsilon > 0$ there exist $c_\epsilon > 0$ and sequences $V_{h,j,\epsilon} \in C^\infty(\Omega)$, $j = \overline{0,1}$ with $\psi_j(hD_y)V_{h,j,\epsilon} = V_{h,j,\epsilon}$ (meaning that $V_{h,j,\epsilon}$ are localized at frequency $1/h$), such that the solution $V_{h,\epsilon}$ to the wave equation with Dirichlet boundary conditions*

$$(\partial_t^2 - \Delta_D)V_{h,\epsilon} = 0, \quad V_{h,\epsilon}|_{[0,1]\times\partial\Omega} = 0, \quad V_{h,\epsilon}|_{t=0} = V_{h,0,\epsilon}, \quad \partial_t V_{h,\epsilon}|_{t=0} = V_{h,1,\epsilon} \tag{4.2}$$

*satisfies*

$$\sup_{h,\epsilon>0} (\|V_{h,0,\epsilon}\|_{\dot{H}^{2(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-2\epsilon}(\Omega)} + \|V_{h,1,\epsilon}\|_{\dot{H}^{2(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-2\epsilon-1}(\Omega)}) \leq 1 \tag{4.3}$$

*and*

$$\lim_{h\to 0} \|V_{h,\epsilon}\|_{L_t^q([0,1],L^r(\Omega))} = \infty. \tag{4.4}$$

*Moreover $V_{h,\epsilon}$ has compact support for $x$ in $(0, h^{\frac{1-\epsilon}{2}}]$ and is well localized at spatial frequency $1/h$; hence, the left hand side in (4.3) is equivalent to*

$$h^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon}(\|V_{h,0,\epsilon}\|_{L^2(\Omega)} + h\|V_{h,1,\epsilon}\|_{L^2(\Omega)}).$$

*Remark* 4.1. In this paper we are rather interested in negative results: Theorem 4.1 shows that for $r > 4$ losses of derivatives are unavoidable for Strichartz estimates, and more specifically a regularity loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ occurs when compared to the free case.

*Remark* 4.2. The key feature of the domain leading to the counterexample is the strict-convexity of the boundary, i.e. the presence of gliding rays, or highly-multiply reflected geodesics. The particular manifold studied in this paper is one for which the eigenmodes are explicitly in terms of Airy's functions and the phases for the oscillatory integrals to be evaluated have precise form. In a forthcoming work [60] we construct examples for general manifolds with a gliding ray, but the heart of the matter is well illustrated by this particular example which generalizes using Melrose's equivalence of glancing hypersurfaces theorem.

We now recall known results in $\mathbb{R}^d$. Let $\Delta_d$ denote the Laplace operator in the flat space $\mathbb{R}^d$. Strichartz estimates read as follows (see [70]):

**Proposition 4.1.** *Let $d \geq 2$, $(q, r)$ be wave-admissible and consider $u$, solution to the wave equation*

$$(\partial_t^2 - \Delta_d)u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \tag{4.5}$$

*for $u_0, u_1 \in C^\infty(\mathbb{R}^d)$; then there is a constant $C$ such that*

$$\|u\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C(\|u_0\|_{\dot{H}^{d(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}}(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{d(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}-1}(\mathbb{R}^d)}). \tag{4.6}$$

**Proposition 4.2.** *Let $d \geq 1$, $(q, r)$ be Schrödinger-admissible pair and $u$, solution to the Schrödinger equation*

$$(i\partial_t + \Delta_d)u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad u|_{t=0} = u_0, \tag{4.7}$$

*for $u_0 \in C^\infty(\mathbb{R}^d)$; then there is a constant $C$ such that*

$$\|u\|_{L^q_t(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{4.8}$$

Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with Strichartz pioneering work [100], where he proved the particular case $q = r$ for the wave and (classical) Schrödinger equations. This was later generalized to mixed $L^q_t L^r_x$ norms by Ginibre and Velo [49] for Schrödinger equations, where $(q, r)$ is sharp admissible and $q > 2$; the wave estimates were obtained independently by Ginibre-Velo [51] and Lindblad-Sogge [75], following earlier work by Kapitanski [67]. The remaining endpoints for both equations were finally settled by Keel and Tao [70].

In the variable coefficients case, even without boundary, the situation is much more complicated: we simply recall here the pioneering work of G.Staffilani and D.Tataru [98], dealing with compact, non trapping perturbations of the flat metric and recent work of J.M.Bouclet and N.Tzvetkov [17] in the context of Schrodinger equation, which considerably weakens the decay of the perturbation (retaining the non trapping character at spatial infinity). On compact manifolds without boundary, due to the finite speed of propagation, it is enough to work in coordinate charts and to establish local Strichartz estimates for variable coefficients wave operators in $\mathbb{R}^d$: we recall here the works by L.Kapitanski [66] and G.Mockenhaupt, A.Seeger and C.Sogge [82] in the case of smooth coefficients when one can use the Lax parametrix construction to obtain the appropriate dispersive estimates. In the case of $C^{1,1}$ coefficients, Strichartz estimates were shown in the works by H.Smith [93] and by D.Tataru [103], the latter work establishing the full range of local estimates; here the lack of smoothness prevents the use of Fourier integral operators and instead wave packets and coherent state methods are used to construct parametrices for the wave operator.

For a manifold with smooth, strictly geodesically concave boundary, the Melrose and Taylor parametrix yields the Strichartz estimates for the wave equation with Dirichlet boundary condition (not including the endpoints) as shown in the paper of Smith and Sogge [95]. If the concavity assumption is removed, however, the presence of multiply reflecting geodesic and their limits, gliding rays, prevent the construction of a similar parametrix!

In [71], Koch, Smith and Tataru obtained "log-loss" estimates for the spectral clusters on compact manifolds without boundary. Recently, Burq, Lebeau and Planchon [28] established Strichartz type inequalities on a manifold with boundary using the $L^r(\Omega)$ estimates for the spectral projectors obtained by Smith and Sogge [96]. The range of indices $(q, r)$ that can be obtained in this manner, however, is restricted by the allowed range of $r$ in the squarefunction estimate for the wave equation, which control the norm of $u$ in the space $L^r(\Omega, L^2(-T, T))$, $T > 0$ (see [96]). In dimension 3, for example, this restricts the indices to $q, r \geq 5$. The work of Blair, Smith and Sogge [15] expands the range of indices $q$ and $r$ obtained in [28]: specifically, they show that if $\Omega$ is a compact manifold with boundary and $(q, r, \beta)$ is a triple satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta,$$

together with the restriction

$$\begin{cases} \frac{3}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}, & d \leq 4 \\ \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, & d \geq 4, \end{cases}$$

then the Strichartz estimates (5.2) hold true for solutions $u$ to the wave equation (4.2) satisfying Dirichlet or Neumann homogeneous boundary conditions, with a constant $C$ depending on $\Omega$ and $T$.

*Remark* 4.3. Notice that Theorem 4.1 states for instance that the scale-invariant Strichartz estimates fail for $\frac{3}{q} + \frac{1}{r} > \frac{15}{24}$, whereas the result of Blair, Smith and Sogge states that such estimates hold if $\frac{3}{q} + \frac{1}{r} \leq \frac{1}{2}$. Of course, the counterexample places a lower bound on the loss for such indices $(q, r)$, and the work [15] would place some upper bounds, but this concise statement shows one explicit gap in our knowledge that remains to be filled.

A very interesting and natural question would be to determine the sharp range of exponents for the Strichartz estimates in any dimension $d \geq 2$!

A classical way to prove Strichartz inequalities is to use dispersive estimates (see (4.18)). The fact that weakened dispersive estimates can *still* imply optimal (and scale invariant) Strichartz estimates for the solution of the wave equation was first noticed by Lebeau: in [74] he proved dispersive estimates with losses (which turned out to be optimal) for the wave equation inside a strictly convex domain from which he deduced Strichartz type estimates without losses but for indices $(q, r)$ satisfying (8.4) with $\alpha = \frac{1}{4}$ in dimension 2.

A natural strategy for proving Theorem 4.1 would be to use the Rayleigh whispering gallery modes which accumulate their energy near the boundary, contributing to large $L^r$ norms. Applying the semi-classical Schrödinger evolution shows that a loss of $\frac{1}{6}(\frac{1}{2} - \frac{1}{r})$ derivatives is necessary for the Strichartz estimates. However, when dealing with the wave operator this strategy fails as the gallery modes satisfy the Strichartz estimates of the free space:

**Theorem 4.2.** *Let $d \geq 2$, let $\Omega = \{x > 0, y \in \mathbb{R}^{d-1}\}$ and consider the following Laplace operator on $\Omega$*

$$\Delta_D = \partial_x^2 + (1+x)\Delta_{d-1}, \quad \textit{where} \quad \Delta_{d-1} = \sum_{j=1}^{d-1} \partial_{y_j}^2. \tag{4.9}$$

*Let $\psi \in C_0^\infty(\mathbb{R}^{d-1} \setminus \{0\})$, $k \geq 1$ and $u_0 \in E_k(\Omega)$, where $E_k(\Omega)$ is to be later defined by* (4.23)*.*

1. *Let $(q, r)$ be a Schrödinger-admissible pair in dimension $d$ with $q > 2$ and consider the semi-classical Schrödinger equation with Dirichlet boundary condition*

$$(\frac{h}{i}\partial_t - h^2\Delta_D)u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = \psi(hD_y)u_0. \tag{4.10}$$

   *Then $u$ satisfies the following Strichartz estimates with a loss,*

$$\|u\|_{L^q([0,T_0],L^r(\Omega))} \lesssim h^{-(\frac{d}{2}+\frac{1}{6})(\frac{1}{2}-\frac{1}{r})} \|u|_{t=0}\|_{L^2(\Omega)}. \tag{4.11}$$

   *Moreover, the bounds* (4.11) *are optimal.*

2. *Let $(q, r)$ be a wave-admissible pair in dimension $d$ with $q > 2$ and consider the wave equation with Dirichlet boundary conditions*

$$(\partial_t^2 - \Delta_D)u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = \psi(hD_y)u_0, \quad \partial_t u|_{t=0} = 0. \tag{4.12}$$

   *Then the solution $u$ of* (4.12) *satisfies*

$$\|u\|_{L^q([0,T_0],L^r(\Omega)} \lesssim h^{-d(\frac{1}{2}-\frac{1}{r})+\frac{1}{q}} \|u|_{t=0}\|_{L^2(\Omega)}. \tag{4.13}$$

*Remark* 4.4. We prove Theorem 4.2 for the model case of the $d$-dimensional half-space together with the metric inherited from the Laplacian $\Delta_D$ defined by (4.9). It is very likely that, using the parametrix introduced by Eskin [48], we could obtain the same result for general operators.

Notice that if the initial data $u_0$ belongs to $E_k(\Omega)$ for some $k \geq 1$ then the solution $u(t,x,y)$ to (4.10) localized in frequency at the level $1/h$ is given by

$$u(t,x,y) = \frac{1}{(2\pi h)^{d-1}} \int e^{\frac{iy\eta}{h}} \hat{u}(t,x,\eta/h) d\eta,$$

therefor

$$\hat{u}(t,x,\eta/h) = e^{ith\lambda_k(\eta/h)} \psi(\eta) \hat{u}_0(x,\eta/h),$$

where $\lambda_k(\eta) = |\eta|^2 + \omega_k |\eta|^{4/3}$ and $\hat{u}_0(x,\eta/h) = Ai(|\eta|^{2/3} x/h^{2/3} - \omega_k)$ is the eigenfunction of $-\Delta_{D,\eta} = -\partial_x^2 + (1+x)\eta^2$ corresponding to the eigenvalue $\lambda_k$.

Theorem 4.2 shows that the method we used for the Schrödinger equation cannot yield Theorem 4.1. We will proceed in a different manner, using co-normal waves with multiply reflected cusps at the boundary (see Figure 4).

The paper is organized as follows: in Section 4.2 we will use gallery modes in order to prove Theorem 4.2; in Section 4.3 we prove Theorem 4.1. Finally, the Appendix collects several useful results.

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## 4.2 Whispering gallery modes

### 4.2.1 Strichartz inequalities

Let $n \geq 2$, $0 < T_0 < \infty$, $\psi(\xi) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth function $G \in C^\infty$ near the support of $\psi$. Let $u_0 \in L^2(\mathbb{R}^n)$ and $h \in (0,1]$ and consider the following semi-classical problem

$$ih\partial_t u - G(\frac{h}{i}D)u = 0, \quad u|_{t=0} = \psi(hD)u_0. \tag{4.14}$$

If we denote by $e^{-\frac{it}{h}G}$ the linear flow, the solution of (4.14) writes

$$e^{-\frac{it}{h}G}\psi(hD)u_0(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(<x,\xi> - tG(\xi))} \psi(\xi) \hat{u_0}(\frac{\xi}{h}) d\xi. \tag{4.15}$$

Let $q \in (2, \infty]$, $r \in [2, \infty]$ and set

$$\frac{1}{q} = \alpha(\frac{1}{2} - \frac{1}{r}), \quad \beta = (n - \alpha)(\frac{1}{2} - \frac{1}{r}). \tag{4.16}$$

*Remark* 4.5. Notice that the pair $(q, r)$ is Schrödinger-admissible in dimension $n$ if $\alpha = \frac{n}{2}$ and wave admissible if $\alpha = \frac{n-1}{2}$.

With the notations in (4.16) the Strichartz inequalities for (4.14) read as follows

$$h^{\beta} \| e^{-\frac{it}{h} G} \psi(hD) u_0 \|_{L^q((0,T_0], L^r(\mathbb{R}^n))} \leq C \| \psi(hD) u_0 \|_{L^2(\mathbb{R}^n)}. \tag{4.17}$$

The classical way to prove (5.2) is to use dispersive inequalities which read as follows

$$\| e^{-\frac{it}{h} G} \psi(hD) u_0 \|_{L^\infty(\mathbb{R}^n)} \lesssim (2\pi h)^{-n} \gamma_{n,h}(\frac{t}{h}) \| \psi(hD) u_0 \|_{L^1(\mathbb{R}^n)} \tag{4.18}$$

for $t \in [0, T_0]$, where we set

$$\gamma_{n,h}(\lambda) = \sup_{z \in \mathbb{R}^n} | \int e^{i\lambda(z\xi - G(\xi))} \psi(\xi) d\xi |. \tag{4.19}$$

In Section 4.6.1 of the Appendix we prove the following:

**Lemma 4.1.** *Let $\alpha \geq 0$ and $(q, r)$ be an $\alpha$-admissible pair in dimension $n$ with $q > 2$. Let $\beta$ be given by* (4.16)*. If the solution $e^{-\frac{it}{h} G}(\psi(hD) u_0)$ of* (4.14) *satisfies the dispersive estimates* (4.18) *for some function $\gamma_{n,h} : \mathbb{R} \to \mathbb{R}_+$, then there exists some $C > 0$ independent of $h$ such that the following inequality holds*

$$h^{\beta} \| e^{-\frac{it}{h} G} \psi(hD) u_0 \|_{L^q((0,T_0], L^r(\mathbb{R}^n))} \leq C \Big( \sup_{s \in (0, \frac{T_0}{h})} s^{\alpha} \gamma_{n,h}(s) \Big)^{\frac{1}{2} - \frac{1}{r}} \| u_0 \|_{L^2(\mathbb{R}^n)}. \tag{4.20}$$

### 4.2.2 Gallery modes

Let $\Omega = \{(x, y) \in \mathbb{R}^d | x > 0, y \in \mathbb{R}^{d-1}\}$ denote the half-space $\mathbb{R}^d_+$ with the Laplacian given by (4.9) with Dirichlet boundary condition on $\partial\Omega$. Taking the Fourier transform in the $y$-variable gives

$$-\Delta_{D,\eta} = -\partial_x^2 + (1 + x)|\eta|^2. \tag{4.21}$$

For $\eta \neq 0$, $-\Delta_{D,\eta}$ is a self-adjoint, positive operator on $L^2(\mathbb{R}_+)$ with compact resolvent. Indeed, the potential $V(x, \eta) = (1 + x)\eta^2$ is bounded from below, it is continuous and $\lim_{x \to \infty} V(x, \eta) = \infty$. Thus one can consider the form associated to $-\partial_x^2 + V(x, \eta)$,

$$Q(u) = \int_{x>0} |\partial_x v|^2 + V(x, \eta)|v|^2 dx, \quad D(Q) = H_0^1(\mathbb{R}_+) \cap \{v \in L^2(\mathbb{R}_+), (1+x)^{1/2} v \in L^2(\mathbb{R}_+))\},$$

which is clearly symmetric, closed and bounded from below. If $c \gg 1$ is chosen such that $-\Delta_{D,\eta} + c$ is invertible, then $(-\Delta_{D,\eta} + c)^{-1}$ sends $L^2(\mathbb{R}_+)$ in $D(Q)$ and we deduce that $(-\Delta_{D,\eta} + c)^{-1}$ is also a (self-adjoint) compact operator. The last assertion follows from the compact inclusion

$$D(Q) = \{v|\partial_x v, (1+x)^{1/2} v \in L^2(\mathbb{R}_+), v(0) = 0\} \hookrightarrow L^2(\mathbb{R}_+).$$

We deduce that there exists a base of eigenfunctions $v_k$ of $-\Delta_{D,\eta}$ associated to a sequence of eigenvalues $\lambda_k(\eta) \to \infty$. From $-\Delta_{D,\eta} v = \lambda v$ we obtain $\partial_x^2 v = (\eta^2 - \lambda + x\eta^2)v$, $v(0,\eta) = 0$ and after a change of variables we find the eigenfunctions

$$v_k(x,\eta) = Ai(|\eta|^{\frac{2}{3}} x - \omega_k), \tag{4.22}$$

where $(-\omega_k)_k$ are the zeros of Airy's function in decreasing order. The corresponding eigenvalues are $\lambda_k(\eta) = |\eta|^2 + \omega_k |\eta|^{\frac{4}{3}}$.

*Definition* 4.2. For $x > 0$ let $E_k(\Omega)$ be the closure in $L^2(\Omega)$ of

$$\{u(x,y) = \frac{1}{(2\pi)^{d-1}} \int e^{iy\eta} Ai(|\eta|^{\frac{2}{3}} x - \omega_k)\hat{\varphi}(\eta) d\eta, \varphi \in \mathcal{S}(\mathbb{R}^{d-1})\}, \tag{4.23}$$

where $\mathcal{S}(\mathbb{R}^{d-1})$ is the Schwartz space of rapidly decreasing functions,

$$\mathcal{S}(\mathbb{R}^{d-1}) = \{f \in C^\infty(\mathbb{R}^{d-1}) | \|z^\alpha D^\beta f\|_{L^\infty(\mathbb{R}^{d-1})} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^{d-1}\}.$$

For $k$ fixed, a function $u \in E_k(\Omega)$ is called whispering gallery mode. Moreover, a function $u \in E_k(\Omega)$ satisfies

$$(\partial_x^2 + x\Delta_{d-1} - \omega_k |\Delta_{d-1}|^{\frac{2}{3}})u = 0. \tag{4.24}$$

*Remark* 4.6. We have the decomposition

$$L^2(\Omega) = \bigoplus_{\perp} E_k(\Omega).$$

Indeed, from the discussion above one can easily see that $(E_k(\Omega))_k$ are closed, orthogonal and that $\cup_k E_k(\Omega)$ is a total family (i.e. that the vector space spanned by $\cup_k E_k(\Omega)$ is dense in $L^2(\Omega)$).

In Section 4.6.3 of the Appendix we prove the following:

**Lemma 4.2.** *Let $\psi$, $\psi_1$, $\psi_2 \in C_0^\infty(\mathbb{R}^{d-1} \setminus \{0\})$ be such that $\psi_1 \psi = \psi_1$ and $\psi\psi_2 = \psi$ and let $\varphi \in C^\infty(\mathbb{R}^{d-1})$. Fix $k \geq 1$ and let $u \in E_k(\Omega)$ be the function associated to $\varphi$ in $E_k(\Omega)$. For $r \in [1, \infty]$ there exist $C_1$, $C_2 > 0$ such that*

$$C_1 \|\psi_1(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})} \leq h^{-\frac{2}{3r}} \|\psi(hD_y)u\|_{L^r(\mathbb{R}_+ \times \mathbb{R}^{d-1})} \leq C_2 \|\psi_2(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})}. \tag{4.25}$$

As a consequence of Lemma 4.2 we have

**Corollary 4.1.** *Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^{d-1})$, $k \geq 1$ and $u_0 \in E_k(\Omega)$ be such that*

$$u_0(x,y) = \frac{1}{(2\pi)^{d-1}} \int e^{iy\eta} Ai(|\eta|^{\frac{2}{3}} x - \omega_k)\hat{\varphi}_0(\eta) d\eta. \tag{4.26}$$

*In order to prove Theorem 4.2 we shall reduce the problem to the study of Strichartz type estimates for a problem with initial data $\varphi_0$. More precisely, from Lemma 4.2 we immediately deduce the following*

1. *if $u$ solves (4.10) with initial data $\psi(hD_y)u_0$, where $u_0$ is given by (4.26) then in order to prove that one can't do better than (4.11) it is enough to establish that the solution $\varphi$ to*

$$\frac{h}{i}\partial_t \varphi - h^2(\Delta_{d-1} - \omega_k |\Delta_{d-1}|^{\frac{2}{3}})\varphi = 0, \quad \varphi|_{t=0} = \psi(hD_y)\varphi_0. \tag{4.27}$$

*satisfies the following Strichartz type estimates*

$$\|\varphi\|_{L^q([0,T_0],L^r(\mathbb{R}^{d-1}))} \leq c h^{-\frac{(d-1)}{2}(\frac{1}{2}-\frac{1}{r})} \|\psi(hD_y)\varphi_0\|_{L^2(\mathbb{R}^{d-1})}. \tag{4.28}$$

2. *if $u$ solves (4.12) with initial data $(\psi(hD_y)u_0, 0)$ then in order to show that the gallery modes give rise to the same Strichartz estimates as in the free case it is sufficient to prove that the solution to*

$$\partial_t^2 \varphi - (\Delta_{d-1} - \omega_k |\Delta_{d-1}|^{\frac{2}{3}})\varphi = 0, \quad \varphi|_{t=0} = \psi(hD_y)\varphi_0, \quad \partial_t \varphi|_{t=0} = 0 \tag{4.29}$$

*satisfies*

$$\|\varphi\|_{L^q([0,T_0],L^r(\mathbb{R}^{d-1}))} \leq c h^{-(\frac{d}{2}-\frac{1}{6})(\frac{1}{2}-\frac{1}{r})} \|\psi(hD_y)\varphi_0\|_{L^2(\mathbb{R}^{d-1})}. \tag{4.30}$$

*Remark* 4.7. Notice that for $\tilde{q} \geq q > 2$ and $f \in C^\infty([0,T])$ we have

$$\|f\|_{L^q([0,T])} \lesssim \|f\|_{L^{\tilde{q}}([0,T])},$$

thus in order to prove Theorem 4.2 it suffices to prove (4.28) (respective (4.30)) with $q$ replaced by some $\tilde{q} \geq q$.

### 4.2.3 Proof of Theorem 4.2

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^{d-1})$, $k \geq 1$, $\omega = \omega_k > 0$ and $u_0 \in E_k(\Omega)$ be such that

$$u_0(x,y) = \frac{1}{(2\pi)^{d-1}} \int e^{iy\eta} Ai(|\eta|^{\frac{2}{3}} x - \omega_k)\hat{\varphi}_0(\eta) d\eta. \tag{4.31}$$

1. **Schrödinger equation**
   Let $\tilde{q}$ be given by

$$\frac{1}{\tilde{q}} = \frac{(d-1)}{2}(\frac{1}{2} - \frac{1}{r}). \tag{4.32}$$

Let $G_s(\eta) = |\eta|^2 + \omega h^{\frac{2}{3}}|\eta|^{\frac{4}{3}}$. Using Corollary 4.1 and Remark 4.7 we are reduced to prove (4.28), with $q$ replaced by $\tilde{q}$, i.e. in order to prove Theorem 4.2 for the Schrödinger operator it will be enough to establish

$$\|e^{-\frac{it}{h}G_s}(\psi(hD_y)\varphi_0)\|_{L^{\tilde{q}}([0,T],L^r(\mathbb{R}^{d-1}))} \leq ch^{-\frac{(d-1)}{2}(\frac{1}{2}-\frac{1}{r})}\|\psi(hD_y)\varphi_0\|_{L^2(\mathbb{R}^{d-1})}, \quad (4.33)$$

where

$$e^{-\frac{it}{h}G_s}(\psi(hD_y)\varphi_0)(t,y) = \frac{1}{(2\pi h)^{d-1}}\int e^{\frac{i}{h}(<y,\eta>-t(|\eta|^2+\omega h^{\frac{2}{3}}|\eta|^{\frac{4}{3}}))}\psi(\eta)\hat{\varphi}_0(\frac{\eta}{h})d\eta.$$

Let $\omega = \omega_k$ and set

$$J(z,\frac{t}{h}) := \int e^{i\frac{t}{h}(<z,\eta>-G_s(\eta))}\psi(\eta)d\eta. \quad (4.34)$$

Recall that $0 \notin \text{supp}(\psi)$, thus the phase function is smooth everywhere on the support of $\psi$. With the notations in (4.19) we have to determine $\gamma_{d-1,h}(\frac{t}{h}) = \sup_{z\in\mathbb{R}^{d-1}}|J(z,\frac{t}{h})|$. Note that if $\frac{|t|}{h}$ is bounded we get immediately that $|J(z,\frac{t}{h})|$ is bounded, thus we can consider the quotient $\frac{t}{h}$ to be large. Let $\lambda = \frac{t}{h} \gg 1$ and apply the stationary phase method. There is one critical point

$$z(\eta) = 2\eta + \frac{4}{3}\omega h^{\frac{2}{3}}\frac{\eta}{|\eta|^{\frac{2}{3}}},$$

non-degenerate since $G_s''(\eta) = 2Id + O(h^{\frac{2}{3}})$ for $\eta$ away from 0 and $h$ small enough, and we can also write $\eta = \eta(z)$. We obtain

$$J(z,\lambda) \simeq \Big(\frac{2\pi}{\sqrt{\lambda}}\Big)^{d-1}\frac{e^{-\frac{i\pi}{4}\text{sign}G_s''(\eta(z))}}{\sqrt{\det G_s''(\eta(z))}}e^{i\lambda\Phi(z,\eta(z))}\sigma(z,\lambda), \quad (4.35)$$

where

$$\Phi(z,\eta) =< z,\eta > -G_s(\eta), \quad \Phi(z,\eta(z)) = |\eta(z)|^2 + \frac{\omega}{3}h^{\frac{2}{3}}|\eta(z)|^{\frac{4}{3}},$$

$$\sigma(z;\lambda) \simeq \sum_{k\geq 0}\lambda^{-k}\sigma_k(z), \quad \sigma_0(z) = \psi(\eta(z)).$$

From the definition (4.19) we deduce that we have $\gamma_{d-1,h}(\lambda) \simeq \lambda^{-\frac{(d-1)}{2}}$. Consequently, for $\lambda = \frac{t}{h} \gg 1$ there exists some constant $C > 0$ such that the following dispersive estimate holds

$$\|e^{-\frac{it}{h}G_s}\psi(hD_y)\varphi_0\|_{L^\infty_y(\mathbb{R}^{d-1})} \leq Ch^{-(d-1)}(\frac{|t|}{h})^{-\frac{(d-1)}{2}}\|\psi(hD_y)\varphi_0\|_{L^1_y(\mathbb{R}^{d-1})}. \quad (4.36)$$

Interpolation between (4.36) and the energy estimate gives

$$\|e^{-\frac{it}{h}G_s}\psi(hD_y)\varphi_0\|_{L^r_y(\mathbb{R}^{d-1})} \leq Ch^{-\frac{(d-1)}{2}(1-\frac{2}{r})}|t|^{-\frac{(d-1)}{2}(1-\frac{2}{r})}\|\psi(hD_y)\varphi_0\|_{L^{r'}_y(\mathbb{R}^{d-1})}. \quad (4.37)$$

Let $\tilde{q}'$ be such that $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and let $T$ be the operator

$$T : L^2(\mathbb{R}^{d-1}) \to L^{\tilde{q}}([0,T_0], L^r_y(\mathbb{R}^{d-1}))$$

which to a given $\varphi_0 \in L^2(\mathbb{R}^{d-1})$ associates $e^{-\frac{it}{h}G_s}\psi(hD_y)\varphi_0 \in L^{\tilde{q}}([0,T_0], L^r_y(\mathbb{R}^{d-1}))$. Then inequality (4.37) implies that for every $g \in L^{\tilde{q}'}([0,T_0], L^{r'}_y(\mathbb{R}^{d-1}))$ we have

$$\|TT^*g\|_{L^{\tilde{q}}(0,T_0]L^r_y} = \| \int_0^T e^{-\frac{i(t-s)}{h}G_s}\psi\psi^* g(s)ds\|_{L^{\tilde{q}}((0,T_0],L^r_y)} \leq \tag{4.38}$$

$$\leq Ch^{-\frac{(d-1)}{2}(1-\frac{2}{r})}\| \int_0^T |t-s|^{-\frac{(d-1)}{2}(1-\frac{2}{r})}\|g(s)\|_{L^{r'}_y} ds\|_{L^{\tilde{q}}((0,T_0])}.$$

If $\tilde{q} > 2$ the application $|t|^{-\frac{2}{\tilde{q}}}* : L^{\tilde{q}'} \to L^{\tilde{q}}$ is bounded by the Hardy-Littlewood-Sobolev theorem and we deduce that the $L^{\tilde{q}}((0,T_0], L^r_y(\mathbb{R}^{d-1}))$ norm of the operator TT$^*$ is bounded from above by $h^{-\frac{(d-1)}{2}(1-\frac{2}{r})}$, thus the norm $\|T\|_{L^2 \to L^{\tilde{q}}((0,T_0],L^r_y(\mathbb{R}^{d-1}))}$ is bounded from above by $h^{-\frac{(d-1)}{2}(\frac{1}{2}-\frac{1}{r})}$.

— **Optimality:**
Let $\eta_0 \in \mathbb{R}^{d-1} \setminus \{0\}$ and for $y \in \mathbb{R}^{d-1}$ set

$$\varphi_{0,h}(y,\eta_0) = h^{-(d-1)/4} e^{\frac{i\Phi_0(y,\eta_0)}{h}}, \quad \Phi_0(y,\eta_0) = < y, \eta_0 > + \frac{i|y|^2}{2} \tag{4.39}$$

that satisfies

$$\|\varphi_{0,h}\|_{L^2_y} = \pi^{(d-1)/4}, \quad \|\varphi_{0,h}\|_{L^\infty_y} = h^{-(d-1)/4}.$$

**Proposition 4.3.** *Let $T_0 > 0$ be small enough and let $t \in [-T_0, T_0]$. Then the (local, holomorphic) solution $\varphi_h(t,y,\eta_0)$ of (4.27) with initial data $\varphi_{0,h}(y,\eta_0)$ has the form*

$$\varphi_h(t,y,\eta_0) \simeq h^{-(d-1)/4} e^{\frac{i\Phi(t,y,\eta_0)}{h}} \sigma(t,y,\eta_0,h),$$

*where the phase function $\Phi(t,y,\eta_0)$ satisfies the eikonal equation (4.40) locally in time $t$ and for $y$ in a neighborhood of 0, and where $\sigma(t,y,\eta_0,h) \simeq \sum_{k\geq 0} h^k \sigma_k(t,y,\eta_0)$ is a classical analytic symbol (see [92, Chps.1,9; Thm.9.1] for the definition and for a complete proof).*

*Proof.* For $t \in [-T_0, T_0]$ small enough we can construct approximatively

$$\varphi_h(t,y,\eta_0) = \exp(-i\frac{t}{h}G_s)\varphi_{0,h}(y,\eta_0)$$

to be the local solution to (4.27) with initial data $\varphi_{0,h}(y,\eta_0)$. In order to solve explicitly (4.27) we use geometric optic's arguments: let first $\Phi(t,y,\eta_0)$ be the (local) solution to the eikonal equation

$$\begin{cases} \partial_t\Phi + |\nabla_y\Phi|^2 + \omega|\nabla_y\Phi|^{4/3} = 0, \\ \Phi|_{t=0} = \Phi_0(y,\eta_0). \end{cases} \tag{4.40}$$

The associated complex Lagrangian manifold is given by

$$\Lambda_\Phi = \{(t, y, \tau, \eta) | \tau = \partial_t \Phi, \eta = \nabla_y \Phi\}.$$

Let $q(t, y, \tau, \eta) = \tau + |\eta|^2 + \omega|\eta|^{4/3}$ and let $H_q$ denote the Hamilton field associated to $q$. Then $\Lambda_\Phi$ is generated by the integral curves of $H_q$ which satisfy

$$\begin{cases} (\dot{t}, \dot{y}, \dot{\tau}, \dot{\eta}) = (1, \nabla_\eta q, 0, 0), \\ (t, y, \tau, \eta)|_0 = (0, y_0, -|\nabla_y \Phi_0|^2 - \omega|\nabla_y \Phi_0|^{4/3}, \nabla_y \Phi_0 = \eta_0 + i y_0). \end{cases} \tag{4.41}$$

We parametrize them by $t$ and write the solution

$$(y(t, y_0, \eta_0), \eta(t, y_0, \eta_0)) = \exp(t H_q(y_0, \eta_0)).$$

The intersection

$$\Lambda_\Phi^{\mathbb{R}} := \{\exp t H_q(y_0, \eta_0) | t \in \mathbb{R}\} \cap T^*\mathbb{R}^d \setminus \{0\}$$

is empty unless $y_0 = 0$, since it is so at $t = 0$ and since $d \exp(t H_q)$ preserves the positivity of the $\mathbb{C}$-Lagrangian $\Lambda_\Phi$ (see Definition 4.5 of the Appendix and Lemma 4.10). Thus on the bicharacteristic starting from $y_0 = 0$ the imaginary part of the phase $\Phi(t, y(t, 0, \eta_0), \eta_0)$ vanishes. Moreover, the following holds:

**Proposition 4.4.** *The phase $\Phi$ satisfies, for $y$ in a neighborhood of $0$,*

$$\Phi(t, y, \eta_0) = \Phi(t, y(t, 0, \eta_0), \eta_0) + (y - y(t, 0, \eta_0))\eta(t, 0, \eta_0)$$
$$+ (y - y(t, 0, \eta_0)) B(t, y, \eta_0)(y - y(t, 0, \eta_0)),$$

*where the phase $\Phi(t, y(t, 0, \eta_0), \eta_0)$ and its derivative $\eta(t, 0, \eta_0)$ with respect to the $y$ variable are real and the imaginary part of $B(t, y, \eta_0) \in \mathcal{M}_{d-1}(\mathbb{C})$ is positive definite.*

*Proof.* Indeed, the initial function $\Phi_0$ is complex valued with Hessian $\mathrm{Im}\nabla_y^2 \Phi_0$ positive definite. Then it follows from [56, Prop.21.5.9] that the complexified tangent plane of $\Lambda_{\Phi_0}$ is a strictly positive Lagrangian plane (see [56, Def.21.5.5]). The tangent plane at $(y(t, 0, \eta_0), \eta(t, 0, \eta_0))$ is the image of the complexified tangent plane of $\Lambda_{\Phi_0}^{\mathbb{R}}$ at $(y_0 = 0, \eta_0)$ under the complexification of a real symplectic map, hence strictly positive. More details for these arguments are given in Section 4.6.2 of the Appendix. □

We look now for $\varphi_h(t, y, \eta_0)$ of the form $h^{-(d-1)/4} e^{\frac{i}{h}\Phi(t, y, \eta_0)} \sigma(t, y, \eta_0, h)$ where $\sigma = \sum h^k \sigma_k$ must be an analytic classical symbol. Substitution in (4.27) yields the following system of transport equations

$$\begin{cases} L\sigma_0 = 0, \quad \sigma_0|_{t=0} = 1, \\ L\sigma_1 + f_1(\sigma_0) = 0, \quad \sigma_1|_{t=0} = 0, \\ . \\ . \\ L\sigma_k + f_k(\sigma_0, .., \sigma_{k-1}) = 0, \quad \sigma_k|_{t=0} = 0, \\ . \end{cases}$$

where $L = \frac{\partial}{\partial t} + \sum_{j=1}^{d-1} q^j(y, \nabla_y \Phi) + s(y, \eta_0)$ with $s(y, \eta_0)$ analytic and $f_k(\sigma_0, .., \sigma_{k-1})$ a linear expression with analytic coefficients of derivatives of $\sigma_0$,.., $\sigma_{k-1}$. It is clear that we can solve this system for $y$ in some complex domain $\mathcal{O}$, independently of $k$; in [92, Chps. 9,10] it is shown that in this way $\sigma$ becomes an analytic symbol there. $\square$

Let us define $\underline{\sigma}_k(t, y, \eta_0) = \sigma_k(t, y, \eta_0)$ for $(t, y) \in (-T_0, T_0) \times \mathcal{O}$ and $\underline{\sigma}_k(t, y, \eta_0) = 0$ otherwise and let $\underline{\sigma}(t, y, \eta_0, h) := \sum_{k \geq 0} h^k \underline{\sigma}_k(t, y, \eta_0)$. Set also

$$\underline{\varphi}_h(t, y, \eta_0) = h^{-(d-1)/4} e^{\frac{i}{h}\Phi(t,y,\eta_0)} \underline{\sigma}(t, y, \eta_0, h),$$

thus $\underline{\varphi}_h$ solves (4.27) for $t \in [-T_0, T_0]$ and $y \in \mathbb{R}^{d-1}$ and we can compute the $L^r(\mathbb{R}^{d-1})$ norm of $\underline{\varphi}_h(t, y, \eta_0)$ globally in $y$:

$$\|\underline{\varphi}_h(t, ., \eta_0)\|_{L^r(\mathbb{R}^{d-1})} =$$

$$h^{-(d-1)/4}\Big(\int e^{-\frac{r}{h}(y-y(t,0,\eta_0)) ImB(t,y,\eta_0)(y-y(t,0,\eta_0))} |\underline{\sigma}(t, y, \eta_0, h)|^r dy\Big)^{1/r} \simeq$$

$$h^{-(d-1)/4+(d-1)/2r}(1 + O(h^{-1}))$$

and consequently for $T_0$ small enough we have

$$\|\underline{\varphi}_h\|_{L^q((0,T_0],L^r(\mathbb{R}^{d-1}))} = h^{-\frac{d-1}{2}(\frac{1}{2}-\frac{1}{r})}(1 + O(h^{-1}))$$

and we conclude using Corollary 4.1.

2. **Wave equation** Let $(q, r)$ be a sharp wave-admissible pair in dimension $d \geq 2$, $q > 2$, and let $\tilde{q}$ be given by (4.32). Using Corollary 4.1 and Remark 4.7 and since $\tilde{q} \geq q$, we are reduced to prove (4.30) with $q$ replace by $\tilde{q}$, i.e.

$$\|e^{i\frac{t}{h}G_w}(\psi(hD_y)\varphi_0)\|_{L^{\tilde{q}}([0,T_0],L^r(\mathbb{R}^{d-1}))} \lesssim h^{-(\frac{d}{2}-\frac{1}{6})(\frac{1}{2}-\frac{1}{r})}\|\psi(hD_y)\varphi_0\|_{L^2(\mathbb{R}^{d-1})},$$

where $G_w(\eta) = \sqrt{|\eta|^2 + \omega h^{\frac{2}{3}}|\eta|^{\frac{4}{3}}}$ and

$$e^{i\frac{t}{h}G_w}(\psi(hD_y)\varphi_0)(t, y) = \frac{1}{(2\pi h)^{d-1}} \int e^{\frac{i}{h}(<y,\eta>-t\sqrt{|\eta|^2+\omega h^{\frac{2}{3}}|\eta|^{\frac{4}{3}}})}\psi(\eta)\hat{\varphi}_0(\frac{\eta}{h})d\eta.$$

In order to obtain dispersive estimates we need the following
**Proposition 4.5.** *Let* $\lambda = \frac{t}{h}$ *and set as before*

$$J(z, \lambda) := \int e^{i\lambda(z\eta - G_w(\eta))}\psi(\eta)d\eta, \quad \gamma_{d-1,h}(\lambda) = \sup_{z \in \mathbb{R}^{d-1}} |J(z, \lambda)|. \tag{4.42}$$

*Then the function* $\gamma_{d-1,h}$ *satisfies*

$$\gamma_{d-1,h}(\lambda) \simeq h^{-1/3}\lambda^{-(d-1)/2}.$$

We postpone the proof of Proposition 4.5 for the end of this section and proceed.
*End of the proof of Theorem 4.2*
From (4.18) the dispersive estimates read as follows

$$\|e^{-\frac{it}{h}G_w}\psi(hD_y)\varphi_0\|_{L^\infty(\mathbb{R}^{d-1})} \lesssim h^{-d+1-\frac{1}{3}}(\frac{t}{h})^{-\frac{(d-1)}{2}}\|\psi(hD_y)\varphi_0\|_{L^1(\mathbb{R}^{d-1})}. \tag{4.43}$$

Lemma 4.1 can be applied at this point of the proof for the $\frac{d-1}{2}$ wave-admissible pair in dimension $d-1$, $(\tilde{q}, r)$, in order to obtain

$$\|e^{-\frac{it}{h}G_w}\psi(hD_y)\varphi_0\|_{L^2\to L^{\tilde{q}}(0,T_0],L^r(\mathbb{R}^{d-1})} \lesssim h^{-(\frac{d}{2}-\frac{1}{6})(\frac{1}{2}-\frac{1}{r})}. \tag{4.44}$$

We conclude using again Corollary 4.1. It remains to prove Proposition 4.5.

*Proof. of Proposition 4.5*
As before, the case of most interest in the study of $\sup_{z\in\mathbb{R}^{d-1}}|J(z,\lambda)|$ will be the one for which $\lambda \gg 1$, since when $\frac{t}{h}$ remains bounded good estimates are found immediately. We shall thus concentrate on the case $\lambda \gg 1$ and we apply the stationary phase lemma. Notice that on the support of $\psi$ the phase function of $J$ is smooth. On the other hand, since $\eta$ stays away from a neighborhood of 0, the critical point of $J(z,\lambda)$ satisfies

$$z = G'_w(\eta) = \frac{\eta}{|\eta|} + O(h^{\frac{2}{3}}).$$

In order to estimate $|J(z,\lambda)|$ it will be thus enough to localize in a $h^{2/3}$ neighborhood of $|z| = 1$. We shall assume without loss of generality that $\psi$ is radial and set $\tilde{\psi}(|\eta|) = \psi(\eta)$ in which case $J(.,\lambda)$ depends also only on $|z|$ and it is enough to estimate

$$J(|z|e_1,\lambda) = \int_0^\infty \int_{\mathbb{S}^{d-2}} e^{i\lambda(|z|\rho\theta_1 - \sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}})}\tilde{\psi}(\rho)\rho^{d-2}d\theta d\rho, \tag{4.45}$$

where $e_1 = (1,0,..,0) \in \mathbb{R}^{d-1}$ and $\mathbb{S}^{d-2}$ is the unit sphere in $\mathbb{R}^{d-1}$. The derivative with respect to $\rho$ gives $|z|\theta_1 = \frac{\rho}{\sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}}} \simeq 1$ and making integrations by parts with respect to $\rho$ one sees that the contribution in the integral in $\eta$ in (4.42) is $O(\lambda^{-\infty})$ if $|z| \ll 1$. Consequently, one may assume $|z| \geq c > 0$. As a consequence $\theta_1$ can be taken close to 1, and since on the sphere $\mathbb{S}^{d-2}$ one has $\theta_1 = \pm\sqrt{1-\theta'^2}$, $\theta = (\theta_1,\theta') \in \mathbb{R}^{d-1}$, we can introduce a cutoff function $b(\theta')$ supported near 0 such that the right hand side in (4.45) writes, modulo $O((\lambda|z|)^{-\infty})$

$$\sum_{\pm}\int_0^\infty \int_{\theta'} e^{i\lambda(\pm\rho|z|\sqrt{1-\theta'^2} - \sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}})}b(\theta')\tilde{\psi}(\rho)\rho^{d-2}d\theta' d\rho.$$

The term corresponding to the critical point $-1$ gives a contribution $O(\lambda^{-\infty})$ in the integral with respect to $\rho$ by non-stationary phase theorem. Using the stationary phase theorem for the integral in $\theta'$ we find

$$J(|z|e_1, \lambda) \simeq \int_0^\infty e^{i\lambda(|z|\rho - \sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}})}\tilde{\psi}(\rho)\sigma_+(\lambda|z|\rho)d\rho + O(\lambda^{-\infty}),$$

where $\sigma_+$ is a symbol of order $-(d-2)/2$. In order to estimate this term we write its phase function as follows

$$|z|\rho - \sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}} = (|z|-1)\rho - (\sqrt{\rho^2+\omega h^{2/3}\rho^{4/3}} - \rho)$$

and set $|z| - 1 = h^{2/3}x$. Hence $J(|z|e_1, \lambda)$ can be estimated by

$$J((1+h^{2/3}x)e_1, \lambda) \simeq \int_0^\infty e^{i\mu(\rho x - \frac{\omega\rho^{1/3}}{1+\sqrt{1+\omega h^{2/3}\rho^{1/3}}})}\tilde{\psi}(\rho)\sigma_+(\lambda\rho(1+h^{2/3}x))d\rho. \quad (4.46)$$

We distinguish two cases, weather $1 \ll \lambda = t/h \lesssim h^{-2/3}$ or $1 \ll \mu = \lambda h^{2/3} = h^{-1/3}t$.
— In the first case $\mu \lesssim 1$ and formula (4.46) give us bounds from above for $\sup_z |J(z,\lambda)|$ of the form $\lambda^{-(d-2)/2}$ (recall that for $|z|$ away from a $h^{2/3}$-neighborhood of 1 the problem was trivial by non-stationary lemma).
— If $1 \ll \mu = h^{-1/3}t$ and $x \neq 0$ we apply the stationary phase lemma in dimension one with phase $\Phi(\rho) = \rho x - \frac{\omega}{2}\rho^{1/3}$ which is smooth since $\rho \neq 0$ and has one critical, non-degenerate ($\Phi''(\rho) = \frac{\omega}{9}\rho^{-5/3} \neq 0$) point satisfying

$$\rho = (6x/\omega)^{-3/2}.$$

For values of $x$ for which $(6x/\omega)^{-3/2}$ belongs to the support of $\tilde{\psi}$ we find

$$\begin{aligned} J((1+h^{2/3}x)e_1, \lambda) &\simeq C(x)\lambda^{-\frac{d-2}{2}}\mu^{-\frac{1}{2}} + O(\lambda^{-\frac{d-2}{2}}\mu^{-3/2}) \quad (4.47)\\ &\simeq C(x)\lambda^{-\frac{d-1}{2}}h^{-\frac{1}{3}} + O(\lambda^{-\frac{d-2}{2}}\mu^{-3/2}), \end{aligned}$$

with $C(x)$ bounded and consequently we can determine $\gamma_{d-1,h}$ defined by (4.19) where $n = d-1$ and $G = G_w$. We find

$$\gamma_{d-1,h}(\frac{t}{h}) \simeq h^{-\frac{1}{3}}\left(\frac{t}{h}\right)^{-\frac{d-1}{2}}, \quad (4.48)$$

thus the proof is complete.

□

## 4.3 Conormal waves with cusp in dimension $d = 2$

In what follows let $0 < \epsilon \ll 1$ be small. We shall construct a sequence $W_{h,\epsilon}$ of approximate solutions of the wave equation

$$(\partial_t^2 - \Delta_D)V(t,x,y) = 0 \quad \text{for} \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^2, \quad V|_{[0,1]\times\partial\Omega} = 0, \quad (4.49)$$

which contradicts the Strichartz estimates of the free space (see Proposition 4.1). Using the approximate solutions $W_{h,\epsilon}$ we shall conclude Theorem 4.1 by showing that one can find (exact) solutions $V_{h,\epsilon}$ of (4.49) which provide losses of derivatives for the $L^q([0,1], L^r(\Omega))$ norms of at least $\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-\epsilon$ for $r>4$ when compared to the free space, $(q,r)$ being a wave-admissible pair in dimension 2.

### 4.3.1 Motivation for the choice of the approximate solution

Let the wave operator be given by $\Box=\partial_t^2-\partial_x^2-(1+x)\partial_y^2$ and let $p(t,x,y,\tau,\xi,\eta)=\xi^2+(1+x)\eta^2-\tau^2$ denote its (homogeneous) symbol. The characteristic set of $\Box$ is the closed conic set $\{(t,x,y,\tau,\xi,\eta)|p(t,x,y,\tau,\xi,\eta)=0\}$, denoted $\mathrm{Char}(p)$. We define the semi-classical wave front set $WF_h(u)$ of a distribution $u$ on $\mathbb{R}^3$ to be the complement of the set of points $(\rho=(t,x,y),\zeta=(\tau,\xi,\eta))\in\mathbb{R}^3\times(\mathbb{R}^3\setminus 0)$ for which there exists a symbol $a(\rho,\zeta)\in\mathcal{S}(\mathbb{R}^6)$ such that $a(\rho,\zeta)\neq 0$ and for all integers $m\geq 0$ the following holds

$$\|a(\rho,hD_\rho)u\|_{L^2}\leq c_m h^m.$$

Let $\rho=\rho(\sigma)$, $\zeta=\zeta(\sigma)$ be a bicharacteristic of $p(\rho,\zeta)$, i.e. such that $(\rho,\zeta)$ satisfies

$$\frac{d\rho}{d\sigma}=\frac{\partial p}{\partial\zeta},\quad \frac{d\zeta}{d\sigma}=-\frac{\partial p}{\partial\rho},\quad p(\rho(0),\zeta(0))=0. \tag{4.50}$$

Assume that the interior of $\Omega$ is given by the inequality $\gamma(\rho)>0$, in this case $\gamma(\rho=(t,x,y))=x$. Then $\rho=\rho(\sigma)$, $\zeta=\zeta(\sigma)$ is tangential to $\mathbb{R}\times\partial\Omega$ if

$$\gamma(\rho(0))=0,\quad \frac{d}{d\sigma}\gamma(\rho(0))=0. \tag{4.51}$$

We say that a point $(\rho,\zeta)$ on the boundary is a gliding point if it is a tangential point and if in addition

$$\frac{d^2}{d\sigma^2}\gamma(\rho(0))<0. \tag{4.52}$$

This is equivalent (see for example [48]) to saying that $(\rho,\zeta)\in T^*(\mathbb{R}\times\partial\Omega)\setminus 0$ is a gliding point if

$$p(\rho,\zeta)=0,\quad \{p,\gamma\}|_{(\rho,\zeta)}=0,\quad \{\{p,\gamma\},p\}|_{(\rho,\zeta)}>0, \tag{4.53}$$

where $\{.,.\}$ denotes the Poisson braket. We say that a point $(\rho,\zeta)$ is hyperbolic if $x=0$ and $\tau^2>\eta^2$, so that there are two distinct nonzero real solutions $\xi$ to $\xi^2+(1+x)\eta^2-\tau^2=0$.

Consider an approximate solution for (4.49) of the form

$$\int e^{\frac{i}{h}(y\eta+t\tau+(x+1-\frac{\tau^2}{\eta^2})\xi+\frac{\xi^3}{3\eta^2})}g(t,\xi/\eta,\tau,h)\Psi(\eta)/\eta\, d\xi d\eta d\tau \tag{4.54}$$

where the symbol $g$ is a smooth function independent of $x$, $y$ and where $\Psi\in C_0^\infty(\mathbb{R}^*)$ is supported for $\eta$ in a small neighborhood of 1, $0\leq\Psi(\eta)\leq 1$, $\Psi(\eta)=1$ for $\eta$ near 1. This

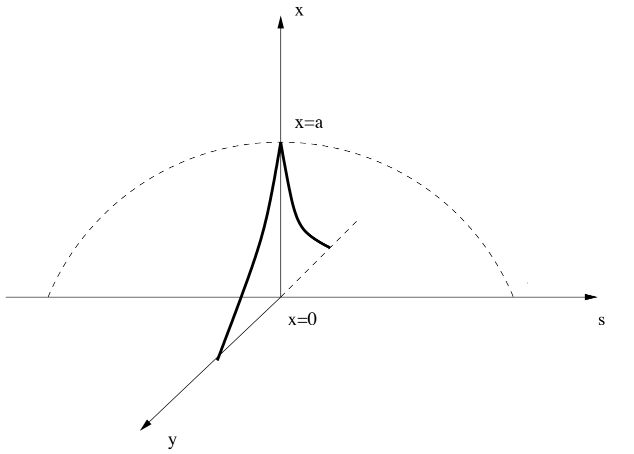


Figure 2 – Bicharacteristics of the half space

choice is motivated by the following: if $v(t, x, y)$ satisfies $(\partial_t^2 - \partial_x^2 - (1 + x)\partial_y^2)v = 0$, then taking the Fourier transform in time $t$ and space $y$ we get $\partial_x^2 \hat{v} = ((1 + x)\eta^2 - \tau^2)\hat{v}$, thus $\hat{v}$ can be expressed using Airy's function (given in Section 4.6.3) and its derivative. After the change of variables $\xi = \eta s$, the Lagrangian manifold associated to the phase function $\Phi$ of (4.54) will be given by

$$\Lambda_\Phi = \{(t, x, y, \tau = \partial_t\Phi, \xi = \partial_x\Phi = \eta s, \eta = \partial_y\Phi)|\partial_s\Phi = 0, \partial_\eta\Phi = 0\} \subset T^*\mathbb{R}^3 \setminus 0. \quad (4.55)$$

Let $\pi : \Lambda_\Phi \to \mathbb{R}^3$ be the natural projection and let $\Sigma$ denote the set of its singular points. The points where the Jacobian of $d\pi$ vanishes lie over the caustic set, thus the fold set is given by $\Sigma = \{s = 0\}$ and the caustic is defined by $\pi(\Sigma) = \{x + (1 - \frac{\tau^2}{\eta^2}) = 0\}$.

If on the boundary we are localized away from the caustic set $\pi(\Sigma)$, $\Lambda_{\Phi|_{x=0}}$ is the graph of a pair of canonical transformations, the billiard ball maps $\delta^\pm$. Roughly speaking, the billiard ball maps $\delta^\pm : T^*(\mathbb{R} \times \partial\Omega) \to T^*(\mathbb{R} \times \partial\Omega)$, defined on the hyperbolic region, continuous up to the boundary, smooth in the interior, are defined at a point of $T^*(\mathbb{R} \times \partial\Omega)$ by taking the two rays that lie over this point, in the hypersurface $\mathrm{Char}(p)$, and following the null bicharacteristic through these points until you pass over $\{x = 0\}$ again, projecting such a point onto $T^*(\mathbb{R} \times \partial\Omega)$ (a gliding point being "a diffractive point viewed from the other side of the boundary", there is no bicharacteristic in $T^*(\mathbb{R} \times \partial\Omega)$ through it, but in any neighborhood of a gliding point there are hyperbolic points).

In our model case the analysis is simplified by the presence of a large *commutative* group of symmetries, the translations in $(y, t)$, and the billiard ball maps have specific formulas

$$\delta^\pm(y, t, \eta, \tau) = \Big(y \pm 4(\frac{\tau^2}{\eta^2} - 1)^{1/2} \pm \frac{8}{3}(\frac{\tau^2}{\eta^2} - 1)^{3/2}, t \mp 4(\frac{\tau^2}{\eta^2} - 1)^{1/2}\frac{\tau}{\eta}, \eta, \tau\Big). \quad (4.56)$$

Away from $\pi(\Sigma)$ these maps have no recurrent points, since under iteration $t((\delta^\pm)^n) \to \pm\infty$

as $n \to \infty$. The composite relation with $n$ factors

$$\Lambda_{\Phi|_{x=0}} \circ ... \circ \Lambda_{\Phi|_{x=0}}$$

has, always away from $\pi(\Sigma)$, $n+1$ components, obtained namely using the graphs of the iterates $(\delta^+)^n, (\delta^+)^{n-2}, .., (\delta^-)^n$,

$$(\delta^\pm)^n(y,t,\eta,\tau) = \Big(y \pm 4n(\frac{\tau^2}{\eta^2}-1)^{1/2} \pm \frac{8}{3}n(\frac{\tau^2}{\eta^2}-1)^{3/2}, t \mp 4n(\frac{\tau^2}{\eta^2}-1)^{1/2}\frac{\tau}{\eta}, \eta, \tau\Big). \quad (4.57)$$

All these graphs, of the powers of $\delta^\pm$, are disjoint away from $\pi(\Sigma)$ and locally finite, in the sense that only a finite number of components meet any compact subset of $\{\frac{\tau^2}{\eta^2}-1>0\}$. Since $(\delta^\pm)^n$ are all immersed canonical relations, it is necessary to find a parametrization of each to get at least microlocal representations of the associated Fourier integral operators. We see that

$$y\eta + t\tau + \frac{4}{3}\eta(\frac{\tau^2}{\eta^2}-1)^{3/2},$$

are parametrizations of $\Lambda_{\Phi|_{x=0}}$, thus the iterated Lagrangians $(\Lambda_{\Phi|_{x=0}})^{\circ n}$ are parametrized by

$$y\eta + t\tau + \frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2},$$

and the corresponding phase functions associated to $(\Lambda_\Phi)^{\circ n}$ will be given by

$$\Phi_n = \Phi + \frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2}.$$

Let us come back to the wave equation (4.49) and describe the approximate solution we want to chose. The domain $\Omega$ being strictly convex, at each point on the boundary there exists a bicharacteristic that intersects the boundary $\mathbb{R}\times\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\mathbb{R}\times\bar{\Omega}$. Here we deal with $\gamma(\rho)=x$ and (4.53) translates into $x=\xi=0$, $|\tau|=|\eta|>0$. Let

$$(\rho_0,\zeta_0) = (0,0,0,0,1,-1) \in T^*(\mathbb{R}\times\partial\Omega).$$

We shall place ourself in the region $\mathcal{V}_a$ near $(\rho_0,\zeta_0)$,

$$\mathcal{V}_a = \{(\rho,\zeta)|\xi^2+(1+x)\eta^2-\tau^2=0, x=0, \tau^2=(1+a)\eta^2\},$$

where $a=h^\delta$, $0<\delta<2/3$ will be chosen later and $\eta$ belongs to a neighborhood of 1. Notice that, in some sense, $a$ measures the "distance" to the gliding point $(\rho_0,\zeta_0)$.

Let $u_h$ be defined by

$$u_h(t,x,y) = \int e^{\frac{i}{h}(y\eta - t(1+a)^{1/2}\eta + (x-a)\xi + \frac{\xi^3}{3\eta^2})} g(t,\xi/\eta,h)\Psi(\eta)/\eta d\xi d\eta, \quad (4.58)$$

where the symbol $g$ is a smooth function independent of $x$, $y$ and where $\Psi \in C_0^\infty(\mathbb{R}^*)$ is supported for $\eta$ in a small neighborhood of 1, $0 \leq \Psi(\eta) \leq 1$, $\Psi(\eta) = 1$ for $\eta$ near 1. We consider the sum

$$U_h(t,x,y) = \sum_{n=0}^{N} u_h^n(t,x,y), \quad u_h^n(t,x,y) = \int e^{\frac{i}{h}\Phi_n} g^n(t,s,\eta,h)\Psi(\eta)d\eta ds,$$

where $\Phi_n = \Phi + \frac{4}{3}n\eta a^{3/2}$ are the phase functions defined above such that $\Lambda_{\Phi_n} = (\Lambda_\Phi)^{\circ n}$ and where the symbols $g^n$ will be chosen such that on the boundary the Dirichlet condition to be satisfied. At $x = 0$ the phases have two critical, non-degenerate points, thus each $u_h^n$ writes as a sum of two trace operators, $Tr_\pm(u_h^n)$, localized respectively for $y - (1+a)^{1/2}t + \frac{4}{3}na^{3/2}$ near $\pm\frac{2}{3}na^{3/2}$, and in order to obtain a contribution $O_{L^2}(h^\infty)$ on the boundary we define the symbols such that $Tr_-(g^n) + Tr_+(g^{n+1}) = O_{L^2}(h^\infty)$. This will be possible by Egorov theorem, as long as $N \ll a^{3/2}/h$. This last condition, together with the assumption of finite time (which implies $0 < N(\frac{\tau^2}{\eta^2} - 1)^{1/2} < \infty$) allows to estimate the number of reflections $N$.

The motivation of this construction comes from the fact that near the caustic set $\pi(\Sigma)$ one notices a singularity of cusp type for which one can estimate the $L^r(\Omega)$ norms. Moreover, if at $t = 0$ one considers symbols localized in a small neighborhood of the caustic set, then one can show that the respective "pieces of cusps" propagate until they reach the boundary but short after that their contribution becomes $O_{L^2}(h^\infty)$, since as $t$ increases, $s$ takes greater values too and thus one quickly quits a neighborhood of the Lagrangian $\Lambda_\Phi$ which contains the semi-classical wave front set $WF_h(u_h)$ of $u_h$. This argument is valid for all $u_h^n$, thus the approximate solutions $u_h^n$ will have almost disjoints supports and the $L^q([0,1], L^r(\Omega))$ norms of the sum $U_h$ will be computed as the sum of the norms of each $u_h^n$ on small intervals of time of size $a^{1/2}$.

### 4.3.2 Choice of the symbol

Let $a = h^\delta$, $0 < \delta < 2/3$ to be chosen and let $u_h$ be given by the formula (5.37). Applying the wave operator $\Box$ to $u_h$ gives:

$$\begin{aligned} h^2\Box u_h &= \int e^{\frac{i}{h}\Phi}\Big(h^2\partial_t^2 g - 2ih\eta(1+a)^{1/2}\partial_t g + \eta^2(x - a + s^2)g\Big)\Psi(\eta)dsd\eta \\ &= \int e^{\frac{i\eta}{h}(y - t(1+a)^{1/2} + s(x-a) + \frac{s^3}{3})}\Big(h^2\partial_t^2 g + ih\eta(\partial_s g - 2(1+a)^{1/2}\partial_t g)\Big)\eta\Psi(\eta)dsd\eta. \quad (4.59) \end{aligned}$$

*Definition* 4.3. Let $\lambda \geq 1$. For a given compact $K \subset \mathbb{R}$ we define the space $\mathcal{S}_K(\lambda)$, consisting of functions $\varrho(z,\lambda) \in C^\infty(\mathbb{R})$ which satisfy

1. $\sup_{z\in\mathbb{R},\lambda\geq 1} |\partial_z^\alpha \varrho(z,\lambda)| \leq C_\alpha$, where $C_\alpha$ are constants independent of $\lambda$,
2. If $\psi(z) \in C_0^\infty$ is a smooth function equal to 1 in a neighborhood of $K$, $0 \leq \psi \leq 1$ then $(1-\psi)\varrho \in O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$.

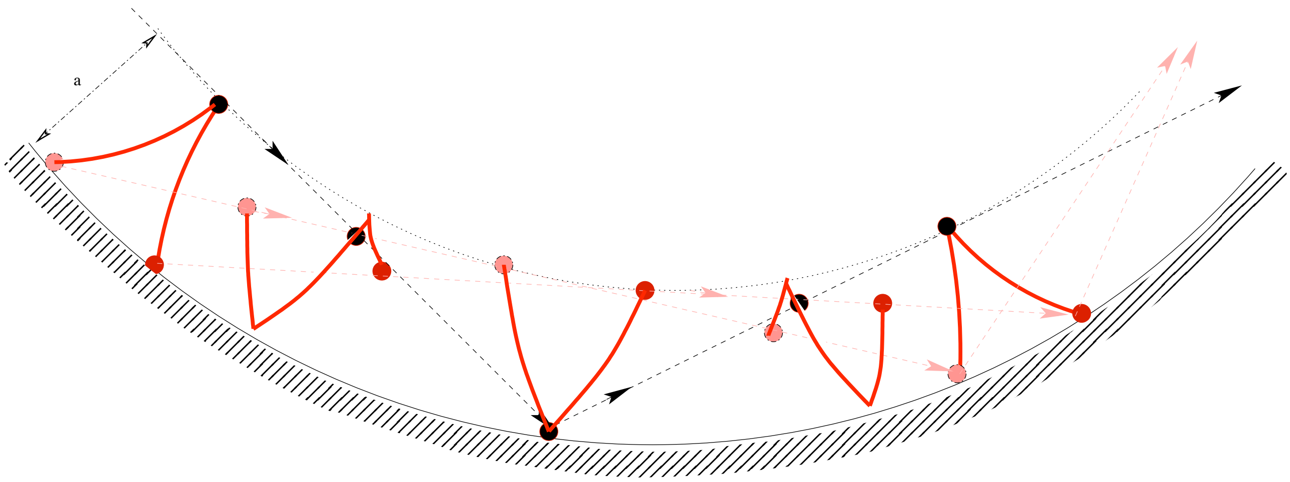


Figure 3 – Propagation of the cusp. *A caustic is defined as the envelope of the rays which appear in a given problem: each ray is tangent to the caustic at a given point. If one assigns a direction on the caustic, it induces a direction on each ray. Each point outside the caustic lies on a ray which has left the caustic and also lies on a ray approaching the caustic. Each curve of constant phase has a cusp where it meets the caustic.*

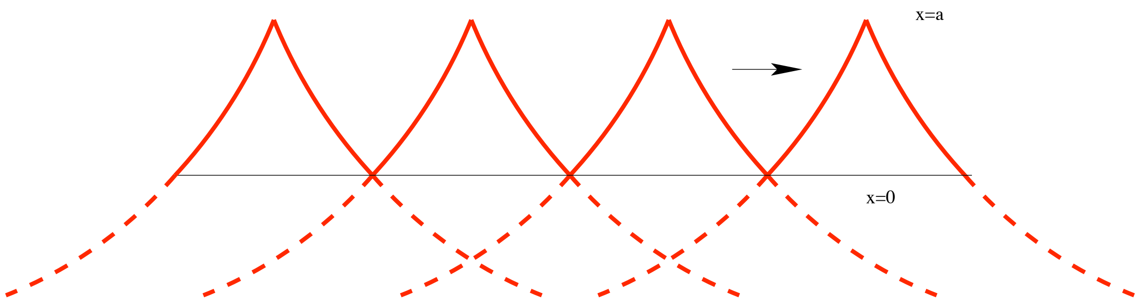


Figure 4 – Periodize

An example of function $\varrho(z,\lambda) \in \mathcal{S}_K(\lambda)$, $K \subset \mathbb{R}$ is the following: let $k(z)$ be the smooth function on $\mathbb{R}$ defined by

$$k(z) = \begin{cases} c\exp\left(-1/(1-|z|^2)\right), & if\ |z|<1, \\ 0, \quad if\ |z| \geq 1, \end{cases}$$

where $c$ is a constant chosen such that $\int_{\mathbb{R}} k(z)dz = 1$. Define a mollifier $k_\lambda(z) := \lambda k(\lambda z)$ and let $\tilde{\varrho} \in C_0^\infty(K)$ be a smooth function with compact support included in $K$. If we set $\varrho(z,\lambda) = (\tilde{\varrho} * k_\lambda)(z)$, then one can easily check that $\varrho$ belongs to $\mathcal{S}_K(\lambda)$.

Let $\lambda = \lambda(h) = h^{3\delta/2-1}$, $K_0 = [-c_0, c_0]$ for some small $0 < c_0 < 1$ and let $\varrho(.,\lambda) \in \mathcal{S}_{K_0}(\lambda)$ be a smooth function. We define

$$g(t,s,h) = \varrho\Big(\frac{t + 2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}}, \lambda\Big). \tag{4.60}$$

Notice that $t + 2(1+a)^{1/2}s$ is an integral curve of the vector field $\partial_s - 2(1+a)^{1/2}\partial_t$, thus inserting (4.60) in (4.59) gives

$$\Box u_h = h^{-\delta}(4(1+a))^{-1}\int e^{\frac{i}{h}\Phi}\partial_1^2\varrho\Big(\frac{t+2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}}, \lambda\Big)\Psi(\eta)dsd\eta. \tag{4.61}$$

### 4.3.3 The boundary condition

We compute $u_h$ on the boundary. We make the change of variables $s = a^{1/2}v$ in the integral defining $u_h(t,0,y)$ and set $z = \frac{t}{2(1+a)^{1/2}a^{1/2}}$. Then

$$\begin{aligned} u_h(t,0,y) = a^{1/2} &\int_\eta \frac{\eta\lambda}{2\pi} e^{\frac{i\eta}{h}(y - t(1+a)^{1/2})} \times \\ &\times \Big(\int_\zeta\int_v e^{i\eta\lambda(\frac{v^3}{3} - v(1-\zeta))}dv \int_{z'} e^{i\eta\lambda(z-z')\zeta}\varrho(z',\lambda)dz'd\zeta\Big)\Psi(\eta)d\eta \\ &= a^{1/2}\int_\eta (\eta\lambda)^{2/3} e^{\frac{i\eta}{h}(y-t(1+a)^{1/2})}\Psi(\eta)\times \\ &\quad \times \int_{\zeta,z'} e^{i\eta\lambda(z-z')\zeta} Ai(-(\eta\lambda)^{2/3}(1-\zeta))\varrho(z',\lambda)dz'd\zeta d\eta. \end{aligned} \tag{4.62}$$

For $\eta \in \text{supp}(\Psi)$ we introduce

$$I(\varrho(.,\lambda))_\eta(z,\lambda) := \frac{(\eta\lambda)^{7/6}}{2\pi}\int_{\zeta,z'} e^{i\eta\lambda(z-z')\zeta} Ai(-(\eta\lambda)^{2/3}(1-\zeta))\varrho(z',\lambda)dz'd\zeta, \tag{4.63}$$

then

$$\Psi(\eta)(I(\varrho(.,\lambda))_\eta)^\wedge(\eta\lambda\zeta,\lambda) = (\eta\lambda)^{1/6}\Psi(\eta)Ai(-(\eta\lambda)^{2/3}(1-\zeta))\hat{\varrho}(\eta\lambda\zeta,\lambda). \tag{4.64}$$

The next Lemma shows that the symbol of the operator defined in (4.63) is localized for $\zeta$ as close as we want to 0.

**Lemma 4.3.** *For $\varrho(.,\lambda)\in\mathcal{S}_K(\lambda)$ for some compact $K$ then $(I(\varrho)_\eta)^\wedge(.,\lambda)$ defined by (4.63), (4.64) is localized near $\zeta=0$, more precisely, if $\chi$ is a smooth function with support included in a small neighborhood $(-2c,2c)$ of $0$, $0<c\leq 1/4$, $\chi|_{[-c,c]}=1$, $0\leq\chi\leq 1$, then we have for $\eta\in supp(\Psi)$*

$$I(\varrho(.,\lambda))_\eta(z,\lambda)=\frac{(\eta\lambda)^{7/6}}{2\pi}\int_{\zeta,z'}e^{i\eta\lambda(z-z')\zeta}Ai(-(\eta\lambda)^{2/3}(1-\zeta))\times \\ \times\chi(\zeta)\varrho(z',\lambda)dz'd\zeta+O_{\mathcal{S}(\mathbb{R})}((\eta\lambda)^{-\infty}). \quad (4.65)$$

*Proof.* Let $\varrho(.,\lambda)\in\mathcal{S}_K(\lambda)$. If we set, for $\eta\in\operatorname{supp}(\Psi)$

$$J(\varrho(.,\lambda))_\eta(z,\lambda):=\frac{(\eta\lambda)^{7/6}}{2\pi}\int_{\zeta,z'}e^{i\eta\lambda(z-z')\zeta}Ai(-(\eta\lambda)^{2/3}(1-\zeta))(1-\chi(\zeta))\varrho(z',\lambda)dz'd\zeta$$

we need to prove the following

$$\Psi(\eta)J(\varrho(.,\lambda))_\eta(z,\lambda)\in\Psi(\eta)O_{\mathcal{S}(\mathbb{R}_z)}((\eta\lambda)^{-\infty})=O_{\mathcal{S}(\mathbb{R}_z)}(\lambda^{-\infty}),$$

which is the same as to show that $(J(\varrho))^\wedge_\eta(\xi,\lambda)\in O_{\mathcal{S}(\mathbb{R}_\xi)}(\lambda^{-\infty})$ or equivalently that

$$\Psi(\eta)J(\varrho)^\wedge_\eta(\eta\lambda\zeta,\lambda)\in\Psi(\eta)O_{\mathcal{S}(\mathbb{R}_\zeta)}((\eta\lambda)^{-\infty}). \quad (4.66)$$

In order to prove (4.66) we first compute $(J(\varrho))^\wedge_\eta(\eta\lambda\zeta,\lambda)$ explicitly:

$$\Psi(\eta)(J(\varrho(.,\lambda)))^\wedge_\eta(\eta\lambda\zeta,\lambda)= \\ =(\eta\lambda)^{1/6}\Psi(\eta)Ai(-(\eta\lambda)^{2/3}(1-\zeta))(1-\chi(\zeta))\hat{\varrho}(\eta\lambda\zeta,\lambda) \quad (4.67)$$

It remains to show that the right hand side of (4.67) belongs to $\Psi(\eta)O_{\mathcal{S}(\mathbb{R}_z)}((\eta\lambda)^{-\infty})$. Notice that this will conclude the proof of the Lemma 4.3. If $\chi(\zeta)\neq 1$ then $\zeta$ lies outside a neighborhood of 0, $|\zeta|\geq c$ and for $\eta\in\operatorname{supp}(\Psi)$ we can perform integrations by parts in the integral defining $\hat{\varrho}(\eta\lambda\zeta,\lambda)$:

$$\Psi(\eta)\hat{\varrho}(\eta\lambda\zeta,\lambda)=\Psi(\eta)\int_{z'}e^{-i\eta\lambda\zeta z'}\varrho(z',\lambda)dz'= \\ =\frac{\Psi(\eta)}{(i\eta\lambda\zeta)^m}\int_{z'}e^{-i\eta\lambda\zeta z'}\partial^m_{z'}\varrho(z',\lambda)dz'. \quad (4.68)$$

Writing $\varrho(z',\lambda)=\psi(z')\varrho(z',\lambda)+(1-\psi(z'))\varrho(z',\lambda)$ for some smooth cutoff function $\psi$ equal to 1 on $K$ and using that $\|\partial^m_{z'}(\psi\varrho)(.,\lambda)\|_{L^\infty(\mathbb{R})}\leq C'_m$ for some constants $C'_m$ independent of $\lambda$ and that, on the other hand $\partial^m_{z'}((1-\psi)\varrho(.,\lambda))\in O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$, we deduce the desired result. □

In what follows we use the results in Section 4.6.3 of the Appendix in order to write, for $\zeta$ close to 0

$$Ai(-(\eta\lambda)^{2/3}(1-\zeta)) = A^{+}(-(\eta\lambda)^{2/3}(1-\zeta)) + A^{-}(-(\eta\lambda)^{2/3}(1-\zeta))$$

where $A^{\pm}$ have the following asymptotic expansions

$$A^{\pm}(-(\eta\lambda)^{2/3}(1-\zeta)) \simeq (\eta\lambda)^{-1/6}(1-\zeta)^{-1/4}\times \\ \times e^{\mp\frac{2i}{3}\eta\lambda(1-\zeta)^{3/2}\pm\frac{i\pi}{2}-\frac{i\pi}{4}}\Big(\sum_{j\geq 0}\frac{a_{\pm,j}(-1)^{-j/2}(1-\zeta)^{-3j/2}}{(\eta\lambda)^{j}}\Big). \quad (4.69)$$

We obtain two contributions in $I(\varrho(.,\lambda))_{\eta}(.,\lambda)$ which we denoted

$$\frac{(\eta\lambda)^{7/6}}{2\pi}\int_{\zeta,z'} e^{i\eta\lambda(z-z')\zeta}A^{\pm}(-(\eta\lambda)^{2/3}(1-\zeta))\chi(\zeta)\varrho(z',\lambda)dz'd\zeta. \quad (4.70)$$

We can summarize the preceding results as follows:

**Proposition 4.6.** *On the boundary $u_h|_{x=0}$ writes (modulo $O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$) as a sum of two trace operators,*

$$u_h(t,0,y) = Tr_{+}(u_h)(t,y;h) + Tr_{-}(u_h)(t,y;h), \quad (4.71)$$

*where, for $z = \frac{t}{2(1+a)^{1/2}a^{1/2}}$,*

$$Tr_{\pm}(u_h)(t,y;h) = 2\pi\sqrt{\frac{a}{\lambda}}\int e^{\frac{i\eta}{h}(y-t(1+a)^{1/2}\mp\frac{2}{3}a^{3/2})}\eta^{-1/2}\Psi(\eta)I_{\pm}(\varrho)_{\eta}(z,\lambda)d\eta, \quad (4.72)$$

$$I_{\pm}(\varrho(.,\lambda))_{\eta}(z,\lambda) = e^{\pm i\pi/2-i\pi/4}\frac{\eta\lambda}{2\pi}\int_{\zeta,z'} e^{i\eta\lambda(\zeta(z-z')\mp\frac{2}{3}((1-\zeta)^{3/2}-1))}\times \\ \times\chi(\zeta)a_{\pm}(\zeta,\eta\lambda)\varrho(z',\lambda)dz'd\zeta, \quad (4.73)$$

*and where $a_{\pm}$ are the symbols of the Airy functions $A^{\pm}$*

$$a_{\pm}(\zeta,\eta\lambda) \simeq (1-\zeta)^{-1/4}\Big(\sum_{j\geq 0}\frac{a_{\pm,j}(-1)^{-j/2}(1-\zeta)^{-3j/2}}{(\eta\lambda)^{j}}\Big), \quad (4.74)$$

*where $a_{\pm,j}$ are given in* (4.141).

We also need the next Lemma:

**Lemma 4.4.** *Let $p\in\mathbb{Z}$ and for some $0 < c_0 < 1$ set $K_p = [-c_0+p, c_0+p]$. Then for $\eta$ belonging to the support of $\Psi$ we have $I_{\pm,\eta} : \mathcal{S}_{K_p}(\lambda) \to \mathcal{S}_{K_{p\mp 1}}(\lambda)$.*

*Proof.* The phase functions in $I_\pm(\varrho(.,\lambda))_\eta(z,\lambda)$ are given by

$$\eta\phi_\pm(z,z',\zeta)=\eta((z-z')\zeta\mp\frac{2}{3}((1-\zeta)^{3/2}-1)),$$

with critical points satisfying

$$\partial_\zeta\phi_\pm(z,z',\zeta)=z-z'\pm(1-\zeta)^{1/2}=0,\quad \partial_{z'}\phi_\pm(z,z',\zeta)=-\zeta=0.$$

Outside small neighborhoods of $\zeta=0$ and $z'=z\pm(1-\zeta)^{1/2}$ we make integrations by parts in order to obtain a small contribution $\Psi(\eta)O_{\mathcal{S}(\mathbb{R})}((\eta\lambda)^{-\infty})$. Indeed, if we write

$$\Psi(\eta)I_\pm(\varrho(.,\lambda))_\eta(z,\lambda)= \tag{4.75}$$

$$=e^{\pm i\pi/2-i\pi/4}\Psi(\eta)\frac{\eta\lambda}{2\pi}\int_{\zeta,z'}e^{i\eta\lambda((z-z')\zeta\mp\frac{2}{3}((1-\zeta)^{3/2}-1))}a_\pm(\zeta,\eta\lambda)\chi(\zeta)\varrho(z',\lambda)dz'd\zeta,$$

where $a_\pm$ are given in (4.74), we have to check that the conditions of Definition 4.3 are satisfied for $I_\pm(\varrho(.,\lambda))_\eta(z,\lambda)$ and $\mathcal{S}_{K_{p\mp1}}(\lambda)$:

— First we prove that for $\eta\in\operatorname{supp}(\Psi)$

$$\sup_{z\in\mathbb{R},\lambda\geq1}|\partial_z^\alpha I_\pm(\varrho)_\eta(z,\lambda)|\leq C_\alpha^1. \tag{4.76}$$

For $\eta\in\operatorname{supp}(\Psi)$ we have

$$\Psi(\eta)\partial_z^\alpha I_\pm(\varrho)_\eta(z,\lambda)=e^{\pm i\pi/2-i\pi/4}\Psi(\eta)\frac{\eta\lambda}{2\pi}\int_{\zeta,z'}e^{i\eta\lambda((z-z')\zeta\mp\frac{2}{3}((1-\zeta)^{3/2}-1))}\times$$
$$\times(i\eta\lambda\zeta)^\alpha a_\pm(\zeta,\lambda)\chi(\zeta)\varrho(z',\lambda)dz'd\zeta, \tag{4.77}$$

and we shall split the integral in $\zeta$ in two parts, according to $\lambda\zeta\leq2$ or $\lambda\zeta>2$ for $\eta$ on the support of $\Psi$: in the first case there is nothing to do, the change of variables $\xi=\eta\lambda\zeta$ allowing to obtain bounds of type (4.76). In case $\lambda\zeta>2$ we make integrations by parts in the integral defining $\hat\varrho(\eta\lambda\zeta,\lambda)$ like in (8.41) in order to conclude.

— Secondly, let $\psi_\pm$ be smooth cutoff functions equal to 1 in small neighborhoods of $K_{p\mp1}$ and such that $0\leq\psi_\pm\leq1$. We prove that

$$(1-\psi_\pm(z))\Psi(\eta)I_\pm(\varrho)_\eta(z,\lambda)=\Psi(\eta)O_{\mathcal{S}(\mathbb{R})}((\eta\lambda)^{-\infty}).$$

Since $\psi_\pm$ equal to 1 on some neighborhoods of $K_{p\mp1}$ there exist $c'>0$ small enough such that $\psi_\pm|_{[-c_0+p\mp1-5c',c_0+p\mp1+5c']}=1$. Since $(I(\varrho)_\eta)^\wedge$ is localized as close as we want to $\zeta=0$ then from the proof of Lemma 4.3 we can find some (other) smooth function $\tilde\chi$ with support included in $(-2c',2c')$, equal to 1 on $[-c',c']$ so that

$$\Psi(\eta)(I(\varrho)_\eta)^\wedge(\eta\lambda\zeta,\lambda)=\Psi(\eta)(I(\varrho)_\eta)^\wedge(\eta\lambda\zeta,\lambda)\tilde\chi(\zeta)+\Psi(\eta)O_{\mathcal{S}(\mathbb{R})}((\eta\lambda)^{-\infty}).$$

Let $\psi \in C_0^\infty$ with support included in $(-c_0 + p - c', c_0 + p + c')$. We split $\varrho = \psi\varrho + (1-\psi)\varrho$ and since $(1-\psi)\varrho(.,\lambda)$ belongs to $O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$ it is enough to prove the preceding assertion with $\varrho$ replaced by $\psi\varrho$. On the support of $\psi\varrho$ we have $|z'-p| \le c_0 + c'$ and on the support of $1-\psi_\pm$ we have $|z-p\pm 1| > c_0 + 5c'$. On the other hand, if $c'$ is chosen small enough then on the support of $\psi$ we have $1-3c' \le (1-\zeta)^{1/2} \le 1+3c'$, thus we can make integrations by parts in the integral in $\zeta$ since in the region we consider we have $|\partial_\zeta \phi_\pm| \ge p \mp 1 + c_0 + c' - p - c_0 - c' \pm 1 - 3c' \ge c'$. From the discussion above and

$$(1-\psi_\pm(z))I_\pm(\varrho)_\eta(z,\lambda) = e^{\pm i\pi/2 - i\pi/4}(1-\psi_\pm(z))\times \\ \times\, \Psi(\eta)\frac{\eta\lambda}{2\pi}\int_{\zeta,z'} e^{i\eta\lambda((z-z')\zeta \mp \frac{2}{3}((1-\zeta)^{3/2}-1))} a_\pm(\zeta,\eta\lambda)\tilde{\chi}(\zeta)\psi(z')\varrho(z',\lambda)dz'd\zeta \quad (4.78)$$

we conclude by performing integrations by parts in $\zeta$. In fact, we could have noticed from the beginning that, inserting under the integral (4.78) a cut-off localized close to $\zeta = 0$, $z' = z$ and performing integrations by parts, one makes appear a factor bounded by $(1+\lambda|\eta||\zeta|)^{-N}$ for all $N \ge 0$.

□

**Construction of the approximate solution:** Let $p \in \mathbb{Z}$ and $K_p = [-c_0 + p, c_0 + p]$. For $\eta \in \mathrm{supp}(\Psi)$, some $\tilde{\lambda} \ge 1$ and $\varrho(.,\tilde{\lambda}) \in \mathcal{S}_{K_0}(\tilde{\lambda})$ write

$$I_\pm(\varrho(.,\tilde{\lambda}))_\eta(z,\lambda) = e^{\pm i\pi/2 - i\pi/4}\frac{\eta\lambda}{2\pi}\int e^{i\eta\lambda(z\zeta - \psi_\pm(z',\zeta))}\chi(\zeta)a_\pm(\zeta,\eta\lambda)\varrho(z',\tilde{\lambda})dz'd\zeta, \quad (4.79)$$

where we set

$$\psi_\pm(z',\zeta) = z'\zeta \pm \frac{2}{3}((1-\zeta)^{3/2}-1). \quad (4.80)$$

We want to apply the Egorov theorem in order to invert the operators $I_{\pm,\eta}$. The symbols $\chi(\zeta)a_\pm(\zeta,\eta\lambda)$ are elliptic at $\zeta = 0$, consequently (eventually shrinking the support of $\chi$) there exists symbols $b_\pm(\zeta,\eta\lambda)$ which are asymptotic expansions in $(\eta\lambda)^{-1}$ for $\eta$ belonging to the support of $\Psi$, such that, if one denotes by $J_\pm(.)_\eta$ the operators defined for $\breve{\varrho} \in \mathcal{S}_{K\mp 1}(\tilde{\lambda})$ by

$$J_\pm(\breve{\varrho}(.,\tilde{\lambda}))_\eta(z',\lambda) = e^{\mp i\pi/2 + i\pi/4}\frac{\eta\lambda}{2\pi}\int e^{i\eta\lambda(\psi_\pm(z',\zeta) - z\zeta)}b_\pm(\zeta,\eta\lambda)\breve{\varrho}(z,\tilde{\lambda})dzd\zeta, \quad (4.81)$$

then one has

$$\breve{\varrho}(.,\tilde{\lambda}) = I_+(J_+(\breve{\varrho}(.,\tilde{\lambda}))_\eta(.,\lambda))_\eta(.,\lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}),$$

$$\varrho(.,\tilde{\lambda}) = J_+(I_+(\varrho(.,\tilde{\lambda}))_\eta(.,\lambda))_\eta(.,\lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}),$$

and also

$$\breve{\varrho}(.,\tilde{\lambda}) = I_-(J_-(\breve{\varrho}(.,\tilde{\lambda}))_\eta(.,\lambda))_\eta(.,\lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}),$$

$$\varrho(.,\tilde{\lambda}) = J_-(I_-(\varrho(.,\tilde{\lambda}))_\eta(.,\lambda))_\eta(.,\lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty}).$$

*Remark* 4.8. A direct computation shows that, for instance

$$J_\pm(I_\pm(\varrho(.,\tilde{\lambda}))_\eta(.,\lambda))_\eta(z,\lambda)=\frac{\eta\lambda}{2\pi}\int e^{i\eta\lambda(z-z')\zeta}\chi(\zeta)a_\pm(\zeta,\eta\lambda)b_\pm(\zeta,\eta\lambda)\varrho(z',\tilde{\lambda})dz'd\zeta$$

and consequently (since the coefficients do not depend on $z'$ and because of the expression of the phase functions $\psi_\pm(z',\zeta)$) one can take $b_\pm(\zeta,\eta\lambda)=\frac{\chi(\zeta)}{a_\pm(\zeta,\eta\lambda)}$.

**Proposition 4.7.** *Let $N\simeq\lambda h^\epsilon$ for some small $\epsilon>0$ and $1\le n\le N$. Let $T_k$ be the translation operator which to a given $\varrho(z)$ associates $\varrho(z+k)$. Then for $\eta\in supp(\Psi)$*

$$(T_1\circ J_+(.)_\eta\circ I_-(.)_\eta\circ T_1)^n:\mathcal{S}_{K_0}(\lambda)\to\mathcal{S}_{K_0}(\lambda/n)\quad \textit{uniformly in}\quad n. \tag{4.82}$$

*Notice that since $\lambda/n\ge h^{-\epsilon}\gg 1$, then one has*

$$O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})=O_{\mathcal{S}(\mathbb{R})}((\lambda/n)^{-\infty})=O_{\mathcal{S}(\mathbb{R})}(h^\infty). \tag{4.83}$$

*Remark* 4.9. Notice that at this point we have a restriction on the number of reflections $N$ which should be much smaller when compared to $\lambda=a^{3/2}/h$. In fact, in the proof of Proposition 4.7 we apply the stationary phase with parameter $\lambda/n$ which should be large enough, more precisely it should be larger than some (positive) power of $h^{-1}$. Using (4.83), this would imply that away from the critical points the contributions of the oscillatory integrals are $O(h^\infty)$.

*Proof.* We start by determining the explicit form of the operator in (4.82).

For $\zeta$ on the support of $\chi$, $|\zeta|\le 2c<1$ for $c$ small enough, set

$$f(\zeta)=2\zeta-\frac{4}{3}(1-(1-\zeta)^{\frac{3}{2}})=\frac{\zeta^2}{2}+O(\zeta^3). \tag{4.84}$$

Let $\varrho(.,\lambda)\in\mathcal{S}_{K_0}(\lambda)$. Then for $\eta$ on the support of $\Psi$ we have

$$T_1(J_+(I_-(T_1(\varrho(.,\lambda)))_\eta)_\eta)(z)=\frac{\eta\lambda}{2\pi}\int_{z',\zeta}e^{i\eta\lambda((z-z')\zeta+f(\zeta))}c(\zeta,\eta\lambda)\varrho(z',\lambda)dz'd\zeta$$
$$=(F_{\eta\lambda}*\varrho(.,\lambda))(z), \tag{4.85}$$

where we set, for $\eta$ on the support of $\Psi$

$$F_{\eta\lambda}(z)=\frac{\eta\lambda}{2\pi}\int_\zeta e^{i\eta\lambda(z\zeta+f(\zeta))}c(\zeta,\eta\lambda)d\zeta, \tag{4.86}$$

$$c(\zeta,\eta\lambda)=\chi(\zeta)a_+(\zeta,\eta\lambda)b_-(\zeta,\eta\lambda)=\chi^2(\zeta)\sum_{j\ge0}c_j(1-\zeta)^{-3j/2}(\eta\lambda)^{-j},\quad c_0=1, \tag{4.87}$$

thus for each $n$ we can write

$$(T_1\circ J_+(.)_\eta\circ I_-(.)_\eta\circ T_1)^n(\varrho(.,\lambda))(z)=(F_{\eta\lambda})^{*n}*\varrho(.,\lambda)(z). \tag{4.88}$$

We can explicitly compute $(F_{\eta\lambda})^{*n}$

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\lambda}{2\pi}\int_\zeta e^{i\eta\lambda z\zeta}\hat{F}^n_{\eta\lambda}(\eta\lambda\zeta)d\zeta = \frac{\eta\lambda}{2\pi}\int_\zeta e^{i\eta\lambda(z\zeta+nf(\zeta))}c^n(\zeta,\eta\lambda)d\zeta. \tag{4.89}$$

Set $\tilde{\zeta} = n\zeta$ and $\tilde{\lambda} = \frac{\lambda}{n}$. The choice we made for $N$ allows to write the right hand side of (4.89) as

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta}} e^{i\eta\tilde{\lambda}(z\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}c^n(\frac{\tilde{\zeta}}{n}, n\eta\tilde{\lambda})d\tilde{\zeta}, \tag{4.90}$$

therefor we obtain

$$(T_1\circ J_+(.)_\eta\circ I_-(.)_\eta\circ T_1)^n(\varrho(.,\lambda))(z) = \\ = \frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta}} e^{i\eta\tilde{\lambda}((z-z')\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}c^n(\frac{\tilde{\zeta}}{n}, n\eta\tilde{\lambda})\varrho(z',\lambda)d\tilde{\zeta}dz'. \tag{4.91}$$

The phase function in (4.91) is given by

$$\eta\phi_n(z,z',\tilde{\zeta}) = \eta((z-z')\tilde{\zeta} + n^2 f(\frac{\tilde{\zeta}}{n})),$$

and its critical points satisfy

$$\partial_{\tilde{\zeta}}\phi_n = z - z' + nf'(\frac{\tilde{\zeta}}{n}) = 0, \quad \partial_{z'}\phi_n = -\tilde{\zeta} = 0. \tag{4.92}$$

— In order to show that for all $\alpha \geq 0$ there exists constants $C^2_\alpha$ independent of $n$, $\lambda$, such that

$$\sup_{z\in\mathbb{R},\tilde{\lambda}=\lambda/n\geq 1} |\partial^\alpha_z(T_1\circ J_+(.)_\eta\circ I_-(.)_\eta\circ T_1)^n(\varrho(.,\lambda))(z)| \leq C^2_\alpha$$

we write

$$\partial^\alpha_z(T_1\circ J_+(.)_\eta\circ I_-(.)_\eta\circ T_1)^n(\varrho(.,\lambda))(z) = (F_{\eta\lambda})^{*n} * \partial^\alpha_z\varrho(.,\lambda)(z).$$

For $\tilde{\zeta}$ outside a small neighborhood of 0, $|\tilde{\zeta}| \geq c$, we perform integrations by parts in $z'$ in the integral (4.90) defining $(F_{\eta\lambda})^{*n}(z)$ and obtain a contribution arbitrarily small. For $|\tilde{\zeta}| < 2c$ small, let $\psi$ be a smooth function with support included in a $c$-neighborhood of $K_0$ and such that $(1-\psi)\varrho(.,\lambda) = O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$. For $z$ away from a $5c$-neighborhood of $K_0$ we saw in the proof of Lemma 4.4 that we have $|\partial_{\tilde{\zeta}}\phi_n(z,z',\tilde{\zeta})| \geq c$ and we conclude again by integrations by parts in $\tilde{\zeta}$. Near the critical points $\tilde{\zeta} = 0$ and $z = z' - nf'(\frac{\tilde{\zeta}}{n})$ we can apply the stationary method lemma in both variables $z'$, $\tilde{\zeta}$, uniformly in $n$: it is crucial here that $f(0) = f'(0) = 0$, $f''(0) = 1$ which gives, setting $g_n(\tilde{\zeta}) = n^2 f(\frac{\tilde{\zeta}}{n})$,

$$|g_n(\tilde{\zeta})| \leq d_0|\tilde{\zeta}|^2, \quad |g'_n(\tilde{\zeta})| \leq d_1|\tilde{\zeta}|, \quad |g^{(k)}_n(\tilde{\zeta})| \leq d_k, \forall k \geq 2,$$

with constants $d_k$ independent of $n$ (we deal with Fourier multipliers), $|g_n''| \geq d_2' > 0$ and that, on the other hand, from (4.87) we have $|\partial_{\tilde{\zeta}}^m \chi^{2n}(\frac{\tilde{\zeta}}{n})| \leq e_m$ for all $m \geq 0$ with constants $e_m$ independent of $n$ and

$$|\partial_{\tilde{\zeta}} c^n(\frac{\tilde{\zeta}}{n}, n\eta\tilde{\lambda})| \leq e_1 |c^n(\frac{\tilde{\zeta}}{n}, n\eta\tilde{\lambda}| + e_0 |\sum_{j\geq 0} c_j \frac{3j}{2} \frac{(1-\frac{\tilde{\zeta}}{n})^{-(3j+2)/2}}{(n\eta\tilde{\lambda})^j}|.$$

— To check that for a smooth function $\tilde{\psi}$ equal to 1 in a neighborhood of $K_0$ we have

$$(1-\tilde{\psi})(T_1 \circ J_+(.)_\eta \circ I_-(.)_\eta \circ T_1)^n(\varrho(.,\lambda)) = O_{\mathcal{S}(\mathbb{R})}(\tilde{\lambda}^{-\infty})$$

we use the same arguments as in the second part of the proof of Lemma 4.4.

□

*Definition* 4.4. Let $\varrho(.,\lambda) \in \mathcal{S}_{K_0}(\lambda)$ and $\eta \in \text{supp}(\Psi)$. For $1 \leq n \leq N$, $N \simeq \lambda h^\epsilon$ set

$$\varrho^n(z,\eta,\lambda) = (-1)^n \Psi(\eta)(T_1 \circ J_+(.)_\eta \circ I_-(.)_\eta T_1)^n(\varrho(.,\lambda))(z),$$

$$\varrho^0(z,\eta,\lambda) = \varrho(z,\lambda)\Psi(\eta).$$

From Proposition 4.7 it follows that $\varrho^n(z,\eta,\lambda) \in \mathcal{S}_{K_0}(\lambda/n)$. Let

$$g^n(t,s,\eta,h) = \varrho^n(\frac{t+2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}} - 2n, \eta, \lambda), \tag{4.93}$$

$$\Phi_n(t,x,y,s,\eta) = \eta(y - t(1+a)^{1/2} + (x-a)s + \frac{s^3}{3} + \frac{4}{3}na^{3/2}), \tag{4.94}$$

and set

$$U_h = \sum_{n=0}^{N} u_h^n, \quad u_h^n(t,x,y) = \int e^{\frac{i}{h}\Phi_n} g^n(t,s,\eta,h) ds d\eta. \tag{4.95}$$

**Proposition 4.8.** *With this choice of the symbols $g^n$ we have for all $0 \leq n \leq N-1$*

$$Tr_-(u_h^n)(t,y;h) + Tr_+(u_h^{n+1})(t,y;h) = O_{L^2}(\lambda^{-\infty}). \tag{4.96}$$

*Proof.* The proof follows from Propositions 4.6, 4.7 and the definition of the symbols $g^n$, since we have

$$e^{\frac{i\pi}{2}} I_-(T_1(\varrho^n(.,\eta,\lambda)))_\eta + e^{-\frac{i\pi}{2}} I_+(T_{-1}(\varrho^{n+1}(.,\eta,\lambda)))_\eta = O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}),$$

where $T_{\pm 1}$ are the translation operators which to a given $\varrho(z)$ associate $\varrho(z \pm 1)$ and since the operators $I_{\pm,\eta}$ are of convolution type so they commute with translations. □

## 4.4 Strichartz estimates for the approximate solution

Let $U_h$ be given by (4.95) and let $(q, r)$ a sharp wave-admissible pair in dimension 2, i.e. such that $\frac{1}{q} = \frac{1}{2}(\frac{1}{2} - \frac{1}{r})$. The "cusp" is reflected with period $a^{1/2}$, $a = h^\delta$ and in order to compute the norm of $U_h$ on a finite interval of time we will take $\delta = \frac{1-\epsilon}{2}$ in order to obtain

$$1 \simeq Na^{1/2} \simeq \lambda h^\epsilon h^{\delta/2}. \tag{4.97}$$

We prove the following

**Proposition 4.9.** *Let $r > 4$, $\beta(r) = \frac{3}{2}(\frac{1}{2} - \frac{1}{r}) + \frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ and let $\beta \leq \beta(r) - \epsilon$ for $\epsilon > 0$ small enough like before. Then the approximate solution of the wave equation* (4.2) *satisfies*

$$h^\beta \|U_h\|_{L^q([0,1],L^r(\Omega))} \gg \|U_h|_{t=0}\|_{L^2(\Omega)}. \tag{4.98}$$

*In particular, the restriction on $\beta$ shows that the Strichartz inequalities of the free case are not valid, there is a loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ derivatives.*

*Proof.* In the construction of $U_h$ we considered an initial "cusp" $u_h^0$ of the form (5.37), with symbol $g$ given by (4.60), with $\varrho^0 \in \mathcal{S}_{[-c_0,c_0]}(a^{3/2}/h)$ depending only on the integral curves of the vector field $Z$ and $\eta$, supported for $\eta$ in a small neighborhood of 1. We introduce the Lagrangian manifold associated to $u_h^n$, with phase function $\Phi_n = \Phi + \frac{4}{3}n\eta a^{3/2}$

$$\Lambda_{\Phi_n} := \{(t, x, y, \tau = -(1+a)^{1/2}\eta, \xi = s\eta, \eta),$$
$$a - x = s^2, y - t(1+a)^{1/2} + \frac{4}{3}na^{3/2} = \frac{2}{3}s^3\} \subset T^*\mathbb{R}^3. \tag{4.99}$$

**Lemma 4.5.** *Let $u_h^n$ be given by* (4.95)*, then $WF_h(u_h^n) \subset \Lambda_{\Phi_n}$.*

*Proof.* If $|\partial_s \Phi_n| \geq c > 0$ we use the operator $L_1 = \frac{h}{i|s^2+a-x|}\partial_s$ in order to gain a power of $h^{1-\frac{\delta}{2}}$ at each integration by parts with respect to $s$, thus the contribution we get in this case is $O_{L^2}(h^\infty)$. Let now $|\partial_\eta \Phi_n| \geq c > 0$ for some positive constant $c$: before making (repeated) integrations by parts using this time the operator $L_2 = \frac{h\partial_\eta\Phi_n}{i|\partial_\eta\Phi_n|^2}\partial_\eta$ we need to estimate the derivatives with respect to $\eta$ for each $g^n$ defined in (4.93). We have

$$u_h^n(t, x, y) = (-1)^m \int e^{\frac{i}{h}\Phi_n} L_2^{*m}(\varrho^n(\frac{t + 2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}} + 2n, \eta, \lambda))d\eta ds$$
$$= (-1)^{m+n}h^m \int e^{\frac{i}{h}\Phi_n} \frac{(\partial_\eta\Phi_n)^m}{|\partial_\eta\Phi_n|^{2m}}\Big(\sum_{k=0}^{m} \partial_\eta^{m-k}\Psi(\eta)\partial_\eta^k(F_{\eta\lambda})^{*n}\Big)*$$
$$* \varrho(,\lambda)(\frac{t + 2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}} + 2n)d\eta ds, \tag{4.100}$$

where $*$ denotes the convolution product. The derivatives of $(F_{\eta\lambda})^{*n}$ with respect to $\eta$ are easily computed using the explicit form of $(F_{\eta\lambda})^{*n}$ that we recall (see the proof of Proposition 4.7):

$$(F_{\eta\lambda})^{*n}(z)=\frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta}}e^{i\eta\tilde{\lambda}(z\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}c^n(\frac{\tilde{\zeta}}{\lambda},\eta n\tilde{\lambda})d\tilde{\zeta},$$

with $c(\zeta,\omega)=\chi^2(\zeta)\sum_{j\geq 0}c_j(1-\zeta)^{-3j/2}\omega^{-j}$ and where we have made the change of variable $\tilde{\zeta}=n\zeta$ and set $\tilde{\lambda}=\lambda/n\geq h^{-\epsilon}\gg 1$. Hence one $\eta$-derivative yields

$$\begin{aligned}\partial_\eta(F_{\eta\lambda})^{*n}(z)=\frac{1}{\eta}(F_{\eta\lambda})^{*n}(z)+\frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta}}e^{i\eta\tilde{\lambda}(z\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}i\tilde{\lambda}(z\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))c^n(\frac{\tilde{\zeta}}{n},\eta n\tilde{\lambda})d\tilde{\zeta}\\+\frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta}}e^{i\eta\tilde{\lambda}(z\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}n\partial_\eta c(\frac{\tilde{\zeta}}{n},\eta n\tilde{\lambda})c^{n-1}(\frac{\tilde{\zeta}}{n},\eta n\tilde{\lambda})d\tilde{\zeta}.\end{aligned}\tag{4.101}$$

The symbol of the third term in the right hand side of (4.101) is $n\partial_\eta c(\zeta,\eta\lambda)c^{n-1}(\zeta,\eta\lambda)$ and we have

$$\partial_\eta c(\zeta,\eta\lambda)=-\eta^{-2}\lambda^{-1}\sum_{j\geq 1}jc_j(1-\zeta)^{-3j/2}(\eta\lambda)^{-(j-1)},$$

and since $n\ll\lambda$, the contribution from this term is easily handled with.

The symbol in the second term in the right hand side of (4.101) equals the symbol of $(F_{\eta\lambda})^{*n}$ multiplied by the factor $i\tilde{\lambda}(z\tilde{\zeta}+\lambda nf(\frac{\tilde{\zeta}}{n}))$. Recall that on the support of $c(\zeta,\eta\lambda)$ we have $\zeta=\tilde{\zeta}/n\in\operatorname{supp}(\chi)$ is as close to zero as we want and there $f(\zeta)=\zeta^2/2+O(\zeta^3)$, hence $n^2f(\frac{\tilde{\zeta}}{n})=\tilde{\zeta}^2/2+O(\tilde{\zeta}^3/n)$. On the other hand, when we take the convolution product of the second term in (4.101) with $\varrho^0(.,\lambda)$ we obtain in the same way as in the proof of Proposition 4.7 that the critical points of the phase in the oscillatory integral obtained in this way,

$$\frac{\eta\tilde{\lambda}}{2\pi}\int_{\tilde{\zeta},z'}e^{i\eta\tilde{\lambda}((z-z')\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))}i\tilde{\lambda}((z-z')\tilde{\zeta}+n^2f(\frac{\tilde{\zeta}}{n}))c^n(\frac{\tilde{\zeta}}{n},\eta n\tilde{\lambda})\varrho^0(z',\lambda)d\tilde{\zeta}dz',$$

are given by $\tilde{\zeta}=0$ and $z=z'$. The phase function which will be denoted again by $\phi_n(z,z',\tilde{\zeta})$ as before satisfies $\phi_n(z,z,0)=0$, $\partial_{z'}\phi_n(z,z,0)=0$ and $\partial_{\tilde{\zeta}}\phi_n(z,z,0)=0$. Applying the stationary phase theorem in $\tilde{\zeta}$ and $z'$, the first term in the asymptotic expansion obtained in this way vanishes, and the next ones are multiplied by strictly negative, integer powers of $\tilde{\lambda}$, hence the contribution from this term will is also bounded.

Notice that when we take higher order derivatives in $\eta$ of $\varrho^n$, we obtain symbols which are products of $\tilde{\lambda}^j(\phi_n)^j\partial_\eta^{k-j}(c^n(\tilde{\zeta}/n,\eta n\tilde{\lambda}))$ and can be dealt with in the same way, taking into account this time that the first $j$ terms in the asymptotic expansion obtained after applying the stationary phase vanish. As a consequence, after each integration by parts in $\eta$ using the operator $L_2$ we gain a factor $h$, meaning that the contribution of $u_h^n$ is $O_{L^2}(h^\infty)$. □

We also need the next results:

**Lemma 4.6.** *If $\varrho(.,\lambda)\in\mathcal{S}_{[-c_0,c_0]}(\lambda)$ with $0<c_0<1$ sufficiently small, then $u_h^n$ have almost disjoint supports in the time variable $t$.*

*Proof.* Let $\mu\in(0,1)$ and $|t-4n(1+a)^{1/2}a^{1/2}|\geq 2(1+a)^{1/2}a^{1/2}(1+\mu)$. Then on the essential support of $\varrho^n(\frac{t+2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}}-2n,\eta,\lambda)$ we must have $|s|\geq a^{1/2}(1+\mu-c_0)$ while on the Lagrangian $\Lambda_{\Phi_n}$ defined in (4.99) we have $|a-x|=s^2\leq a$. Consequently, if $\mu\geq c_0+\epsilon_0$ for some $\epsilon_0>0$ as small as we want, we are not anymore on the Lagrangian $\Lambda_{\Phi_n}$. Since outside any neighborhood of $\Lambda_{\Phi_n}$ the contribution in the integral defining $u_h^n$ is $O_{L^2}(h^\infty)$, we conclude that $u_h^n$ "lives" essentially on a time interval

$$[4n(1+a)^{1/2}a^{1/2}-2(1+a)^{1/2}a^{1/2}(1+c_0),4n(1+a)^{1/2}a^{1/2}+2(1+a)^{1/2}a^{1/2}(1+c_0)].$$

□

Since $a=h^\delta\ll 1$ and therefor $(1+a)^{1/2}\simeq 1$ we claim that $u_h^n$ is in fact essentially supported for $t$ in the time interval

$$[4na^{1/2}-2a^{1/2}(1+c_0),4na^{1/2}+2a^{1/2}(1+c_0)]$$

**Lemma 4.7.** *Let $0<c_0<1/3$ and let $I_k$ be small neighborhoods of $4a^{1/2}k$ of size $a^{1/2}$,*

$$I_k=[4ka^{1/2}-a^{1/2}c_0,4ka^{1/2}+a^{1/2}c_0].$$

*If $t\in I_k$ then in the sum $U_h(t,.)$ there is only one cusp that appears, $u_h^k(t,.)$, the contribution from all the others $u_h^n(t,.)$ with $n\neq k$ being $O_{L^2}(h^\infty)$.*

*Proof.* On the essential support of $\varrho^n(.,\eta,\lambda)$ one has

$$|t+2(1+a)^{1/2}s-4n(1+a)^{1/2}a^{1/2}|\leq 2(1+a)^{1/2}a^{1/2}c_0.$$

Suppose $n\neq k$: we have to show that the contribution from $u_h^n$ is $O_{L^2}(h^\infty)$. Write

$$2(1+a)^{1/2}a^{1/2}c_0\geq 4|n-k|(1+a)^{1/2}a^{1/2}-|t-4k(1+a)^{1/2}a^{1/2}|-2(1+a)^{1/2}|s|$$

$$\geq 4(1+a)^{1/2}a^{1/2}-(1+a)^{1/2}a^{1/2}c_0-2(1+a)^{1/2}|s|,$$

which yields $|s|\geq 3a^{1/2}/2$ since $c_0<1/3$ and as in the proof of Lemma 4.6 we see that we are localized away from a neighborhood of $\Lambda_{\Phi_n}$ (on which $|s|\leq a^{1/2}$), thus the contribution is $O_{L^2}(h^\infty)$. Consequently, the only nontrivial part comes from $n=k$ in which case we find $|s|\leq 3c_0a^{1/2}/2\leq a^{1/2}/2$, thus the $k$-th "piece of cusp" does not reach the boundary $\{x=0\}$ (since on the Lagrangian $\Lambda_{\Phi_k}$ we have $a-x=s^2$ and outside any neighborhood of $\Lambda_{\Phi_k}$ the contribution is $O_{L^2}(h^\infty)$). □

We turn to the proof of Proposition 4.9. We use Lemma 4.7 and Proposition 4.14 from the Appendix to estimate from below the $L^q([0,1],L^r(\Omega))$ norm of $U_h$:

$$\|U_h\|^q_{L^q([0,1],L^r(\Omega))} = \int_0^1 \|U_h\|^q_{L^r(\Omega)}dt = \int_0^1 \|\sum_{n=0}^N u_h^n\|^q_{L^r(\Omega)}dt \tag{4.102}$$

$$\geq \sum_{k\leq N/5}\int_{t\in I_k} \|\sum_{n=0}^N u_h^n\|^q_{L^r(\Omega)}dt + O(h^\infty) \tag{4.103}$$

$$\simeq \sum_{k\leq N/5} |I_k|\|u_h^0\|^q_{L^r(\Omega)} + O(h^\infty) \tag{4.104}$$

$$\simeq \|u_h^0\|^q_{L^r(\Omega)} + O(h^\infty). \tag{4.105}$$

Indeed, we have shown in Lemma 4.7 that for $t$ belonging to sufficiently small intervals of time $I_k$ there is only $u_h^k$ to be considered in the sum since the supports of $u_h^n$ will be disjoints. On the other hand, for $t\in I_k$, $u_h^k(t,.)$ admits a cusp singularity at $x=a$ which guarantees that the piece of cusp does not "live" enough to reach the boundary. Moreover, we see from Proposition 4.14 that for $t\in I_k$ the $L^r(\Omega)$ norms of $u_h^k(t,.)$ are equivalent to the $L^r(\Omega)$ norms of $u_h^0$. Using Corollary 4.3 we deduce that there are constants $C$ independent of $h$ such that for $r=2$

$$\|U_h|_{t=0}\|_{L^2(\Omega)} = \|u_h|_{t=0}\|_{L^2(\Omega)} \simeq h^{1+\frac{\delta}{4}}, \tag{4.106}$$

while for $r>4$

$$\|U_h\|_{L^q([0,1],L^r(\Omega))} \geq Ch^{\frac{1}{3}+\frac{5}{3r}} \tag{4.107}$$

and since $\delta = \frac{1-\epsilon}{2}$ we deduce that (4.98) holds for $\beta\leq\beta(r)-\epsilon$ since we have

$$h^\beta\|U_h\|_{L^q([0,1],L^r(\Omega))} \geq Ch^{\beta(r)-\epsilon}h^{\frac{1}{3}+\frac{5}{3r}} = Ch^{-7\epsilon/8}h^{1+(1-\epsilon)/8} \gg h^{1+\frac{\delta}{4}} \simeq \|U_h|_{t=0}\|_{L^2(\Omega)}. \tag{4.108}$$

*Remark* 4.10. Notice that for $2\leq r<4$

$$\|U_h\|_{L^q([0,1],L^r(\Omega))} \geq Ch^{\frac{1}{r}+\frac{1}{2}+\delta(\frac{1}{r}-\frac{1}{4})}, \tag{4.109}$$

therefor in this case the previous construction doesn't provide a contradiction to the Strichartz inequalities when compared to the free case.

□

**Proposition 4.10.** *The approximate solution $U_h$ defined in* (4.95) *satisfies the Dirichlet boundary condition*

$$U_h|_{[0,1]\times\partial\Omega} = O(h^\infty). \tag{4.110}$$

*Proof.* Using Propositions 4.6 and 4.8, the contribution of $U_h$ on the boundary writes

$$U_h(t,0,y)=\sum_{n=0}^{N}\sum_{\pm}Tr_{\pm}(u_h^n)(t,y;h)=Tr_+(u_h^0)(t,y;h)+Tr_-(u_h^N)(t,y;h). \tag{4.111}$$

The first term in the right hand side of (4.111) is easy to handle since $Tr_+(u_h^0)(t,y;h)$ is essentially supported for

$$t\in[-2(1+c_0)a^{1/2},-2(1-c_0)a^{1/2}].$$

Since we consider only the restriction to $[0,1]\times\partial\Omega$, the contribution from this term will be $O_{L^2}(h^\infty)$. To deal with the second term in the right hand side of (4.111) we first study the essential support of $Tr_-(u_h^N)(t,y;h)$ for $t\in[0,1]$. We distinguish two situations:

— If $(4h^{-\delta/2})^{-1}-[(4h^{-\delta/2})^{-1}]<1/2$, where we denoted by $[z]$ the integer part of $z$ we take

$$N:=[(4h^{-\delta/2})^{-1}]$$

and we deduce that $Tr_-(u_h^N)(t,y;h)$ is essentially supported for $t$ in an interval strictly contained in $[0,1]$ while $Tr_+(u_h^N)(t,y;h)$ has a nontrivial contribution only on

$$[4Na^{1/2}-2(1+c_0)a^{1/2},4Na^{1/2}-2(1-c_0)a^{1/2}].$$

A direct computation shows that for this choice of $N$

$$4Na^{1/2}-2(1+c_0)a^{1/2}\simeq 4h^{-\delta}[(4h^{-\delta})^{-1}]+\frac{1}{2}(4h^{-\delta})^{-1}>1.$$

Therefor, on $[0,1]$ the contribution of $Tr_+(u_h^N)(t,y;h)$ is canceled by $Tr_-(u_h^{N-1})(t,y;h)$, while the contribution of $Tr_-(u_h^N)$ equals $O_{L^2}(h^\infty)$ since it is essentially supported outside $[0,1]$.

— If $(4h^{-\delta/2})^{-1}-[(4h^{-\delta/2})^{-1}]\geq 1/2$, we set

$$N:=[(4h^{-\alpha/2})^{-1}]+1$$

and we conclude using the same arguments as in the preceding case.

□

## 4.5 End of the proof of Theorem 4.1

Let $U_h$ be the approximate solution to the wave equation (4.49) defined by (4.95). In (4.106) we obtained $\|U_h|_{t=0}\|_{L^2(\Omega)}\simeq h^{1+\delta/4}$. We now consider the $L^2$-normalized approximate solution $W_h=\frac{1}{\|U_h|_{t=0}\|_{L^2(\Omega)}}U_h$. We also let $V_h=W_h+w_h$, where $V_h$ solves

$$\Box V_h=0,\quad V_h|_{[0,1]\times\partial\Omega}=0, \tag{4.112}$$

with initial data

$$V_h|_{t=0}=W_h|_{t=0},\quad \partial_tV_h|_{t=0}=\partial_tW_h|_{t=0}. \tag{4.113}$$

**Proposition 4.11.** *Under the preceding assumptions $w_h$ satisfies*

$$\|\Box w_h\|_{L^2(t\in[0,1],L^2(\Omega))} = O(h^{-\delta}), \quad w_h|_{\partial\Omega} = O_{L^2}(h^\infty), \tag{4.114}$$

$$w_h|_{t=0} = 0, \quad \partial_t w_h|_{t=0} = 0. \tag{4.115}$$

*Proof.* If we set $\alpha(h) := \|U_h|_{t=0}\|_{L^2(\Omega)} \simeq h^{1+\delta/4}$, then one has

$$\|\Box w_h\|^2_{L^2(t\in[0,1],L^2(\Omega))} = \alpha(h)^{-2}\|\sum_{n=0}^{N}\Box u_h^n\|^2_{L^2(t\in[0,1])L^2(\Omega)} \tag{4.116}$$

$$\lesssim \alpha(h)^{-2}\sum_{k\leq N/4}\int_{J_k}\|\sum_{n=0}^{N}\Box u_h^n\|^2_{L^2(\Omega)}dt + O(h^\infty) \tag{4.117}$$

$$\lesssim 8\alpha(h)^{-2}\sum_{k\leq N/4}\int_{J_k}\|\Box u_h^k\|^2_{L^2(\Omega)} + O(h^\infty), \tag{4.118}$$

where

$$J_k := [4a^{1/2}k - 2a^{1/2}, 4a^{1/2}k + 2a^{1/2}],$$

and where we used the fact that for each $n$ there are at most three cusps to consider for $t \in J_k$ as shown in Lemma 4.7. Let us estimate $\|\Box u_h^k(t,.)\|_{L^2(\Omega)}$ for $t \in J_k$. The proof of Proposition 4.14 of the Appendix applied to $\Box u_h^k$ (computed in (4.61)) yields

$$\|\Box u_h^k(t,.)\|_{L^2(\Omega)} \lesssim h^{-\delta+1+\delta/4},$$

since the assumption $\varrho \in \mathcal{S}_{[-c_0,c_0]}(\lambda)$ implies that $\sup_z|\partial_z^2\varrho| \leq C$ for some constant $C$ independent of $\lambda$ and one can bound from above the $L^2(\Omega)$ norm of $\Box u_h^k$ (notice that the only difference between the estimates concerning $u_h^k$ is that instead of $\varrho^n$ we now have $\partial^2\varrho^n$ which we handle in the same way). Consequently we obtain

$$\|\Box w_h\|^2_{L^2(t\in[0,1],L^2(\Omega))} \lesssim a(h)^{-2}\sum_{k\leq N/4}|J_k|h^{-2\delta+2+\delta/2} \lesssim h^{-2\delta},$$

since $|J_k|$ are of size $a^{1/2}$, $k \leq N/4$ and $Na^{1/2} \simeq 1$. □

**Corollary 4.2.** *If $(q,r)$ is a sharp wave-admissible pair in dimension two then $w_h$ satisfies*

$$\|w_h\|_{L^q([0,1],L^r(\Omega))} \leq Ch^{1-\delta-2(\frac{1}{2}-\frac{1}{r})}, \tag{4.119}$$

*where $C$ is some constant independent of $h$.*

*Proof.* Write the Duhamel formula for $w_h$,

$$w_h(t,x,y) = \int_0^t \frac{\sin(t-\tau)\sqrt{-\Delta_D}}{\sqrt{-\Delta_D}}(\Box w_h(\tau,.))d\tau. \tag{4.120}$$

Using the Minkowsky inequality and Proposition 4.11 we find

$$\|w_h\|_{L^\infty([0,1],H^1(\Omega))} = \|\int_0^t \frac{\sin(t-\tau)\sqrt{-\Delta_D}}{\sqrt{-\Delta_D}}(\Box w_h(\tau,.))d\tau\|_{L^\infty([0,1],H^1(\Omega))}$$
$$\leq \int_0^1 \|\frac{\Box w_h(\tau,.)}{\sqrt{-\Delta_D}}\|_{H^1(\Omega)}d\tau \simeq \|\Box w_h\|_{L^1([0,1],L^2(\Omega))} \leq Ch^{-\delta}. \quad (4.121)$$

*Remark* 4.11. Notice that since we are dealing with the Dirichlet Laplace operator $\Delta_D$ inside a bounded domain there is no problem in estimating $\|(\sqrt{-\Delta_D})^{-1}f\|_{H^1(\Omega)}$ by $\|f\|_{L^2(\Omega)}$. Indeed, let $(e_{\nu_j})_{j\geq 0}$ be the eigenbasis of $L^2(\Omega)$ consisting in eigenfunctions of $-\Delta_D$ associated to the eigenvalues $\nu_j^2$ considered in non-decreasing order and decompose $f = \sum_{j\geq 0} f_j e_{\nu_j}$, $f_j =< f, e_{\nu_j} >$. Then

$$(\sqrt{-\Delta_D})^{-1}f \simeq \sum_j \frac{1}{\nu_j} f_j e_{\nu_j}$$

and since $\nu_1 \geq c > 0$ for some fixed constant $c > 0$ we can estimate

$$\|(\sqrt{-\Delta_D})^{-1}f\|^2_{H^1(\Omega)} \simeq \sum_{j\geq 0} \frac{(1+\nu_j^2)}{\nu_j^2}\|f_j\|^2_{L^2(\Omega)}.$$

Take now $C = \sup_j(1+1/\nu_j^2) \leq 1+1/c^2$, then

$$\|(\sqrt{-\Delta_D})^{-1}f\|_{H^1(\Omega)} \leq \sqrt{C}\|f\|_{L^2(\Omega)}. \quad (4.122)$$

If, instead, we were considering the Neumann Laplacian $\Delta_N$ inside the domain $\Omega$, in order to obtain bounds like in (4.122) we had to introduce a cut-off function $\Psi \in C_0^\infty(\mathbb{R})$ equal to 1 close to 0 and decompose a function $f$

$$f = \Psi(-\Delta_N)f + (1-\Psi(-\Delta_N))f$$

and treat separately the contribution $\Psi(-\Delta_N)f$ obtained for small frequencies of $f$.

In order to obtain estimates for the $L^\infty([0,1],L^r(\Omega))$ norms of $w_h$ we also need to establish bounds from above for its $L^\infty([0,1],L^2(\Omega))$ norms. We need the next result:

**Proposition 4.12.** *Let $f(x,y) : \Omega \to \mathbb{R}$ be localized at frequency $1/h$ in the $y \in \mathbb{R}^{d-1}$ variable, i.e. such that there exists $\psi \in C_0^\infty(\mathbb{R}^{d-1}\setminus 0)$ with $\psi(hD_y)f = f$. Then there exists a constant $C > 0$ independent of $h$ such that one has*

$$\|f\|_{H^{-1}(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

*Proof.* Since $\chi(hD_y)f = f$ we have

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\|g\|_{H^1(\Omega)}\leq 1} \int \psi f\bar{g} \leq \|f\|_{L^2(\Omega)} \times \sup_{\|g\|_{H^1(\Omega)}\leq 1} \|\psi(hD_y)g\|_{L^2(\Omega)}$$
$$\leq h\|f\|_{L^2(\Omega)}\|\tilde{\psi}(hD_y)\nabla_y g\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)},$$

where we set $\tilde{\psi}(\eta) = |\eta|^{-1}\psi(\eta)$. □

Using again Duhamel's formula written above, we have

$$\|w_h\|_{L^\infty([0,1],L^2(\Omega))} \lesssim \|\Box w_h\|_{L^1([0,1],H^{-1}(\Omega))} \tag{4.123}$$

and from Proposition 4.12 applied to $f = \Box w_h$ we deduce

$$\|w_h\|_{L^\infty([0,1],L^2(\Omega))} \lesssim h\|\Box w_h\|_{L^1([0,1],L^2)} \lesssim Ch^{1-\delta}. \tag{4.124}$$

Interpolation between (4.121) and (4.124) with weights $\sigma$ and $1-\sigma$ yields

$$\|w_h\|_{L^\infty([0,1],H^\sigma(\Omega))} \leq Ch^{1-\delta-\sigma}. \tag{4.125}$$

We take $\sigma = 2(\frac{1}{2} - \frac{1}{r})$ and use the Sobolev inequality in order to obtain

$$\|w_h\|_{L^q([0,1],L^r(\Omega))} \leq Ch^{1-\delta-2(\frac{1}{2}-\frac{1}{r})}. \tag{4.126}$$

□

*End of the proof of Theorem* 4.1

From Corollary 4.2 we see that the norm $\|w_h\|_{L^q(([0,1],L^r(\Omega))}$ is much smaller then the norm of $\|W_h\|_{L^q(([0,1],L^r(\Omega))}$: in fact we have to check that the following inequality holds for $r > 4$

$$h^{1-\delta-2(\frac{1}{2}-\frac{1}{r})} \ll h^{\frac{1}{3}+\frac{5}{3r}-1-\frac{\delta}{4}} \tag{4.127}$$

which is obviously true. Let $\beta < \beta(r) = \frac{3}{2}(\frac{1}{2} - \frac{1}{r}) + \frac{1}{6}(\frac{1}{4} - \frac{1}{r})$. We have

$$h^\beta\|V_h\|_{L^q([0,1],L^r(\Omega))} \geq h^\beta(\|W_h\|_{L^q([0,1],L^r(\Omega))} - \|w_h\|_{L^q([0,1],L^r(\Omega))}) \tag{4.128}$$

$$\geq \frac{1}{2}h^\beta\|W_h\|_{L^q([0,1],L^r(\Omega))} \gg 1. \tag{4.129}$$

On the other hand $\|V_h\|_{L^2(\Omega)} \simeq 1$, $h\|\partial_t V_h|_{t=0}\|_{L^2(\Omega)} \simeq 1$ , thus for $\beta < \beta(r)$ the (exact) solution $V_h$ satisfies

$$h^\beta\|V_h\|_{L^q([0,1],L^r(\Omega))} \gg \|V_h|_{t=0}\|_{L^2(\Omega)}. \tag{4.130}$$

The proof of Theorem 4.1 is complete.

## 4.6 Appendix

### 4.6.1 Proof of Lemma 4.1 ( TT* argument)

*Proof.* Let $0 < T_0 < \infty$ and denote by $T$ the operator which to a given $u_0 \in L^2(\mathbb{R}^n)$ associates $U(t)\psi(hD)u_0 \in L^q([0,T_0], L^r(\mathbb{R}^n))$, where by $U(t) = e^{-\frac{it}{h}G}$ we denoted the linear flow. Its adjoint $T^* : L^{q'}([0,T_0], L^{r'}(\mathbb{R}^n)) \to L^2(\mathbb{R}^n)$ is given by

$$(T^*g)(x) = \int_0^{T_0} \psi^* U(-t)g(t,x)dt \tag{4.131}$$

thus we can write

$$(TT^*g)(t,x) = \int_0^{T_0} U(t)\psi\psi^* U(-s)g(s,x)ds = \int_0^{T_0} U(t-s)\psi\psi^* g(s,x)ds \tag{4.132}$$

since $\psi$ has constant coefficients. Suppose that the dispersive estimate

$$\|e^{-\frac{it}{h}G}\psi(hD)u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim (2\pi h)^{-n}\gamma_{n,h}(\frac{t}{h})\|\psi(hD)u_0\|_{L^1(\mathbb{R}^n)} \tag{4.133}$$

holds for a function $\gamma_{n,h} : \mathbb{R} \to \mathbb{R}_+$. Interpolation between (4.133) and the energy estimates gives

$$\|e^{-\frac{it}{h}G}\psi(hD)u_0\|_{L^r(\mathbb{R}^n)} \leq Ch^{-n(1-\frac{2}{r})}\gamma_{n,h}(\frac{t}{h})^{1-\frac{2}{r}}\|u_0\|_{L^{r'}(\mathbb{R}^n)}, \tag{4.134}$$

and from (4.132) and (4.134) we deduce

$$\|TT^*g\|_{L^q((0,T_0],L^r(\mathbb{R}^n))} \leq Ch^{-n(1-\frac{2}{r})}\|\int_0^{T_0}\gamma_{n,h}(\frac{t-s}{h})^{1-\frac{2}{r}}\|g(s)\|_{L^{r'}(\mathbb{R}^n)}ds\|_{L^q[0,T_0]}. \tag{4.135}$$

The application $|t|^{-\frac{2}{q}}* : L^{q'} \to L^q$ is bounded for $q > 2$ by Hardy-Littlewood-Sobolev theorem, thus we obtain (4.20),

$$\|T\|^2_{L^2\to L^q((0,T_0],L^r(\mathbb{R}^n))} \leq h^{-n(1-\frac{2}{r})}\sup_{t\in(0,T_0]} t^{\frac{2}{q}}\gamma(\frac{t}{h})^{1-\frac{2}{r}} \leq Ch^{-2\beta}\Big(\sup_{s\in(0,\frac{T_0}{h}]} s^\alpha\gamma(s))\Big)^{1-\frac{2}{r}}. \tag{4.136}$$

□

### 4.6.2 Propagation of positivity

On $\mathbb{C}^{2m} = \mathbb{C}^m_z \times \mathbb{C}^m_\zeta$ one considers the symplectic 2-form $\sigma = dz \wedge d\zeta =: \sigma_\mathbb{R} + i\sigma_\mathbb{I}$.

*Definition* 4.5. Let $\Lambda$ be a smooth manifold of $\mathbb{C}^{2m}$. It is called

1. $\mathbb{R}$ (resp. $\mathbb{I}$, $\mathbb{C}$)-Lagrangian if its dimension on $\mathbb{R}$ is $2m$ and $\sigma_\mathbb{R}|_\Lambda = 0$ (resp. $\sigma_\mathbb{I}|_\Lambda = 0$, $\sigma_\mathbb{C}|_\Lambda = 0$);
2. $\mathbb{R}$ (resp. $\mathbb{I}$)-symplectic if $\sigma_\mathbb{R}|_{T\Lambda}$ (resp. $\sigma_\mathbb{I}|_{T\Lambda}$) is nondegenerate;
3. positive at some point $\rho \in \Lambda$ if the (real-valued) quadratic form $Q : u \to \frac{1}{i}\sigma(u,\bar{u})$ is positive definite on the tangent space $T_\rho\Lambda$ of $\Lambda$ at $\rho$.

**Lemma 4.8.** *Assume that the projection $\Lambda \ni (z,\zeta) \to z \in \mathbb{C}^m$ is a local diffeomorphism. Then $\Lambda$ is a $\mathbb{C}$-Lagrangian if and only if it is locally described by an equation of the type $\zeta = \frac{\partial\Phi}{\partial z}$, where $\Phi$ is a holomorphic function of $z$ and we write (locally) $\Lambda = \Lambda_\Phi$.*

**Lemma 4.9.** *A $\mathbb{C}$-Lagrangian $\Lambda$ is positive at some point $\rho$ if and only if near $\rho$ it is of the form $\Lambda_\Phi$, where $\Phi$ is a holomorphic function such that the real symmetric matrix $(Im\frac{\partial^2\Phi}{\partial z_j\partial z_k})_{j,k=\overline{1,m}}$ is positive definite.*

Let $q = q(z, \zeta)$ be a holomorphic function on an open subset $U \subset \mathbb{C}^{2m}$. Then as in the real domain one defines the Hamilton field of $q$ by the identity $\sigma(u, H_q(z,\zeta)) = dq(z,\zeta)u$. One also defines the Hamilton flow $\exp sH_q(z,\zeta)$ for $s$ real, by

$$\frac{\partial}{\partial s} \exp sH_q(z,\zeta) = H_q(\exp sH_q(z,\zeta)) \tag{4.137}$$

and one can easily prove that for any open subset $U' \subset\subset U$ and for any $s \in \mathbb{R}$ such that $\cup_{s'\in[0,s]} \exp s'H_q(U') \subset U$, the application $U' \ni (z,\zeta) \to \exp sH_q(z,\zeta)$ is a complex canonical transformation.

**Lemma 4.10.** *Let $\Lambda$ be a $\mathbb{C}$-Lagrangian submanifold of $\mathbb{C}^{2m}$ and assume that there exists $\rho \in \Lambda \cap \mathbb{R}^{2m}$ such that $\Lambda$ is positive at $\rho$. Moreover assume that there exists a complex canonical transformation $\kappa$ defined on a complex domain containing $\mathbb{R}^{2m}$ such that $\kappa(\mathbb{R}^{2m}) \subset \mathbb{R}^{2m}$ and $\kappa(\rho) \in \Lambda$. Then $\Lambda$ is positive at $\kappa(\rho)$.*

*Proof.* Observe that if $u \in T_{\kappa(\rho)}\Lambda$, then $u = d\kappa(\rho)v$ with $v \in T_\rho\Lambda$ and $\bar{u} = \overline{d\kappa(\rho)\bar{v}} = d\kappa(\bar{\rho})\bar{v} = d\kappa\rho\bar{v}$. Take now $\kappa = \exp sH_q$. For the proofs see [76], [92]. □

### 4.6.3 Airy functions

We give below some of the basic properties of the function $Ai(z)$ which are used in this work. For $z \in \mathbb{R}$, $Ai(z)$ is defined by

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\frac{u^3}{3}+zu)}du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\frac{u^3}{3} + zu)du. \tag{4.138}$$

This integral is not absolutely convergent, but is well defined as the Fourier transform of a temperate distribution. For pozitive $z > 0$, $z \to \infty$ we have

$$Ai(z) = O(z^{-\infty}), \tag{4.139}$$

$$Ai(-z) = A^+(-z) + A^-(-z)(\simeq \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \cos(\frac{2}{3}z^{3/2} - \frac{\pi}{4})), \tag{4.140}$$

where

$$A^{\pm}(-z) \simeq z^{-1/4} e^{\mp\frac{2i}{3}z^{3/2} \pm \frac{i\pi}{2} - \frac{i\pi}{4}} (\sum_{j=0}^{\infty} a_{\pm,j}(-1)^{-j/2} z^{-3j/2}), \quad a_{\pm,0} = \frac{1}{4\pi^{3/2}}. \tag{4.141}$$

**Proposition 4.13.** *All the zeroes of $Ai(z)$ are real and negative, say*

$$Ai(-\omega_j) = 0, \quad 0 > -\omega_0 > -\omega_1 > ... \to -\infty. \tag{4.142}$$

**Proof of Lemma 4.2:**

*Proof.* Let $k \geq 0$ be fixed. After the change of variables $x = h^{2/3}\zeta$ it remains to show that

$$\|\psi_1(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})} \lesssim \|\psi(hD_y)u(h^{2/3}\zeta,.)\|_{L^r(\mathbb{R}_+\times\mathbb{R}^{d-1})} \lesssim \|\psi_2(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})} \quad (4.143)$$

for some cut-off functions. From the multiplier theorem it follows that for $\zeta > 0$ fixed, the operator $\psi(hD_y)Ai(|hD_y|^{2/3}\zeta - \omega_k)$ is bounded on $L^r(\mathbb{R}^{d-1})$ for all $r \in (1,\infty)$. Indeed, for $\zeta \in [0, M]$ for some $M \gg 1$ large, we have

$$|\partial_\eta^\alpha(\psi(\eta)Ai(|\eta|^{2/3}\zeta - \omega_k))| \leq C(d,k,M), \quad \forall 0 \leq |\alpha| \leq [(d-1)/2]+1,$$

while for $|\zeta| > M$ we bound the above derivatives by $C(d,k,l)|\zeta|^{-l}$ for every $l \geq 0$. Since $\psi = \psi\psi_2$, we find

$$\|\psi(hD_y)Ai(|hD_y|^{2/3}\zeta - \omega_k)\varphi\|_{L^r(\mathbb{R}^{d-1})} \leq C(r,d,k,\zeta)\|\psi_2(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})}, \quad (4.144)$$

where $C(r,d,k,\zeta) = \max(r,(r-1)^{-1}) \times \begin{cases} C(d,k,M), & |\zeta| \leq M, \\ C(d,k,l)|\zeta|^{-l}, \quad \forall l \geq 0, & |\zeta| > M. \end{cases}$

Let now $\eta_0$, $\eta_1$ such that $|\eta_0|^{-2/3}(\omega_k - \omega_0) < |\eta_1|^{-2/3}(\omega_k + \frac{1}{2})$ and let $\operatorname{supp}(\psi) \subset \{0 < |\eta_0| \leq |\eta| \leq |\eta_1|\}$. Let $\epsilon > 0$ be fixed, small and define $\zeta_0 = (\omega_k - \omega_0 + \epsilon)|\eta_0|^{-2/3}$, $\zeta_1 = (\omega_k + 1 - \epsilon)|\eta_1|^{-2/3}$. For $\zeta \in [\zeta_0, \zeta_1]$ we have $-\omega_0 + \epsilon \leq z = \zeta|\eta|^{2/3} - \omega_k \leq 1 - \epsilon$. For these values of $z$, $Ai(z)$ is bounded from below and the operator $\psi(hD_y)Ai(|hD_y|^{2/3}\zeta - \omega_k)$ is invertible. Setting $b(\zeta,\eta) = \frac{\psi(\eta)}{Ai(|\eta|^{2/3}\zeta - \omega_k)}$ and using the assumption $\psi_1 = \psi\psi_1$, we have

$$\psi_1(hD_y)\varphi = \frac{1}{2\pi h}\int e^{iy\eta/h} b(\zeta,\eta)(\widehat{\psi(hD_y)u(h^{2/3}\zeta,.)})(\frac{\eta}{h})d\eta, \quad (4.145)$$

with $\widehat{\psi(hD_y)u(h^{2/3}}\zeta,.)(\frac{\eta}{h}) = \psi(\eta)Ai(|\eta|^{2/3}\zeta - \omega_k)\psi_1(\eta)\varphi(\frac{\eta}{h})$. Using again the multiplier theorem this time in (4.145), we obtain, uniformly for $\zeta \in [\zeta_0, \zeta_1]$,

$$\begin{aligned} \|\psi_1(hD_y)\varphi\|_{L^r(\mathbb{R}^{d-1})} &\leq C(r,d,k,\zeta)\|\psi(hD_y)u(h^{\frac{2}{3}}\zeta,.)\|_{L^r(\mathbb{R}^{d-1})} \\ &\leq C(r,d,k)\|\psi(hD_y)u(h^{\frac{2}{3}}\zeta,.)\|_{L^r(\mathbb{R}_+\times\mathbb{R}^{d-1})}. \end{aligned} \quad (4.146)$$

□

### 4.6.4 $L^r$ norms of the phase integrals associated to a cusp type Lagrangian

**Proposition 4.14.** *For $t \in [4na^{1/2} - 2a^{1/2}(1+c_0), 4na^{1/2} + 2a^{1/2}(1+c_0)]$, the $L^r(\Omega)$ norm of a cusp $u_h^n(t,.)$ of the form (5.37) are estimated (uniformly in $t$) by*

$$\|u_h^n(t,.)\|_{L^r(\Omega)} \simeq \begin{cases} h^{\frac{1}{r}+\frac{1}{2}}a^{\frac{1}{r}-\frac{1}{4}}, & 2 \leq r < 4, \\ h^{\frac{1}{3}+\frac{5}{3r}}, & r > 4. \end{cases} \quad (4.147)$$

From Proposition 4.14 we deduce the following

**Corollary 4.3.** *For $t \in [4na^{1/2} - 2a^{1/2}(1+c_0), 4na^{1/2} + 2a^{1/2}(1+c_0)]$, the $L^r(\Omega)$ norms of a cusp $u_h^n(t,.)$ satisfy*

— *for $2 \leq r < 4$*

$$\|u_h^n(t,.)\|_{L^r(\Omega)} \simeq h^{\frac{1}{r}+\frac{1}{2}+\delta(\frac{1}{r}-\frac{1}{4})}, \tag{4.148}$$

$$\|u_h^n(0,.)\|_{L^2(\Omega)} \simeq h^{1+\frac{\delta}{4}}. \tag{4.149}$$

— *for $r > 4$*

$$\|u_h^n(t,.)\|_{L^r(\Omega)} \simeq h^{\frac{1}{3}+\frac{5}{3r}}. \tag{4.150}$$

*Proof.* Let $0 \leq n \leq N \simeq \lambda h^{\epsilon}$ be fixed and let

$$t \in [4na^{1/2} - 2a^{1/2}(1+c_0), 4na^{1/2} + 2a^{1/2}(1+c_0)].$$

We compute the $L^r(\Omega)$ norms of

$$u_h^n(t,x,y) = \int_{\mathbb{R}^2} e^{\frac{i\eta}{h}(y-(1+a)^{1/2}t+(x-a)s+s^3/3-\frac{4}{3}na^{3/2})} \times$$
$$\times \Psi(\eta)\varrho^n(\frac{t+2(1+a)^{1/2}s}{2(1+a)^{1/2}a^{1/2}} - 2n, \eta, \lambda) ds d\eta, \tag{4.151}$$

where the symbol $\varrho^n(.,\eta,\lambda) \in \mathcal{S}_{[-c_0,c_0]}(\lambda/(n+1))$ defined in (4.4) is essentially supported for the first variable in $[-c_0, c_0]$ and where $\eta$ close to 1 on the support of $\Psi$. Notice that due to the translation $y \to (y - t(1+a)^{1/2} + \frac{4}{3}na^{3/2})$ and the change of variable $x \to (a-x)$ we are reduced to estimate the norm of

$$v_h^n(z,x,y) := \int e^{\frac{i\eta}{h}(y+\frac{s^3}{3}-sx)} \varrho^n(z + \frac{s}{h^{\frac{\delta}{2}}}, \eta, \lambda)\Psi(\eta) ds d\eta,$$

for $z = \frac{t}{2(1+a)^{1/2}a^{1/2}} - 2n \in [-(1+c_0), 1+c_0]$. We distinguish several regions:

1. For $|x| \leq Mh^{2/3}$ where $M$ is a constant, we make the changes of variables $x = \zeta h^{2/3}$ and $s = h^{1/3}u$ which gives

$$I(z,x,\eta,h) := \int e^{\frac{i\eta}{h}(\frac{s^3}{3}-sx)} \varrho^n(z + h^{-\delta/2}s, \eta, \lambda) ds \tag{4.152}$$
$$= h^{1/3} \int e^{i\eta(\frac{u^3}{3}-u\zeta)} \varrho^n(z + h^{1/3-\delta/2}u, \eta, \lambda) du.$$

Let $Q(\zeta, u) = \frac{u^3}{3} - \zeta u$ and for $\theta : \mathbb{R} \to [0,1]$, set

$$F_\theta(w,\zeta,z,h) = \int e^{iw\eta}\Psi(\eta) f_\theta(\zeta,\eta,z,h) d\eta, \tag{4.153}$$

$$f_\theta(\zeta,\eta,z,h)=\int e^{i\eta Q(\zeta,u)}\theta(u)\varrho^n(z+h^{1/3-\delta/2}u,\eta,\lambda)du. \tag{4.154}$$

We make integrations by parts in order to compute

$$w^kF_\theta(w,\zeta,z,h)=i^k\int e^{iw\eta}\partial_\eta^k(\Psi(\eta)f_\theta(\zeta,\eta,z,h))d\eta,$$

$$\partial_\eta^k f_\theta(\zeta,\eta,z,h)=\int e^{i\eta Q(\zeta,u)}\theta(u)\sum_{j=0}^k C_k^j(iQ)^{k-j}\partial_\eta^j\varrho^n(z+h^{1/3-\delta/2}u,\eta,\lambda)du.$$

Let $\theta(u)=1_{|u|\leq\sqrt{1+M}}$. Since we integrate for $\eta$ in a neighborhood of 1, for all $k\geq 0$ we estimate

$$\|w^kF_\theta(w,\zeta,z,h)\|_{L^\infty_w}\leq \tag{4.155}$$

$$\sum_{j=0}^k C_k^j\sup_{|u|\leq\sqrt{1+M}}|Q(\zeta,u)|^{k-j}\int|\partial_\eta^j\varrho^n(z+h^{1/3-\delta/2}u,\eta,\lambda)|d\eta\leq C_{k,M},$$

where $C_{k,M}$ are constants and where we used the fact that $\varrho^n$ writes as a convolution product $\varrho^n(z,\eta,\lambda)=(F_{\eta\lambda})^{*n}*\varrho^0(.,\lambda)(z)$ and the derivatives in $\eta$ of $(F_{\eta\lambda})^{*n}$ were computed in Lemma 4.5.

For $\sqrt{1+M}\leq|u|\lesssim h^{\frac{\delta}{2}-\frac{1}{3}}$ we integrate by parts using the operator $L=\frac{\partial_u}{i\eta\partial_uQ}$ which satisfies $L(e^{i\eta Q})=e^{i\eta Q}$. If we denote, for fixed $k$, $j\in\{0,..,k\}$

$$Q_0^{k,j}:=(1-\theta(u))Q^{k-j}\partial_\eta^j\varrho^n(z+h^{1/3-\delta/2}u,\eta,\lambda),$$

and for $l\geq 0$, $Q_{l+1}^{k,j}=\partial_u(\frac{Q_l^{k,j}}{\partial_uQ})$, then we can write

$$\int L^l(e^{i\eta Q})(1-\theta(u))Q_0^{k,j}du=\frac{(-1)^l}{(i\eta)^l}\int e^{i\eta Q}Q_l^{k,j}du, \tag{4.156}$$

where

$$Q_l^{k,j}=\sum_{m=0}^l c_{l,m}^{k,j}(\zeta,\varrho^n(z+.,\eta,\lambda))|u|^{3(k-j)-3l+m}h^{m(\frac{1}{3}-\frac{\delta}{2})},$$

where $c_{l,m}^{k,j}$ depends on the derivatives $\partial_u^{l-m}\partial_\eta^j\varrho^n(z+.,\eta,\lambda)$. The principal term is obtained for $j=0$ and $m=0$ and it equals $|u|^{3k-3l}$. It's enough to take $l=2k$ to obtain similar bounds for $\|w^kF_{1-\theta}(w,\zeta,z,h)\|_{L^\infty_w}$ as in (4.155). We find

$$\|v_h^n(z,.)\|_{L^r(|x|\leq Mh^{2/3},y)}=h^{2/3r}(\int_y\int_0^M|v_h^n(z,h^{2/3}\zeta,y)|^rd\zeta dy)^{1/r}$$
$$=h^{5/3r+1/3}\|F_1(w,\zeta,z,h)\|_{L^r(\zeta\leq M,w)}\simeq h^{5/3r+1/3}. \tag{4.157}$$

2. For $x\in(Mh^{2/3},a]$ with $M\gg 1$ big enough we apply the stationary phase theorem:

**Proposition 4.15.** *([56, Thm.7.7.5]) Let $K \subset \mathbb{R}$ be a compact set, $f \in C_0^\infty(K)$, $\phi \in C^\infty(\mathring{K})$ such that $\phi(0) = \phi'(0) = 0$, $\phi''(0) \neq 0$, $\phi' \neq 0$ in $\mathring{K} \setminus \{0\}$. Let $\omega \gg 1$, then for every $k \geq 1$ we have*

$$|\int e^{i\omega\phi(u)} f(u)du - \frac{(2\pi i)^{\frac{1}{2}} e^{i\omega\phi(0)}}{(\omega\phi''(0))^{\frac{1}{2}}} \sum_{j<k} \omega^{-j} L_j f| \leq C\omega^{-k} \sum_{|\alpha|\leq 2k} \sup |\partial^\alpha f|. \qquad (4.158)$$

*Here $C$ is bounded when $\phi$ stays in a bounded set in $C^\infty(\mathring{K})$, $|u|/|\phi'(u)|$ has a uniform bound and*

$$L_j f = \sum_{\nu-\mu=j} \sum_{2\nu\geq 3\mu} \frac{i^{-j}2^{-\nu}}{\mu!\nu!} (\phi''(0))^{-\nu} \partial^{2\nu}(\kappa^\mu f)(0). \qquad (4.159)$$

*where $\kappa(u) = \phi(u) - \phi(0) - \frac{\phi''(0)}{2}u^2$ vanishes of third order at 0.*
We make the change of variable $s = \sqrt{x}(\pm 1 + u)$ to compute the integral in $s$ in the expression of $v_h$. Using Proposition 4.15 with $\phi_\pm(u) = \frac{u^3}{3} \pm u^2$, $\omega = \eta\frac{x^{3/2}}{h} \gg 1$, $\kappa_\pm(u) = u^3/3$ we write $I(z, x, \eta, h)$ as a sum $I(z, x, \eta, h) \simeq \sum_{\pm, j\geq 0} I_\pm^j(z, x, \eta, h)$, where

$$I_\pm^j(z, x, \eta, h) := (i\pi)^{1/2} h^{1/2+j} \eta^{-1/2-j} e^{\mp\frac{2}{3}i\eta x^{3/2}/h} x^{-1/4-3j/2} \times$$
$$\times L_j(\varrho^n(z + h^{-\frac{\delta}{2}}\sqrt{x}(\pm 1 + u), \eta, \lambda))|_{u=0}. \qquad (4.160)$$

We compute each $L^r$ norm of $\int e^{\frac{iy\eta}{h}} \Psi(\eta) I_\pm^j(z, x, \eta, h) d\eta$:

$$\| \int e^{\frac{iy\eta}{h}} \Psi(\eta) I_\pm^j(z, x, \eta, h) d\eta \|_{L^r(x\in(Mh^{2/3},a],y)} \simeq$$
$$h^{1/2+j} \|x^{-1/4-3j/2} \int e^{\frac{i\eta}{h}(y\mp\frac{2}{3}x^{3/2})} \Psi(\eta) \eta^{-1/2-j} L_j(\varrho^n(z \pm h^{-\delta/2}x^{1/2}, \eta, \lambda)) d\eta \|_{L^r(x\in(Mh^{2/3},a],y)}. \qquad (4.161)$$

Using again the fact that $\varrho^n(z, \eta, \lambda) = (F_{\eta\lambda})^{*n} * \varrho^0(., \lambda)(z)$ we introduce the map $F^{n,j}(z, \eta) := \Psi(\eta)\eta^{-1/2-j}(F_{\eta\lambda})^{*n}(z)$ which is compactly supported in $\eta$; if $\widehat{F^{n,j}}(z, .)$ denotes its Fourier transform with respect to $\eta$, (4.161) reads

$$h^{1/2+j} \|x^{-1/4-3j/2} \widehat{F^{n,j}}(., \frac{(y \mp \frac{2}{3}x^{3/2})}{h}) * L_j(\varrho^n(., \eta, \lambda)(z \pm h^{-\delta/2}x^{1/2}))\|_{L^r(x\in(Mh^{2/3},a],y)}.$$

Setting $y = hw$, $x = h^{2/3}\zeta$ and translating $w \to w \mp \frac{2}{3}\zeta^{3/2}$ we can estimate from above and from below each one of the above norms. For $j \geq 0$, $L_j$ is a differential operator of order $2j$ and each derivative on $\varrho$ gives a factor $\sqrt{x}/h^{\delta/2} \leq 1$. We estimate the $L^r$ norm of $\int e^{\frac{iy\eta}{h}} \Psi(\eta) I(z, x, \eta, h) d\eta$ from above and from below by the sum over $j$ of

$$Ch^{r(1/2+j+5/3r-1/6-j)} \int_M^{Ah^{-2/3}} \zeta^{-r(1/4+3j/2)} d\zeta$$

where $C > 0$ are constants, and since the operators $L_j$ are of order $2j$, for each $j$ there will be $2j$ terms in the sum: summing up over $j \geq 0$ (taking $M \geq 2$ for example) and using the assumption $\varrho^n \in \mathcal{S}_{[-c_0,c_0]}(\lambda/(n+1))$ which assures uniform bounds for the derivatives $\partial^j \varrho^n(.,\eta,\lambda)$ for each $n, j \geq 0$, we obtain for $r > 4$

$$\| \int e^{\frac{iy\eta}{h}} \Psi(\eta) I(z,x,\eta,h) d\eta \|^r_{L^r(x\in(Mh^{2/3},a],y)} \simeq h^{r/3+5/3} \sum_{j\geq 0} \frac{jM^{1-r(1/4+3j/2)}}{(r(1/4+3j/2)-1)},$$

and taking $M \geq 2$ sufficiently big we can sum over $j$ and we deduce (4.157) for $r > 4$.

For $r \in [2,4)$ and $j = 0$ we have $r/4 - 1 < 0$ and

$$\int_M^{ah^{-2/3}} \zeta^{-r/4} d\zeta \simeq \frac{(ah^{-2/3})^{1-r/4}}{1-r/4}.$$

For $r \in [2,4)$ and $j \geq 1$ we have $r(1/4+3j/2) - 1 > 0$ and

$$\int_M^{ah^{-2/3}} \zeta^{-r(1/4+3j/2)} d\zeta \simeq \frac{M^{1-r(1/4+3j/2)}}{r(1/4+3j/2)-1}.$$

If $M \geq 2$ is sufficiently large the sum of the $L^r$ norms over $j \geq 0$ is small compared to the norm for $j = 0$, hence (4.157) follows for $r \in [2,4)$ too.

3. In the last case $x > a$ the $L^r(\Omega)$ norms are as small as we want since the contribution of $u_h^n$ in this case is $O_{L^2}(h^\infty)$, because this region is localized away from a neighborhood of the Lagrangian $\Lambda_{\Phi_n}$ and we use Lemma 4.5.

□

# 5 Counterexamples to Strichartz inequalities for the wave equation on general bounded domains of dimension $D \geq 2$

The key feature of the manifold leading to the counter-example to the Stricharz estimates for the wave equation with Dirichlet boundary condition is the presence of a point where the boundary is microlocally geodesically strictly convex (i.e. the presence of gliding rays, or highly-multipliy reflected geodesics). Indeed, in the opposite extreme of strict geodesic-concavity, the usual Strichartz estimates do hold by the work of H.Smith and C.Sogge [95]. The particular manifold studied in the previous paper is one for which the eigenmodes are explicitly expressed in terms of Airy's function, and the phases for the oscillatory integrals to be evaluated have precise form.

In this section we extend the previous result to the case of a general manifold of dimension $d \geq 2$ with a gliding ray; the heart of the matter is well illustrated by the particular example of the Friedlander's model and here we will only sketch the generalization of the proof which is possible using Melrose's equivalence of glancing hypersurfaces theorem.

## 5.1 Introduction

Let $\Omega$ be a smooth manifold of dimension $d \geq 2$ with $C^\infty$ boundary $\partial\Omega$, equipped with a Riemannian metric $g$. Let $\Delta_g$ be the Laplace-Beltrami operator associated to $g$ on $\Omega$, acting on $L^2(\Omega)$ with Dirichlet boundary condition. Let $0 < T < \infty$ and consider the wave equation with Dirichlet boundary conditions:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0 \text{ on } \Omega \times [0,T], \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.1}$$

Strichartz estimates are a family of dispersive estimates on solutions $u : \Omega \times [0,T] \to \mathbb{C}$ to the wave equation (6.21). In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([0,T],L^r(\Omega))} \leq C(\|u_0\|_{\dot{H}^\gamma(\Omega)} + \|u_1\|_{\dot{H}^{\gamma-1}}), \tag{5.2}$$

where $\dot{H}^\gamma(\Omega)$ denotes the homogeneous Sobolev space over $\Omega$ and where the pair $(q,r)$ is wave admissible in dimension $d$, i.e. it satisfies $2 \leq q \leq \infty$, $2 \leq r < \infty$ and moreover

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad \frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}. \tag{5.3}$$

When equality holds in (5.3) the pair $(q,r)$ is called sharp wave admissible in dimension $d$. Estimates involving $r = \infty$ hold when $(q,r,d) \neq (2,\infty,3)$, but typically require the use of Besov spaces.

In $\mathbb{R}^d$ and for $g_{ij} = \delta_{ij}$, global Strichartz estimates in the context of the wave and Schrödinger equations are well established (for more references see Section 4 or [61]). However, for general manifolds phenomena such as trapped geodesics or finiteness of volume can preclude the development of global estimates, leading us to consider local in time estimates.

For manifolds with smooth, strictly geodesically concave boundary, the usual Strichartz estimates do hold by the work of H.Smith and C.Sogge [95]. If the concavity assumption is removed, however, the presence of multiply reflecting geodesic and their limits, the gliding rays, prevent the construction of a similar parametrix!

Note that on an exterior domain a source point does not generate caustics and that the presence of caustics generated in small time near a source point is the one which makes things difficult inside a strictly convex set.

*Assumption* 5.1. We assume that there exists a point $(\rho_0, \vartheta_0) \in T^*(\partial\Omega \times \mathbb{R})$ and a bicharacteristic which is tangential to $\partial\Omega \times \mathbb{R}$ at $(\rho_0, \vartheta_0)$ having exactly second order contact with the boundary. We call such a point a *gliding point*.

In this section we give the main steps which allow to generalize the proof of Theorem 4.1 of Section 4 to the case of a domain satisfying the Assumptions 5.1. Precisely, for any

$$\gamma < \frac{(d+1)}{2}(\frac{1}{2} - \frac{1}{r}) + \frac{1}{6}(\frac{1}{4} - \frac{1}{r})$$

we construct solutions $V_h$ to (5.1) which satisfy for any sharp admissible pair $(q, r)$ in dimension $d \geq 2$

$$\|V_h\|_{L^q([0,T],L^r(\Omega))} \geq h^{-\gamma}(\|V_h|_{t=0}\|_{L^2(\Omega)} + h\|\partial_t V_{h,}|_{t=0}\|_{L^2(\Omega)}). \tag{5.4}$$

*Remark* 5.1. Notice that this result is meaningful only in dimensions $d \in \{2, 3, 4\}$ since all the admissible pairs $(q, r)$ in higher dimension satisfy $r \leq 4$.

A classical way to prove Strichartz inequalities is to use dispersive estimates: the fact that weakened dispersive estimates can still imply optimal (and scale invariant) Strichartz estimates for the solution to the wave equation was first noticed by G.Lebeau; he proved in [74] that a loss of $\frac{1}{4}$ derivatives is unavoidable for the wave equation inside a strictly convex domain, and this appears because of swallowtail type caustics in the wave front set of the solution. However, these estimates, although optimal for the dispersion, imply only Strichartz type inequalities without losses (!) but with indices satisfying

$$\frac{1}{q} \leq (\frac{d-2}{2} + \frac{1}{4})(\frac{1}{2} - \frac{1}{r}).$$

The estimates for the spectral projectors obtained by H.Smith and C.Sogge [96] have been recently used by N.Burq, G.Lebeau and F.Planchon in [28] to establish Strichartz estimates for a certain range of triples $(q, r, \gamma)$ on a manifold with boundary; the range of triples that can be obtained in this way, however, is restricted by the allowed range of $r$ in the squarefunction estimate. The range of indices $(q, r)$ obtained there is expanded in the work of M.Blair, H.Smith and C.Sogge [15] where they show sharp Strichartz estimates for indices restricted to

$$\frac{1}{q} \leq \frac{(d-1)}{3}(\frac{1}{2} - \frac{1}{r}).$$

What's the importance of this counter-example? The very little work showing Stricharz without losses (but with different indices) have encouraged the belief in results similar to those of the Euclidian space, but the strongest (heuristic) argument is given by the following results concerning the equation (5.1): if one considers the initial data to be supported away from the boundary, then, by finite speed of propagation, sharp Strichartz estimates follow naturally; this is what happens although the data is very close to the boundary, but if the wave is transverse to it, so there is only one reflection. On the other hand, while considering a travelling wave concentrated near the boundary which propagates along the boundary (an example of such a wave is provided by the gallery modes) it is shown in Section 4 (see also [61]) that this kind of initial data yields Stricharz without losses too.

The counter-example for a domain satisfying the Assumptions 5.1 is constructed, essentially, as a superposition of travelling cusp-solutions to the wave equation. The reduction to the model case studied in Section 4 relies essentially on Melrose's Theorem [81] of glancing surfaces.

## 5.2 Reduction to the two dimensional case

Let $\Omega$ satisfy the Assumption 5.1. Write local coordinates on $\Omega$ as $(x, y_1, .., y_{d-1})$ with $x > 0$ on $\Omega$, $\partial\Omega = \{(0, y) | y = (y_1, .., y_{d-1}) \in \mathbb{R}^{d-1}\}$ and local coordinates induced by the product $X = \Omega \times \mathbb{R}_t$, as $(x, y, t)$.

Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on $T^*X$ near $\pi^{-1}(q)$, $q \in T^*\partial X$, where $\pi : T^*X \to^b T^*X$ is the canonical inclusion from the cotangent bundle into the $b$-cotangent bundle defined by ${}^bT^*X = T^*\mathring{X} \cup T^*\partial X$. The corresponding local coordinates on the boundary are denoted $(y, t, \eta, \tau)$ (on a neighborhood of a point $q$ in $T^*\partial X$). The metric function in $T^*\Omega$ has the form

$$g(x, y, \xi, \eta) = A(x, y)\xi^2 + 2\sum_{j=1}^{d-1} C_j(x, y)\xi\eta_j + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k,$$

with $A$, $B_{j,k}$, $C_j$ smooth. Moreover, these coordinates can be chosen so that $A(x, y) = 1$ and $C_j(x, y) = 0$ (see Hörmander [56, Appendix C]). Thus, in this coordinates chart the metric on the boundary writes

$$g(0, y, \xi, \eta) = \xi^2 + R(0, y, \eta), \quad R(x, y, \eta) = \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k.$$

On $T^*\partial\Omega$ the metric $g$ takes even a simpler form, since introducing geodesic coordinates we can assume moreover that, locally,

$$B_{1,1}(0,y) = 1, \quad B_{1,j}(0,y) = 0 \quad \forall j \in \{2,..,d-1\}.$$

The Assumption 5.1 on the domain $\Omega$ is equivalent to saying that there exists a point $(0, y_0, \xi_0, \eta_0)$ on $T^*\Omega$ where the boundary is microlocally strictly convex, i.e. that there exists a bicharacteristic passing through this point that intersects $\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. If $p \in C^\infty(T^*X \setminus o)$ (where we write $o$ for the "zero" section) denotes the principal symbol of the wave operator $\partial_t^2 - \Delta_g$, this last condition translates into

$$\tau^2 = R(0, y_0, \eta_0), \quad \{p, x\} = \frac{\partial p}{\partial \xi} = 2\xi_0 = 0, \tag{5.5}$$

$$\{\{p, x\}, p\} = \{\frac{\partial p}{\partial \xi}, p\} = 2\partial_x R(0, y_0, \eta_0) > 0, \tag{5.6}$$

where $\{f_1, f_2\}$ denotes the Poisson bracket $\{f_1, f_2\} = \frac{\partial f_1}{\partial \vartheta}\frac{\partial f_2}{\partial \rho} - \frac{\partial f_1}{\partial \rho}\frac{\partial f_2}{\partial \vartheta}$. Denote the gliding point (in $T^*(\Omega \times \mathbb{R})$) of the Assumption 5.1 by

$$(\rho_0, \vartheta_0) = (0, y_0, 0, 0, \eta_0, \tau_0 = -\sqrt{R(0, y_0, \eta_0)}).$$

We start the proof by reducing the problem to the study of the two dimensional case. Consider the following assumptions:

*Assumption* 5.2. Let $\tilde{\Omega}$ be a smooth manifold of dimension 2 with $C^\infty$ boundary and with a Riemannian metric $\tilde{g}$. Suppose that in a chart of local coordinates $\tilde{\Omega} = \{(x, \tilde{y}) | x > 0, \tilde{y} \in \mathbb{R}\}$ and that the Laplace-Beltrami operator associated to $\tilde{g}$ is given by

$$\partial_x^2 + (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2,$$

where $b(\tilde{y})$ is a smooth function. Suppose in addition that there exists a point $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0) \in T^*\tilde{\Omega}$ and a bicharacteristic intersecting the boundary tangentially at this point and having exactly second order contact with the boundary. This is equivalent to saying that at $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0)$ the following holds

$$\tilde{\xi}_0 = 0, \quad 2b(\tilde{y}_0) > 0.$$

We suppose $b(\tilde{y}_0) = 1$ and we let $0 < c < 1$ be small enough such that for $\tilde{y}$ in a neighborhood of $\tilde{y}_0$ to have $|b(\tilde{y}) - 1| \le c$.

**Theorem 5.1.** *Under the Assumption 5.2, given $T > 0$, for every $\epsilon > 0$ small enough there exist sequences $\tilde{V}_{h,j,\epsilon}$, $j \in \{0, 1\}$, such that the approximate solutions $\tilde{V}_{h,\epsilon}$ to the wave equation on $\tilde{\Omega}$ with Dirichlet boundary condition*

$$\begin{cases} \partial_t^2 V - \partial_x^2 V - (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2 V = 0, & on \quad \tilde{\Omega} \times \mathbb{R} \\ V|_{t=0} = \tilde{V}_{h,0,\epsilon}, \quad \partial_t V|_{t=0} = \tilde{V}_{h,1,\epsilon}, \\ V|_{\partial\Omega \times [0,T]} = 0, \end{cases} \tag{5.7}$$

*write as a sum*

$$\tilde{V}_{h,\epsilon}(x,\tilde{y},t) = \sum_{n=0}^{N} v^n_{h,\epsilon}(x,\tilde{y},t), \tag{5.8}$$

*where the functions $v^n_{h,\epsilon}(x,\tilde{y},t)$ satisfy the following conditions:*

— *for $4 < r < \infty$:*

$$\begin{cases} \|v^n_{h,\epsilon}(.,t)\|_{L^r(\tilde{\Omega})} \geq Ch^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon}, \\ \sup_{\epsilon>0} \|v^n_{h,\epsilon}(.,t)\|_{L^2(\tilde{\Omega})} \leq 1, \end{cases} \tag{5.9}$$

*where the constants $C > 0$ are independent of $h$ and $n$;*

— *$v^n_{h,\epsilon}(x,\tilde{y},t)$ are essentially supported for the time variable $t$ in almost disjoint intervals of time and for the tangential variable $\tilde{y}$ in almost disjoint intervals.*

— *$\tilde{V}_{h,\epsilon}$ are supported for the normal variable $x \in [0, \tilde{C}_\epsilon h^{(1-\epsilon)/2}]$ with $\tilde{C}_\epsilon > 0$ independent of $h$ and localized at spatial frequency $\frac{1}{h}$ in the tangential variable $\tilde{y}$. Moreover,*

$$\sup_{\epsilon>0} \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim 1, \quad \sup_{\epsilon>0} \|\partial_{\tilde{y}} \tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h}, \quad \sup_{\epsilon>0} \|\partial^2_{\tilde{y}} \tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h^2}; \tag{5.10}$$

—

$$\partial_t^2 \tilde{V}_{h,\epsilon} - \partial_x^2 \tilde{V}_{h,\epsilon} - (1 + xb(\tilde{y}))\partial^2_{\tilde{y}})\tilde{V}_{h,\epsilon} = O_{L^2(\tilde{\Omega})}(1/h), \quad \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \leq 1.$$

A first step is to prove that in order to construct a counter-example in dimension $d \geq 2$ it is enough to prove Theorem 5.1. Suppose we have already did that. Let $(\Omega, g)$ be a Riemannian manifold of dimension $d > 2$ satisfying the Assumptions 5.1 and let $(0, y_0, \xi_0, \eta_0) \in T^*\Omega$ be a point satisfying (5.5), (5.6). If $e_1$ is the eigenfunction corresponding to the strictly positive eigenvalue of the second fundamental form associated to the metric $g$ then it follows that local coordinates can be chosen such that $y_0 = 0 \in \mathbb{R}^{d-1}$, $\eta_0 = (1, 0, .., 0) \in \mathbb{R}^{d-1}$ and such that the Laplace-Beltrami operator $\Delta_g$ be given by

$$\Delta_g = \partial_x^2 + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_j \partial_k,$$

where for $x$ and $|y'|$ close to zero

$$B_{1,1}(x,y) = 1 + x\partial_x B_{1,1}(0, y_1, 0) + O(x|y'|) + O(x^2),$$

and for $j \in \{2, .., d-1\}$

$$B_{1,j}(x,y) = x\partial_x B_{1,j}(0,y) + O(x^2).$$

Define $\tilde{\Omega} = \{(x, y_1) | x > 0, y_1 \in \mathbb{R}\}$ the two dimensional manifold equipped with the metric

$$\tilde{g}(x, y_1, \xi, \eta_1) = \xi^2 + (1 + xb(y_1))\eta_1^2, \quad b(y_1) := \partial_x B_{1,1}(0, y_1, 0), \quad b(0) = 1.$$

Applying Theorem 5.1 near $(0, y_1 = 0, 0, \eta_1 = 1) \in T^*\tilde{\Omega}$ we obtain, for $\epsilon > 0$ small enough, sequences $\tilde{V}_{h,\epsilon,j}$, $j \in \{0, 1\}$ such that the solution $\tilde{V}_{h,\epsilon}$ to (5.7) satisfies (5.8), (5.9) and

(5.10). Let $\chi \in C_0^\infty(\mathbb{R}^{d-2})$ be a cut-off function supported in the coordinate chart such that $\chi = 1$ in a neighborhood of $0 \in \mathbb{R}^{d-2}$ and for $j \in \{0,1\}$ set

$$V_{h,\epsilon,j}(x, y_1, y') := h^{-(d-2)/4}\tilde{V}_{h,\epsilon/3,j}(x, y_1)e^{-\frac{|y'|^2}{2h}}\chi(y'). \tag{5.11}$$

**Proposition 5.1.** *Let $(q, r)$ be a sharp wave admissible pair in dimension $d$ and let*

$$\gamma \leq \frac{(d+1)}{2}(\frac{1}{2} - \frac{1}{r}) + \frac{1}{6}(\frac{1}{4} - \frac{1}{r}) - \epsilon.$$

*Then the solution $V_{h,\epsilon}$ to the wave equation (4.2) with Dirichlet boundary condition and initial data $(V_{h,\epsilon,0}, V_{h,\epsilon,1})$ defined in (5.11) satisfies (5.4).*

*Proof.* Let $(q, r)$ be a sharp wave admissible pair in dimension $d > 2$ with $r > 4$ and set

$$\beta(r, d) = \frac{(d+1)}{2}(\frac{1}{2} - \frac{1}{r}) + \frac{1}{6}(\frac{1}{4} - \frac{1}{r}).$$

If $\tilde{V}_{h,\epsilon/3}$ is the approximate solution to (5.7) with initial data $(\tilde{V}_{h,\epsilon/3,j})_{j=0,1}$ satisfying all the conditions in Theorem 5.1, we define

$$W_{h,\epsilon}(x, y, t) := h^{-(d-2)/4}\tilde{V}_{h,\epsilon/3}(x, y_1, t)e^{-\frac{|y'|^2}{2h}}\chi(y').$$

Using Theorem 5.1 we can prove the following:

**Lemma 5.1.** *There exists constants $c_j$, $j = 0, 1$, independent of $h$ such that $W_{h,\epsilon}$ satisfies*

$$\|W_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} \geq c_0 h^{-\beta(r,d)+2\epsilon/3}, \quad \|W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} \leq c_1. \tag{5.12}$$

Let $V_{h,\epsilon}$ be the solution to the wave equation (4.2) with initial data $(V_{h,\epsilon,j})_{j=0,1}$ and write $V_{h\epsilon} = W_{h,\epsilon} + w_{h,\epsilon,err}$. The following holds:

**Lemma 5.2.** *The "error" $w_{h,\epsilon,err}$ satisfies*

$$(\partial_t^2 - \Delta_g)w_{h,err} \simeq O_{L^2(\Omega)}(h^{-2(1-(1-\epsilon/3)/2)}) \geq O_{\dot{H}^{-1}(\Omega)}(h^{-\epsilon/3}). \tag{5.13}$$

*Moreover,*

$$\|w_{h,\epsilon,err}\|_{L^q([0,T],L^r(\Omega))} \leq C_\epsilon h^{-\beta(r,d)+2\epsilon-\epsilon/3}. \tag{5.14}$$

In the rest of the proof we proceed by contradiction. Precisely, we suppose that the operator

$$\sin t\sqrt{-\Delta_g} : L^2(\Omega) \to L^q([0,T], L^r(\Omega))$$

is bounded by $h^{-\beta(r,d)+2\epsilon}$. This last assumption implies

$$\|V_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} \leq C_{0,\epsilon}h^{-\beta(r,d)+2\epsilon}(\|V_{h,\epsilon,0}\|_{L^2(\Omega)} + \|V_{h,\epsilon,1}\|_{\dot{H}^{-1}(\Omega)}) \leq C_{1,\epsilon}h^{-\beta(r,d)+2\epsilon}, \tag{5.15}$$

where $C_{j,\epsilon} > 0$ are independent of $h$. If (5.15) were true, together with (5.12) it would yield

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim \|W_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} \lesssim (\|V_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} + \|w_{h,\epsilon,err}\|_{L^q([0,T],L^r(\Omega))}) \tag{5.16}$$

and from (5.14) and (5.15) we obtain a contradiction, since we should have

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim h^{-\beta(r,d)+2\epsilon} + h^{-\beta(r,d)+2\epsilon-\epsilon/3}$$

which is obviously not true. The proof is complete. □

## 5.3 Construction of an approximate solution in $2D$

We are reduced to prove Theorem 5.1 where we take $T = 1$. In what follows we fix $\epsilon > 0$ small enough and we do not mention anymore the dependence on $\epsilon$ of the solution of the wave equation (4.2) we shall construct. Let therefore $\Omega$ be a Riemannian manifold of dimension $d = 2$ with smooth boundary $\partial\Omega$ satisfying the assumptions of Theorem 5.1 and let $g$ denote its Riemannian metric. Let local coordinates be chosen such that $\Omega$ be given by

$$\Omega = \{(x, y)|x > 0, y \in \mathbb{R}\},$$

and the Laplace-Beltrami operator $\Delta_g$ associated to the metric $g$ be given by

$$\Delta_g = \partial_x^2 + (1 + xb(y))\partial_y^2,$$

where $b$ is a smooth function. Set $X = \Omega \times \mathbb{R}_t$, let $\Box = \partial_t^2 - \Delta_g$ denote the wave operator on $X$ and let $p \in C^\infty(T^*X \setminus o)$ be the principal symbol of $\Box$, which is homogeneous of degree 2 in $T^*X \setminus o$ (where we write $o$ for the "zero section" of $T^*X$). The characteristic set $P := \text{Char}(p) \subset T^*X \setminus o$ of $\Box$ is defined by $p^{-1}(\{0\})$.

The canonical local coordinates on $T^*X$ will be denoted $(x, y, t, \xi, \eta, \tau)$, so one forms are $\alpha = \xi dx + \eta dy + \tau dt$. Let $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on $T^*X$ and corresponding coordinates $(y, t, \eta, \tau)$ on a neighborhood $\mathcal{U}$ of a point in $T^*\partial X$. The elliptic, glancing and hyperbolic sets are defined by

$$\mathcal{E} \cap \mathcal{U} = \{(y, t, \eta, \tau)|\tau^2 < \eta^2\},$$

$$\mathcal{G} \cap \mathcal{U} = \{(y, t, \eta, \tau)|\tau^2 = \eta^2\},$$

$$\mathcal{H} \cap \mathcal{U} = \{(y, t, \eta, \tau)|\tau^2 > \eta^2\}.$$

Let $\rho = \rho(s) = (x, y, t)(s)$, $\vartheta = \vartheta(s) = (\xi, \eta, \tau)(s)$ be a bicharacteristic of $p(\rho, \vartheta)$, i.e. such that $(\rho, \vartheta)$ satisfies

$$\frac{d\rho}{ds} = \frac{\partial p}{\partial \vartheta}, \quad \frac{d\vartheta}{ds} = -\frac{\partial p}{\partial \rho}, \quad p(\rho(0), \vartheta(0)) = 0. \tag{5.17}$$

We say that $(\rho(s), \vartheta(s))|_{s=0}$ on the boundary $\partial X$ is a gliding point if it satisfies

$$x(\rho(0)) = 0, \quad \frac{d}{ds}x(\rho(0)) = 0, \quad \frac{d^2}{ds^2}x(\rho(0)) < 0. \tag{5.18}$$

This is equivalent to saying that $(\rho, \vartheta) \in T^*X \setminus o$ is a gliding point if

$$p(\rho, \vartheta) = 0, \quad \{p, x\}|_{(\rho,\vartheta)} = 0, \quad \{\{p, x\}, p\}|_{(\rho,\vartheta)} > 0. \tag{5.19}$$

The assumption on the domain $\Omega$ is equivalent to saying that there exists a point $(0, y_0, \xi_0, \eta_0)$ on $T^*\Omega$ through which there exists a bicharacteristic passing tangentially and having exactly second order contact with $\partial\Omega$. From (5.19) we see that this last condition writes

$$\tau^2 = (1 + xb(y))\eta^2|_{x=0}, \quad \{p, x\} = \frac{\partial p}{\partial \xi} = 2\xi_0 = 0, \tag{5.20}$$

$$\{\{p, x\}, p\} = \{\frac{\partial p}{\partial \xi}, p\} = 2b(y_0)\eta_0^2 > 0. \tag{5.21}$$

We can suppose that $b(y_0) = 1$ and we let $c > 0$ small so that we have $|b(y) - 1| \leq c$ for $y$ in a neighborhood of $y_0$. Denote the gliding point (in $T^*\partial X$) by

$$\pi(\rho_0, \vartheta_0) = (y_0, 0, \eta_0, \tau_0 = -\eta_0).$$

Suppose without loss of generality that $y_0 = 0$, $\eta_0 = 1$, thus $\pi(\rho_0, \vartheta_0) = (0, 0, 1, -1) \in \mathcal{G}$. We define the semi-classical wave front set $WF_h(u)$ of a distribution $u$ on $\mathbb{R}^3$ to be the complement of the set of points $(\rho = (x, y, t), \zeta = (\xi, \eta, \tau)) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus 0)$ for which there exists a symbol $a(\rho, \zeta) \in \mathcal{S}(\mathbb{R}^6)$ such that $a(\rho, \zeta) \neq 0$ and for all integers $m \geq 0$ the following holds

$$\|a(\rho, hD_\rho)u\|_{L^2} \leq c_m h^m.$$

## 5.4 Geometric reduction

We look for an approximate solution to the equation (5.1) of the form

$$u_h(x, y, t) = \int_{\xi,\eta,\tau} e^{\frac{i}{h}(\theta + \zeta\xi + \frac{\xi^3}{3})} g(x, y, t, \eta, \tau, \xi, h) d\xi d\eta d\tau, \tag{5.22}$$

where the functions $\theta(x, y, t, \eta, \tau)$, $\zeta(x, y, \eta, \tau)$ are real valued and homogeneous in $(\eta, \tau)$ of degree 1 and 2/3, respectively, and where we have, moreover,

$$\zeta_0(\eta, \tau) := \zeta(0, y, \eta, \tau) = -\frac{(\tau^2 - \eta^2)}{\eta^2}\eta^{2/3}. \tag{5.23}$$

The functions $\theta$, $\zeta$ must solve an eikonal equation that we derive in what follows. Denote by $< .,. >$ the symmetric bilinear form obtained by polarization of the second order homogeneous principal symbol $p$ of the wave operator $\Box$,

$$p(x, y, t, \xi, \eta, \tau) = \xi^2 + (1 + xb(y))\eta^2 - \tau^2, \tag{5.24}$$

$$< da, db >= \partial_x a \partial_x b + (1 + xb(y))\partial_y a \partial_y b - \partial_t a \partial_t b. \tag{5.25}$$

We compute, for $\alpha, \beta \in \{t, x, y\}$

$$e^{-\frac{i\Phi}{h}} \partial^2_{\alpha,\beta}(g e^{\frac{i}{h}(\theta + \zeta\xi + \frac{\xi^3}{3})}) = -\frac{1}{h^2}\partial_\alpha\Phi\partial_\beta\Phi g + \frac{i}{h}(\partial_\alpha\Phi\partial_\beta g + \partial_\beta\Phi\partial_\alpha g + \partial^2_{\alpha,\beta}\Phi g) + \partial^2_{\alpha\beta} g,$$

where we set

$$\Phi = \theta + \zeta\xi + \frac{\xi^3}{3}. \tag{5.26}$$

Applying the wave operator $\Box$ to $u_h$, the term multiplied by $\frac{1}{h^2}$ becomes

$$(\partial_x\theta+\xi\partial_x\zeta)^2+(1+xb(y))(\partial_y\theta+\xi\partial_y\zeta)^2-(\partial_t\theta+\xi\partial_t\zeta)^2= \tag{5.27}$$
$$=< d\theta, d\theta > -2\xi < d\theta, d\zeta > +\xi^2 < d\zeta, d\zeta > .$$

In order to eliminate this term we ask that for some non-vanishing function $H$, the functions $\theta$, $\zeta$ satisfy the following system

$$\begin{cases} < d\theta, d\theta >= H\zeta, \\ < d\theta, d\zeta >= 0, \\ < d\zeta, d\zeta >= H. \end{cases} \tag{5.28}$$

This is equivalent to determine $\theta$, $\zeta$ solutions to

$$\begin{cases} < d\theta, d\theta > -\zeta < d\zeta, d\zeta >= 0, \\ < d\theta, d\zeta >= 0. \end{cases} \tag{5.29}$$

The system (5.29) is a nonlinear system of partial differential equations, which is elliptic where $\zeta > 0$ (shadow region), hyperbolic where $\zeta < 0$ (illuminated region) and parabolic where $\zeta = 0$ (caustic curve or surface). It is crucial that there is a solution of the form

$$\phi^{\pm} = \theta \mp \frac{2}{3}(-\zeta)^{3/2} \tag{5.30}$$

with $\theta$, $\zeta$ smooth. In terms of (5.30), the eikonal equation takes the form

$$p(x, y, t, d\phi^{\pm}) = 0 \tag{5.31}$$

by taking the sum and the difference of the equations (5.31). It is easy, by Hamilton-Jacobi theory, to find many smooth solutions to the eikonal equation (5.31). Solutions with the singularity (5.30) arise from solving the initial value problem for (5.31) off an initial surface which does not have the usual transversality condition, corresponding to the fact that there are bicharacteristics tangent to the boundary.

For the model problem, with Laplace operator defined by $\partial_x^2 + (1 + x)\partial_y^2$, the equation (5.31) has the solution

$$\phi_F^{\pm} = \theta_F \mp \frac{2}{3}(-\zeta_F)^{3/2}, \tag{5.32}$$

where

$$\theta_F(x, y, t, \eta, \tau) = y\eta + t\tau, \quad \zeta_F(x, y, \eta, \tau) = (x - \frac{\tau^2 - \eta^2}{\eta^2})\eta^{2/3}, \tag{5.33}$$

as can be seen by direct computation. This solution serves very much as a guide to the general construction.

*Definition* 5.1. Two hypersurfaces $P$, $Q$ in the symplectic manifold $T^*X$ are homogeneous glancing surfaces at $(\rho, \vartheta) \in T^*X \setminus o$ provided they are conic, in terms of defining functions

1. $\{p,q\}((\rho,\vartheta))=0$,
2. $\{p,\{p,q\}\}((\rho,\vartheta))\neq 0$ and $\{q,\{q,p\}\}((\rho,\vartheta))\neq 0$

and they satisfy the transversally condition: $dp$, $dq$ and the fundamental 1-form $\alpha=\xi dx+\eta dy+\tau dt$ are linearly independent at $(\rho,\vartheta)$.

A model case of a pair of (homogeneous) glancing surfaces is given by

$$Q_F=\{q_F(x,y,\xi,\eta,\tau)=x=0\},\quad P_F=\{p_F=\xi^2+(1+x)\eta^2-\tau^2=0\}, \tag{5.34}$$

which have a second order intersection at the point

$$(\bar\rho_0,\bar\vartheta_0)=(0,y_0=0,t_0=0,0,\eta_0=1,\tau_0=-1)\in T^*X\setminus o,$$

**Theorem 5.2.** *Let $P$ and $Q$ be two hypersurfaces in $T^*X\setminus o$ satisfying the glancing conditions in Definition 5.1 at $(\rho_0,\vartheta_0)\in P\cap Q\subset T^*X\setminus o$. Then there exist real functions $\theta$ and $\zeta$ which are $C^\infty$ in a conic neighborhood $\mathcal{U}$ of $(\rho_0,1,-1)\in X\times\mathbb{R}^2$, are homogeneous of degrees one and two-thirds, respectively, and have the following properties*

— *$\zeta_0:=\zeta|_{x=0}=-(\tau^2-\eta^2)\eta^{-4/3}$ and $\partial\zeta|_{\partial X}>0$ on $\mathcal{U}\cap\partial X\times\mathbb{R}^2$,*
— *$d_{y,t}(\partial_\eta\theta,\partial_\tau\theta)$ are linearly independent on $\mathcal{U}$,*
— *the system* (5.29) *holds in $\zeta\leq 0$ and in Taylor series on $\partial X$.*

*Moreover, $\zeta$ is a defining function for the fold set. By translation invariance in time $\zeta$ it is independent of $t$ while the phase function $\theta$ is linearly in the time variable.*

*Remark* 5.2. Theorem 5.2 determines the phase functions $\theta$, $\zeta$, solutions to the eikonal equations (5.29) corresponding to the glancing surfaces $P=\mathrm{char}(p)$ and $Q=\{x=0\}$. In what follows we use the construction of the model case in order to determine the symbol $g$ in (5.22). They will be defined first on the boundary $\partial X$ using the symplectomorphism $\chi_\partial$ which is generated by the restriction $\theta_0:=\theta|_{\partial X}$,

$$\chi_\partial^{-1}:(y,t,d_y\theta_0,d_t\theta_0)\to(d_\eta\theta_0,d_\tau\theta_0,\eta,\tau),\quad \chi_\partial^{-1}(\pi(\rho_0,\vartheta_0))=\pi(\bar\rho_0,\bar\vartheta_0), \tag{5.35}$$

where $\pi$ denotes the natural inclusion $\pi:T^*X\to^b T^*X:=T^*\mathring{X}\sqcup T^*\partial X$. Because of the choice of the normalization (5.23) and the construction of the phase function $\theta$ in Theorem 5.2, the canonical transformation $\chi_\partial$ conjugates the billiard ball maps to normal form.

## 5.5 Construction of an approximate solution

### 5.5.1 An approximate solution in the Friedlander's case

We start by recalling the construction of an approximate solution for the wave equation with Dirichlet conditions in the model case of the model wave operator defined by $\Box_F=\partial_t^2-\partial_x^2-(1+x)\partial_y^2$. If we denote it by $U_{F,h}$, from Section 4 $U_{F,h}$ writes as a sum

$$U_{F,h}(x,y,t)=\sum_{n=0}^{N}u^n_{F,h}(x,y,t), \tag{5.36}$$

$$u^n_{F,h}(x,y,t) = \int_{\xi,\eta} e^{\frac{i}{h}(y\eta - t\eta(1+a)^{1/2} + \xi\eta^{2/3}(x-a) + \frac{\xi^3}{3} + \frac{4}{3}n\eta a^{3/2})} \times \varrho^n\Big(\frac{t + 2\eta^{-1/3}(1+a)^{1/2}\xi}{2(1+a)^{1/2}a^{1/2}} - 2n), \eta, \lambda\Big) d\xi d\eta, \quad (5.37)$$

where $a = h^{(1-\epsilon)/2}$, $\epsilon > 0$ and where $\lambda = \lambda(h) = a^{3/2}/h$ and the symbols $\varrho^n(.,\eta,\lambda)$ belong to the space $\mathcal{S}_{K_0}(\lambda/(n+1))$ introduced in Definition 4.3 of Section 4, where $K_0 = [-c_0, c_0]$ for some small $0 < c_0 < 1$. Precisely, $\varrho^n$ are introduced in Definition 4.4.

**Proposition 5.2.** *(see Section 4, Prop.4.6) On the boundary $u^n_{F,h}|_{x=0}$ writes (modulo $O_{L^2}(\lambda^{-\infty})$) as a sum of two trace operators $Tr_\pm(u^n_{F,h})$, where*

$$Tr_\pm(u^n_{F,h}) = h^{1/3} \int_\eta e^{\frac{i}{h}(y\eta - t\eta(1+a)^{1/2} + \frac{2}{3}(2n\mp 1)\eta a^{3/2})} \Psi(\eta)(\eta\lambda)^{-1/6} \times I_\pm(\varrho^n(.,\eta,\lambda))_\eta\Big(\frac{t}{2(1+a)^{1/2}a^{1/2}}, \lambda\Big) d\eta, \quad (5.38)$$

*where $I_\pm(\varrho^n(.,\eta,\lambda))_\eta(z,\lambda)$ are defined modulo $O_{\mathcal{S}(\mathbb{R})}((\eta\lambda)^{-\infty})$ by*

$$I_\pm(\varrho^n(.,\eta,\lambda))_\eta(z,\lambda) = e^{\pm i\pi/2 - i\pi/4}\frac{\eta\lambda}{2\pi}\int_w e^{i\eta\lambda(w(z-z')\mp\frac{2}{3}((1-w)^{3/2}-1))}\kappa(w) a_\pm(w,\eta\lambda)\varrho^n(z',\lambda) dw, \quad (5.39)$$

*where $\kappa$ is a smooth function supported for $w$ as close as we want to $0$ and where $a_\pm(w,\eta\lambda)$ are the asymptotic expansions of the symbols of the Airy functions $A^\pm$ introduced in (4.74) of Section 4. Moreover, the symbols $k(w)a_\pm(w,\eta\lambda)$ are elliptic at $w=0$.*

**Proposition 5.3.** *(see Section 4) This choice of the symbols gives for all $0 \leq n \leq N-1$*

$$Tr_-(u^n_{F,h})(y,t,h) + Tr_+(u^{n+1}_{F,h})(y,t,h) = O_{L^2}(\lambda^{-\infty}). \quad (5.40)$$

### 5.5.2 Construction of an approximate solution to the equation (5.7)

Recall the form of the approximate solution $u_h$ we considered in (5.22). Away from the caustic set defined by the locus where $\xi = \zeta = 0$, there are two main contributions in $u_h$ denoted $Tr_\pm(u_h)$ with phase functions $\phi^\pm = \theta \mp \frac{2}{3}(-\zeta)^{3/2}$ given in (5.30). These are the phases corresponding to the Airy functions $A_\pm(\zeta)$ and one can think (at least away from the boundary $x = 0$) of the part $Tr_-(u_{h,})$ corresponding to $A_-(\zeta)$ as a free wave or the "incoming piece": after hitting the boundary it gives rise to the outgoing one which corresponds to $A_+(\zeta)\frac{A_-(\zeta_0)}{A_+(\zeta_0)}$ with phase $-\frac{2}{3}(-\zeta)^{3/2} + \frac{4}{3}(-\zeta_0)^{3/2}$. The oscillatory part $\frac{4}{3}(-\zeta_0)^{3/2}$ corresponds to the billiard ball map shift corresponding to reflection. The phase $\zeta_0$ is called an interpolating Hamiltonian for the billiard ball maps $\delta^\pm$ and we have $\delta^\pm(y,t,\eta,\tau) = \exp(\pm\frac{4}{3}H_{(-\zeta_0)^{3/2}})$.

Inspired from the model case we construct an approximate solution $U_n$ to (5.7) as a sum over $n \in \{0,..,N\}$ of $u_h^n(x,y,t)$, each of the form (5.22),

$$u_h^n(x,y,t) = \int_{\xi,\eta,\tau} e^{\frac{i}{h}\Phi^n(x,y,t,\xi,\eta,\tau)} g^n(x,y,t,\eta,\tau,\xi,h) d\xi d\eta d\tau, \tag{5.41}$$

for some symbols $g^n(.,h)$ suitably chosen and where

$$\Phi^n(x,y,t,\xi,\eta,\tau) := \theta(x,y,t,\eta,\tau) + \xi\zeta(x,y,\eta,\tau) + \frac{\xi^3}{3} + \frac{4}{3}n(-\zeta_0)^{3/2}(\eta,\tau).$$

In order to obtain solution to (5.7) satisfying the Dirichlet boundary condition we construct the symbols $g^n$ of $u_h^n$ such that on the boundary

$$Tr_-(u_h^n) + Tr_+(u_h^{n+1}) = O_{L^2}(h^\infty). \tag{5.42}$$

In order to do this we introduce an operator

$$J(f)(y,t) := \frac{1}{(2\pi h)^2} \int_{\eta,\tau} e^{\frac{i}{h}\theta_0(y,t,\eta,\tau)} a_h(y,\eta,\tau) \widehat{f}(\eta/h,\tau/h) d\eta d\tau, \tag{5.43}$$

where $a_h(y,\eta,\tau) = a(y,\eta/h,\tau/h)$ for some elliptic symbol $a$ of order 0 and type $(1,0)$, compactly supported in a conic neighborhood of the glancing point $\pi(\rho_0,\vartheta_0)$. Defined in this way $J$ is an elliptic FIO in a neighborhood of $(\pi(\bar\rho_0,\bar\vartheta_0),\pi(\rho_0,\vartheta_0))$, with canonical relation $\chi_\partial$ given by the symplectomorphism generated by $\theta_0$ and $\chi_\partial(\pi(\bar\rho_0,\bar\vartheta_0)) = \pi(\rho_0,\vartheta_0)$.

For $n \in \{0,..N\}$ we want to define the symbols $g^n$ such that

$$u_h^n(0,y,t) := \sum_{\pm} Tr_\pm(u_h^n)(y,t), \quad Tr_\pm(u_h^n)(y,t) := J(Tr_\pm(u_{F,h}^n))(y,t).$$

In this way, using Proposition 5.3 and the fact that $J$ is elliptic the Dirichlet boundary condition (5.42) will be satisfied on $\partial\Omega \times [0,1]$. An explicit computation yields

**Lemma 5.3.** *(Section 4, [62]) On the boundary $J \circ Tr_\pm(u_{F,h}^n)$ write*

$$J(Tr_\pm(u_{F,h}^n))(y,t) = h^{1/3} \int e^{\frac{i}{h}(\theta_0(y,t,\eta,-\eta(1+a)^{1/2}) + \frac{2}{3}(2n\mp 1)(-\zeta_0)^{3/2}(\eta,-\eta(1+a)^{1/2}))}$$
$$\times (\eta\lambda)^{-1/6}\Psi(\eta) I_\pm(\sigma^n(.,y,\eta,h))_\eta \Big( \frac{\partial_\tau\theta_0}{\partial_\tau\zeta_0(-\zeta_0)^{1/2}}(y,t,\eta,-\eta(1+a)^{1/2}) - 2n, \lambda \Big) d\eta, \tag{5.44}$$

*where*

$$\sigma_n(z,y,\eta,h) \simeq \Big( \sum_{k\geq 0} h^{k/2} a^{-k/2} \mu_k(y,\eta,h) \partial^k \varrho^n(.,\eta,\lambda) \Big)(z,\eta,\lambda), \tag{5.45}$$

*where $\mu_k(y,\eta,h)$ are symbols of order 0 and type $(1,0)$ with $\mu_0(y,\eta,h) = a_h(y,\eta,-\eta(1+a)^{1/2})$ independent of $n$ and where we recall that $\lambda = a^{3/2}/h$, $a = h^{(1-\epsilon)/2}$. Moreover, if $\eta \in supp(\Psi)$ and $0 \leq n \leq N \simeq \lambda h^\epsilon$ then $\sigma^n(.,y,\eta,h) \in \mathcal{S}_{K_0}(\lambda/(n+1))$.*

*Definition* 5.2. For $n \in \{0, .., N\}$ we define

$$g^n(x, y, t, \eta, \tau, h) := \frac{1}{h}\Psi(\eta)\sigma_n(\frac{\partial_\tau\theta + \xi\partial_\tau\zeta}{2a^{1/2}(1+a)^{1/2}} - 2n, y, \eta, h)\delta(\tau/\eta = -(1+a)^{1/2}), \quad (5.46)$$

where $\delta$ denotes the Dirac distribution. Notice that for $g^n$ defined in this way the two contributions of $u_h^n$ on the boundary coincide with (5.44).

**Proposition 5.4.**

$$\|\Box u_h^n(., t)\|_{L^2(\Omega)} = O(h^{-1})\|u_h^n(., t)\|_{L^2(\Omega)}. \quad (5.47)$$

Using that the wave front set $WF_h(u_h^n) \subset \Lambda_{\Phi^n}$, where

$$\Lambda_{\Phi^n} := \{(x, y, t, \xi, \eta, -\eta(1+a)^{1/2})|\zeta + \xi^2 = 0, \partial_\eta\Phi^n(x, y, t, \xi, \eta, -\eta(1+a)^{1/2}) = 0\},$$

we can show that if $c_0$ is small enough then $u_h^n(., t)$ have almost disjoint supports in the time variable. The rest of the proof then follows like in the model case, the only difference being the fact that we haven't solved the first transport equation and we cannot proceed like in Section 4. Using Lemma 5.4 we achieve the proof of Theorem (5.1) using the same contradiction argument as in Lemma 5.2. The $L^r(\Omega)$ norms of the cusps $u_h^n$ are computed in a similar manner as in Section 4.6.4 in the model case.

# Part III

# Équation de Schrödinger

## Summary

# 6 On the Schrödinger equation outside strictly convex obstacles

We prove sharp Strichartz estimates for the semi-classical Schrödinger equation on a compact Riemannian manifold with smooth, strictly geodesically concave boundary. We deduce classical Strichartz estimates for the Schrödinger equation outside a strictly convex obstacle, local existence for the $H^1$-critical (quintic) Schrödinger equation and scattering for the sub-critical Schrödinger equation in $3D$ domains.

## 6.1 Introduction

Let $(M,g)$ be a Riemannian manifold of dimension $n\geq 2$. Strichartz estimates are a family of dispersive estimates on solutions $u(x,t): M\times[-T,T]\to\mathbb{C}$ to the Schrödinger equation

$$i\partial_t u+\Delta_g u=0,\quad u(x,0)=u_0(x), \tag{6.1}$$

where $\Delta_g$ denotes the Laplace-Beltrami operator on $(M,g)$. In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([-T,T],L^r(M))}\leq C\|u_0\|_{H^s(M)}, \tag{6.2}$$

where $H^s(M)$ denotes the Sobolev space over $M$ and $2\leq q,r\leq\infty$ satisfy $(q,r,n)\neq(2,\infty,2)$ (for the case $q=2$ see [70]) and are given by the scaling admissibility condition

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2}. \tag{6.3}$$

In $\mathbb{R}^n$ and for $g_{ij}=\delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with Strichartz pioneering work [100], where he proved the particular case $q=r$ for the wave and (classical) Schrödinger equations. This was later generalized to mixed $L^q_tL^r_x$ norms by Ginibre and Velo [49] for Schrödinger equations, where $(q,r)$ is sharp admissible and $q>2$; the wave estimates were obtained independently by Ginibre-Velo [51] and Lindblad-Sogge [75], following earlier work by Kapitanski [67]. The remaining endpoints for both equations were finally settled by Keel and Tao [70]. In that case $s=0$ and $T=\infty$; (see also Kato [68], Cazenave-Weissler [33]). Estimates for the flat 2-torus were shown by Bourgain [19] to hold for $q=r=4$ and any $s>0$.

In the variable coefficients case, even without boundaries, the situation is much more complicated: we simply recall the pioneering work of Staffilani and Tataru [98], dealing with compact, non trapping pertubations of the flat metric, and recent work of Bouclet and Tzvetkov [17] which considerably weakens the decay of the pertubation (retaining the non-trapping character at spatial infinity). On compact manifolds without boundaries, Burq, Gerard and Tzvetkov [27] established Strichartz estimates with $s=1/p$, hence with a loss of derivatives when compared to the case of flat geometries. Recently, Blair, Smith and Sogge [15] improved on the current results for compact $(M, g)$ where either $\partial M \neq \emptyset$, or $\partial M = \emptyset$ and $g$ Lipschitz, by showing that Strichartz estimates hold with a loss of $s = 4/3p$ derivatives. This appears to be the natural analog of the estimates of [27] for the general boundaryless case.

In this paper we prove that Strichartz estimates for the semi-classical Schrödinger equation also hold on compact Riemannian manifolds with smooth, strictly geodesically concave boundaries. By the last condition we understand that the second fundamental form on the boundary of the manifold is strictly positive definite.

*Assumption* 6.1. Let $(M, g)$ be a smooth $n$-dimensional compact Riemannian manifold with $C^\infty$ boundary. We shall assume that $n \geq 2$ and that $\partial M$ is strictly geodesically concave throughout. Let $\Delta_g$ be the Laplace-Beltrami operator associated to $g$ on $M$, acting on $L^2(M)$, with domain $H^2(M) \cap H^1_0(M)$. We assume that we can globally write

$$\Delta_g = \sum_{j,k=2}^{n} g^{j,k}(x)\partial_j\partial_k + \sum_{j=1}^{n} a_j(x)\partial_j, \tag{6.4}$$

where the coefficients belong to a bounded set of $C^\infty$ and the principal part is uniformly elliptic.

Let $0 < \alpha_0 \leq 1/2$, $2 \leq \beta_0$, $\Psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be compactly supported in the interval $(\alpha_0, \beta_0)$. We introduce the operator $\Psi(-h^2\Delta_g)$ using the Dynkin-Helffer-Sjöstrand formula [43] and refer to [83], [43] or [63] for a complete overview of its properties (see also [27] for compact manifolds without boundaries).

*Definition* 6.1. Given $\Psi \in C_0^\infty(\mathbb{R})$ we have

$$\Psi(-h^2\Delta_g) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\Psi}(z)(z + h^2\Delta_g)^{-1}dL(z),$$

where $dL(z)$ denotes the Lebesque measure on $\mathbb{C}$ and $\tilde{\Psi}$ is an almost analytic extension of $\Psi$, e.g., with $< z >= (1 + |z|^2)^{1/2}$, $N \geq 0$,

$$\tilde{\Psi}(z) = \Big(\sum_{m=0}^{N} \partial^m\Psi(\mathrm{Re}z)(i\mathrm{Im}z)^m/m!\Big)\tau(\mathrm{Im}z/ < \mathrm{Re}z >),$$

where $\tau$ is a non-negative $C^\infty$ function such that $\tau(s) = 1$ if $|s| \leq 1$ and $\tau(s) = 0$ if $|s| \geq 2$.

Our main result reads as follows:

**Theorem 6.1.** *Under the Assumptions 6.1, given $(q, r)$ satisfying the scaling condition (6.3), $q > 2$ and $T > 0$ sufficiently small, there exists a constant $C = C(T) > 0$ such that the solution $v(x, t)$ of the semi-classical Schrödinger equation on $M \times \mathbb{R}$ with Dirichlet boundary condition*

$$\begin{cases} ih\partial_t v + h^2\Delta_g v = 0, \text{ on } M \times \mathbb{R}, \\ v(x, 0) = \Psi(-h^2\Delta_g)v_0(x), \\ v|_{\partial M} = 0 \end{cases} \tag{6.5}$$

*satisfies*

$$\|v\|_{L^q((-T,T),L^r(M))} \leq Ch^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})}\|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}. \tag{6.6}$$

*Remark* 6.1. An example of compact manifold with smooth, strictly concave boundary is given by the Sinaï billiard (defined as the complementary of a strictly convex obstacle on a cube of $\mathbb{R}^n$ with periodic boundary conditions).

We deduce from Theorem 6.1 and [63, Thm.1.1] (see also Lemma 6.9), as in [27], the following Strichartz estimates with derivative loss:

**Corollary 6.1.** *Under the Assumptions 6.1, given $(q,r)$ satisfying the scaling condition (6.3), $q > 2$ and $I$ any finite time interval, there exists a constant $C = C(I) > 0$ such that the solution $u(x,t)$ of the (classical) Schrödinger equation on $M \times \mathbb{R}$ with Dirichlet boundary condition*

$$\begin{cases} i\partial_t u + \Delta_g u = 0, \text{ on } M \times \mathbb{R}, \\ u(x,0) = u_0(x), \quad u|_{\partial M} = 0 \end{cases} \tag{6.7}$$

*satisfies*

$$\|u\|_{L^q((I,L^r(M))} \leq C(I)\|u_0\|_{H^{\frac{1}{q}}(M)}. \tag{6.8}$$

The proof of Theorem 6.1 is based on the finite speed of propagation of the semi-classical flow (see Lebeau [73]) and the energy conservation which allow us to use the arguments of Smith and Sogge [95] for the wave equation: using the Melrose and Taylor parametrix for the stationary wave (see [79], [80] or Zworski [111]) we obtain, by Fourier transform in time a parametrix for the Schrödinger operator near a "glancing" point. Since in the elliptic and hyperbolic regions the solution of (6.9) will clearly satisfy the same Strichartz estimates as on a manifold without boundary, we need to restrict our attention only on the glancing region.

As an application of Theorem 6.1 we prove classical, global Strichartz estimates for the Schrödinger equation outside a strictly convex domain in $\mathbb{R}^n$.

*Assumption* 6.2. Let $\Omega = \mathbb{R}^n \setminus \Theta$, where $\Theta$ is a compact with smooth boundary. We assume that $n \geq 2$ and that $\partial\Omega$ is strictly geodesically concave throughout. Let $\Delta_D = \sum_{j=1}^n \partial_j^2$ denote the Dirichlet Laplace operator (with constant coefficients) on $\Omega$.

**Theorem 6.2.** *Under the Assumptions 6.2, given $(q,r)$ satisfying the scaling condition (6.3), $q > 2$ and $u_0 \in L^2(\Omega)$, there exists a constant $C > 0$ such that the solution $u(x,t)$ of the Schrödinger equation on $\Omega \times \mathbb{R}$ with Dirichlet boundary condition*

$$\begin{cases} i\partial_t u + \Delta_D u = 0, \text{ on } \Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{6.9}$$

*satisfies*

$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{6.10}$$

The proof of Theorem 6.2 combines several arguments: firstly, we perform a time rescaling, first used by Lebeau [73] in the context of control theory, which transforms the equation into a semi-classical problem for which we can use the local in time semi-classical Strichartz estimates proved in Theorem 6.1. Secondly, we adapt a result of Burq [23] which provides Strichartz estimates without loss for a non-trapping problem, with a metric that equals the identity outside a compact set. The proof relies on a local smoothing effect for the free evolution $\exp(it\Delta_D)$, first observed in the case of the flat space in the works of Constantin and Saut [38], Doi [45], Burq, Gérard and Tzvetkov [26] in the non-trapping case. Following a strategy suggested by Staffilani and Tataru [98], we prove that away from the obstacle the free evolution enjoys the Strichartz estimates exactly as for the free space.

We give two applications of Theorem 6.2 : the first one is a local existence result for the quintic Schrödinger equation in $3D$, while the second one is a scattering result for the subcritical (sub-quintic) Schrödinger equation in $3D$ domains.

**Theorem 6.3.** *(Local existence for the quintic Schrödinger equation) Let $\Omega$ be a three dimensional Riemannian manifold satisfying the Assumptions 6.2. Let $T > 0$ and $u_0 \in H_0^1(\Omega)$. Then there exists a unique solution $u \in C([0,T], H_0^1(\Omega)) \cap L^5((0,T], W^{1,30/11}(\Omega))$ of the quintic nonlinear equation*

$$i\partial_t u + \Delta_D u = \pm |u|^4 u \text{ on } \Omega \times \mathbb{R}, \quad u|_{t=0} = u_0 \text{ on } \Omega, \quad u|_{\partial\Omega} = 0. \tag{6.11}$$

*Moreover, for any $T > 0$, the flow $u_0 \to u$ is Lipschitz continuous from any bounded set of $H_0^1(\Omega)$ to $C([-T,T), H_0^1(\Omega))$. If the initial data $u_0$ has sufficiently small $H^1$ norm, then the solution is global in time.*

**Theorem 6.4.** *(Scattering for subcritical Schrödinger equation) Let $\Omega$ be a three dimensional Riemannian manifold satisfying the Assumptions 6.2. Let $3 \leq p < 5$ and $u_0 \in H_0^1(\Omega)$. Then the global in time solution of the defocusing Schrödinger equation*

$$i\partial_t u + \Delta_D u = |u|^{p-1} u, \quad u|_{t=0} = u_0 \quad \text{on} \quad \Omega, \quad u|_{\partial\Omega} = 0 \tag{6.12}$$

*scatters in $H_0^1(\Omega)$. If $p = 5$ and the gradient $\nabla u_0$ of the initial data has sufficiently small $L^2$ norm, then the global solution of the critical Schrödinger equation scatters in $H_0^1(\Omega)$.*

Results for the Cauchy problem associated to the critical wave equation outside a strictly convex obstacle were obtained by Smith and Sogge [95]. Their result was a consequence of the fact that the Strichartz estimates for the Euclidian wave equation also hold on Riemannian manifolds with smooth, compact and strictly concave boundaries.

In [28], Burq, Lebeau and Planchon proved that the defocusing quintic wave equation with Dirichlet boundary conditions is globally wellposed on $H^1(M) \times L^2(M)$ for any smooth, compact domain $M \subset \mathbb{R}^3$. Their proof relies on $L^p$ estimates for the spectral projector obtained by Smith and Sogge [96]. A similar result for the defocusing critical wave equation with Neumann boundary conditions was obtained in [30].

Recall that in $\mathbb{R}^3 \times \mathbb{R}_t$, Colliander, Keel, Staffilani, Takaoka and Tao [36] established global well-posedness and scattering for energy-class solutions to the quintic defocusing Schrödinger equation (9.2), which is energy-critical. In the case of the Schrödinger equation outside a obstacle in $\mathbb{R}^3$, non-trapping but not convex, the counterexamples constructed in [60, 62] for the wave equation suggest that losses are likely to occur in the Strichartz estimates for the Schrödinger equation too. In this context Burq, Gerard and Tzvetkov [26] proved global existence for subcubic defocusing nonlinearities and Anton [6] for the cubic case. Recently Planchon and Vega [87] improved the local well-posedness theory to $H^1$-subcritical (subquintic) nonlinearities for $n = 3$. Theorem 6.4 is proved in [87] in the case of the exterior of a star-shaped domain in the particular case $p = 3$, using

$$\|e^{it\Delta_D} u_0\|_{L^4_{t,x}} \lesssim \|u_0\|_{\dot{H}^{1/4}},$$

but since this estimate fails to provide control of $L^4_t L^\infty_x$ one has to use local smoothing estimates close to the boundary, and Strichartz estimates for the usual Laplacian on $\mathbb{R}^3$ away from it, the sub-criticality with respect to $H^1$ compensating the weakness of the local smoothing estimate. Here we give a simpler proof in the exterior of a strictly convex domain and for all $3 < p < 5$ using the Strichartz estimates (6.10).

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## 6.2 Estimates for semi-classical Schrödinger equation in a compact domain with strictly concave boundary

In this section we prove Theorem 6.1. In what follows Assumptions 6.1 are supposed to hold. We may assume that the metric $g$ is extended smoothly across the boundary, so that $M$ is a geodesically concave subset of a complete, compact Riemannian manifold $\tilde{M}$. By the free semi-classical Schrödinger equation we mean the semi-classical Schrödinger equation on $\tilde{M}$, where the data $v_0$ has been extended to $\tilde{M}$ by an extension operator preserving the Sobolev spaces. By a broken geodesic in $M$ we mean a geodesic that is allowed to reflect off $\partial M$ according to the reflection law for the metric $g$.

### 6.2.1 Restriction in a small neighborhood of the boundary. Elliptic and hyperbolic regions

We consider $\delta > 0$ a small positive number and for $T > 0$ small enough we set

$$S(\delta, T) := \{(x,t) \in M \times [-T,T] | \mathrm{dist}(x, \partial M) < \delta\}.$$

On the complement of $S(\delta, T)$ in $M \times [-T,T]$, the solution $v(x,t)$ equals the solution of the semi-classical Schrödinger equation on $\mathbb{R}^n$ for which Strichartz estimates are known by the work of Staffillani and Tataru [98], thus it suffices to establish Strichartz estimates for the norm of $v$ over $S(\delta, T)$.

We show that in order to prove Theorem 6.1 it will be sufficient to consider only data $v_0$ supported outside a small neighborhood of the boundary. Recall that in [73] Lebeau proved that if $\Psi$ is supported in an interval $[\alpha_0, \beta_0]$ and if $\varphi \in C_0^\infty(\mathbb{R})$ is equal to 1 near the interval $[-\beta_0, -\alpha_0]$, then for $t$ in a bounded set (and for $D_t = \frac{1}{i}\partial_t$) one has

$$\forall N \geq 1, \quad \exists C_N > 0 \quad |(1-\varphi)(hD_t) \exp{(ith\Delta_g)} \Psi(-h^2\Delta_g) v_0| \leq C_N h^N. \tag{6.13}$$

For $\delta$ and $T$ sufficiently small, let $\chi(x,t) \in C_0^\infty$ be compactly supported and be equal to 1 on $S(\delta, T)$. Let $t_0 > 0$ be such that $T = t_0/4$ and let $A \in C^\infty(\mathbb{R}^n)$, $A = 0$ near $\partial M$, $A = 1$ outside a neighborhood of the boundary be such that every broken bicharacteristic $\gamma$ starting at $t = 0$ from the support of $\chi(x,t)$ and for $-\tau \in [\alpha_0, \beta_0]$ satisfies

$$\mathrm{dist}(\gamma(t), \mathrm{supp}(1-A)) > 0, \quad \forall t \in [-2t_0, -t_0]. \tag{6.14}$$

Let $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 0$ for $t \leq -2t_0$, $\psi(t) = 1$ for $t > -t_0$ and set

$$w(x,t) = \psi(t) \exp{(ith\Delta_g)} \Psi(-h^2\Delta_g) v_0.$$

Then $w$ satisfies

$$\begin{cases} ih\partial_t w + h^2\Delta_g w = ih\psi'(t) e^{ith\Delta_g} \Psi(-h^2\Delta_g) v_0, \\ w|_{\partial\Omega\times\mathbb{R}} = 0, \quad w|_{t\leq -2t_0} = 0, \end{cases}$$

and writing Duhamel formula we have

$$w(x,t) = \int_{-2t_0}^{t} e^{i(t-s)h\Delta_g} \psi'(s) e^{ish\Delta_g} \Psi(-h^2\Delta_g) v_0 ds.$$

Notice that $w(x,t) = v(x,t)$ if $t \geq -t_0$, hence for $t \in [-t_0, T]$ we can write

$$v(x,t) = \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) e^{ish\Delta_g} \Psi(-h^2\Delta_g) v_0 ds. \tag{6.15}$$

In particular, for $t \in [-T, T]$, $T = t_0/4$, $v(x,t) = w(x,t)$ is given by (6.15). We want to estimate the $L_t^q L_x^r$ norms of $v(x,t)$ for $(x,t)$ on $S(\delta, T)$ where $v = \chi v$. Let

$$v_Q(x,t) = \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) Q(x) e^{ish\Delta_g} \Psi(-h^2\Delta_g) v_0 ds, \quad Q \in \{A, 1-A\},$$

then $v = v_A + v_{1-A}$, where $v_{1-A}$ solves

$$\begin{cases} ih\partial_t v_{1-A} + h^2\Delta_g v_{1-A} = ih\psi'(t)(1-A)e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0, \\ v_{1-A}|_{\partial M\times\mathbb{R}} = 0, \quad v_{1-A}|_{t<-2t_0} = 0. \end{cases}$$

We apply Proposition 6.10 from the Appendix with $Q = 1-A$, $\tilde{\psi} = \psi'$ to deduce that if $\rho_0 \in WF_b(v_{1-A})$ then the broken bicharacteristic starting from $\rho_0$ must intersect $WF_b((1-A)v) \cap \{t \in [-2t_0, -t_0]\}$. Since we are interested in estimating the norm of $v$ on $S(\delta, T)$ it is enough to consider only $\rho_0 \in WF_b(\chi v_{1-A})$. Thus, if $\gamma$ is a broken bicharacteristic starting at $t = 0$ from $\rho_0$, $-\tau \in [\alpha_0, \beta_0]$, then Proposition 6.10 implies that for some $t \in [-2t_0, -t_0]$, $\gamma(t)$ must intersect $WF_b((1-A)v)$. On the other hand from (6.14) this implies (see Definition 6.3) that for every $\sigma \geq 0$

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|\chi v_{1-A}\|_{H^\sigma(M\times\mathbb{R})} \leq C_N h^N. \tag{6.16}$$

We are thus reduced to estimating $v(x,t)$ for initial data supported outside a small neighborhood of the boundary. Indeed, suppose that the estimates (6.6) hold true for any initial data compactly supported where $A \neq 0$. It follows from (6.15), (6.16) that

$$\begin{aligned} \|\chi v_A\|_{L^q((-T,T),L^r(M))} &\leq \|\psi'(s)A(x)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\|_{L^1(s\in(-2t_0,-t_0),L^2(M))} \\ &\lesssim \Big(\int_{-2t_0}^{-t_0} |\Psi'(s)|ds\Big)\|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)} \\ &= \|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}, \end{aligned} \tag{6.17}$$

where we used the fact that the semi-classical Schrödinger flow $\exp(ihs\Delta_g)\Psi(-h^2\Delta_g)$, which maps data at time 0 to data at time $s$, is an isomorphisme on $H^\sigma(M)$ for every $\sigma \geq 0$.

*Remark* 6.2. Notice that when dealing with the wave equation, since the speed of propagation is equal to 1, one can take $\psi(t) = 1_{\{t\geq -t_0\}}$ for some small $t_0 \geq 0$ and reduce the problem to proving Strichartz estimates for the flow $\exp(ih(t_0+.)\Delta_g)\Psi(-h^2\Delta_g)$ and initial data compactly supported outside a small neighborhood of $\partial M$.

Let $\Delta_0$ denote the Laplacian on $\tilde{M}$ coming from the extension of the metric $g$ smoothly across the boundary $\partial M$. We let $\mathcal{M}$ denote the outgoing solution to the Dirichlet problem for the semiclassical Schrödinger operator on $M \times \mathbb{R}$. Thus, if $g$ is a function on $\partial M \times \mathbb{R}$ which vanishes for $t \leq -2t_0$, then $\mathcal{M}g$ is the solution on $M \times \mathbb{R}$ to

$$\begin{cases} ih\partial_t \mathcal{M}g + h^2\Delta_g \mathcal{M}g = 0, \\ \mathcal{M}g|_{\partial M\times\mathbb{R}} = g. \end{cases} \tag{6.18}$$

Then, for $t \in [-t_0, T]$ and data $f$ supported outside a small neighborhood of the boundary and localized at frequency $1/h$, i.e. such that $f = \Psi(-h^2\Delta_g)f$, we have

$$\chi v_A(x,t) = \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)A(x)e^{ish\Delta_g} f ds$$
$$= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0} f ds$$
$$- \mathcal{M}\Big(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0} f ds|_{\partial M\times\mathbb{R}}\Big). \quad (6.19)$$

The cotangent bundle of $\partial M \times \mathbb{R}$ is divided into three disjoint sets: the hyperbolic and elliptic regions where the Dirichlet problem is respectively hyperbolic and elliptic, and the glancing region which is the boundary between the two.

Let local coordinates be chosen such that $M = \{(x', x_n)|x_n > 0\}$ and $\Delta_g = \partial^2_{x_n} - r(x, D_{x'})$. A point $(x', t, \eta', \tau) \in T^*(\partial M \times \mathbb{R})$ is classified as one of three distinct types. It is said to be *hyperbolic* if $-\tau + r(x', 0, \eta') > 0$, so that there are two distinct nonzero real solutions $\eta_n$ to $\tau - r(x', 0, \eta') = \eta_n^2$. These two solutions yield two distinct bicharacteristics, one of which enters $M$ as $t$ increases (the *incoming ray*) and one which exits $M$ as $t$ increases (the *outgoing ray*). The point is *elliptic* if $-\tau + r(x', 0, \eta') < 0$, so there are no real solutions $\eta_n$ to $\tau - r(x', 0, \eta') = \eta_n^2$. In the remaining case $-\tau + r(x', 0, \eta') = 0$, there is a unique solution which yields a glancing ray, and the point is said to be a *glancing point*. We decompose the identity operator into

$$\mathrm{Id}(x,t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}((x'-y')\eta' + (t-s)\tau)}(\chi_h + \chi_e + \chi_{gl})(y', \eta', \tau) d\eta' d\tau,$$

where at $(y', \eta', \tau)$ we have

$$\chi_h := 1_{\{-\tau + r(y',0,\eta') \geq c\}}, \quad \chi_e := 1_{\{-\tau + r(y',0,\eta') \leq -c\}}, \quad \chi_h := 1_{\{-\tau + r(y',0,\eta') \in [-c,c]\}},$$

for some $c > 0$ sufficiently small. The corresponding operators with symbols $\chi_h$, $\chi_e$, denoted $\Pi_h$, $\Pi_e$, respectively, are pseudo-differential cutoffs essentially supported inside the hyperbolic and elliptic regions, while the operator with symbol $\chi_{gl}$, denoted $\Pi_{gl}$, is essentially supported in a small set around the glancing region. Thus, on $S(\delta, T)$ we can write $\chi v_A$ as the sum of four terms

$$\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)A(x)e^{ish\Delta_g} f ds = \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0} f ds$$
$$- \sum_{\Pi \in \{\Pi_e, \Pi_h, \Pi_{gl}\}} \mathcal{M}\Pi\Big(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0} f ds|_{\partial M\times\mathbb{R}}\Big). \quad (6.20)$$

*Remark* 6.3. For the first term in the right hand side, $\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0} f ds$, the desired estimates follow as in the boundaryless case by the results of Staffilani and Tataru [98] (since we considered the extension of the metric $g$ across the boundary to be smooth).

**Elliptic region:** Using the compactness argument of the proof of Propositions 6.8, 6.9 from the Appendix, together with the inclusion

$$WF_b\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f ds|_{\partial M\times\mathbb{R}}\Big)\subset\mathcal{H}\cup\mathcal{G},$$

where $\mathcal{H}$ and $\mathcal{G}$ denote the hyperbolic and the glancing regions, respectively, it follows that the elliptic part satisfies for all $\sigma\geq 0$

$$\mathcal{M}\Pi_e\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f ds|_{\partial M\times\mathbb{R}}\Big)=O(h^\infty)\|f\|_{H^\sigma(M)}.$$

For the definition and properties of the $b$-wave front set see Appendix.

**Hyperbolic region:** If local coordinates are chosen such that $M=\{(x',x_n)|x_n>0\}$, on the essential support of $\Pi_h$ the forward Dirichlet problem can be solved locally, modulo smoothing kernels, on an open set in $\tilde{M}\times\mathbb{R}$ around $\partial M$. Precisely, near a hyperbolic point, the solution $v$ to (6.5) can be decomposed modulo smoothing operators into an incoming part $v_-$ and an outgoing part $v_+$ where

$$v_\pm(x,t)=\frac{1}{(2\pi h)^d}\int e^{\frac{i}{h}\varphi_\pm(x,t,\xi)}\sigma_\pm(x,t,\xi,h)d\xi,$$

where the phases $\varphi_\pm$ satisfy the eikonal equations

$$\begin{cases}\partial_s\varphi_\pm+<d\varphi_\pm,d\varphi_\pm>_g=0,\\ \varphi_+|_{\partial M}=\varphi_-|_{\partial M},\quad \partial_{x_n}\varphi_+|_{\partial M}=-\partial_{x_n}\varphi_-|_{\partial M},\end{cases}$$

where $<.,.>_g$ denotes the inner product induced by the metric $g$. The symbols are asymptotic expansions in $h$ and write $\sigma_\pm(.,h)=\sum_{k\geq 0}h^k\sigma_{\pm,k}$, where $\sigma_0$ solves the linear transport equation

$$\partial_s\sigma_{\pm,0}+(\Delta_g\varphi_\pm)\sigma_{\pm,0}+<d\varphi_\pm,d\sigma_{\pm,0}>_g=0,$$

while for $k\geq 1$, $\sigma_{\pm,k}$ satisfies the non-homogeneous transport equations

$$\partial_s\sigma_{\pm,k}+(\Delta_g\varphi_\pm)\sigma_{\pm,k}+<d\varphi_\pm,d\sigma_{\pm,k}>_g=i\Delta_g\sigma_{\pm,k-1}.$$

A direct computation shows that

$$\|\sum_\pm v_\pm\|^2_{H^\sigma(M\times\mathbb{R})}\simeq\sum_\pm\|v_\pm\|^2_{H^\sigma(M\times\mathbb{R})}\simeq\|v\|^2_{H^\sigma(M\times\mathbb{R})}\simeq\|v\|^2_{L^\infty(\mathbb{R})H^\sigma(M)}.$$

Each component $v_\pm$ is a solution of linear Schrödinger equation (without boundary) and consequently satisfies the usual Strichartz estimates (see Burq, Gérard and Tzvetkov [27]).

Note that $\sum_\pm v_\pm$ contains the contribution from

$$\mathcal{M}\Pi_h\Big(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0 ds|_{\partial M\times\mathbb{R}}\Big)$$

and a contribution from $\chi\int_{-2t_0}^{-t_0}e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0 ds$.

### 6.2.2 Glancing region

Near a diffractive point we use the Melrose and Taylor construction for the wave equation in order to write, following Zworski [111], the solution to the wave equation as a finite sum of pseudo-differential cutoffs, each essentially supported in a suitably small neighborhood of a glancing ray. Using the Fourier transform in time we obtain a parametrix for the semi-classical Schrödinger equation (6.5) microllocally near a glancing direction and modulo smoothing operators.

**Preliminaries. Parametrix for the wave equation near the glancing region:** We start by recalling the results by Melrose and Taylor [79], [80], Zworski [111, Prop.4.1] for the wave equation near the glancing region. Let $w$ solve the (semi-classical) wave equation on $M$ with Dirichlet boundary conditions

$$\begin{cases} h^2D_t^2 w + h^2\Delta_g w = 0, \quad M\times\mathbb{R}, \quad w|_{\partial M\times\mathbb{R}} = 0, \\ w(x,0) = f(x), \quad D_t w(x,0) = g(x), \end{cases} \tag{6.21}$$

where $f$, $g$ are compactly supported in $M$ and localized at spacial frequency $1/h$, and where $D_t = \frac{1}{i}\partial_t$.

**Proposition 6.1.** *Near a glancing direction the solution to* (6.21) *writes, modulo smoothing operators*

$$\begin{aligned} w(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} & e^{\frac{i}{h}(\theta(x,\xi)+it\xi_1)}\widehat{K(f,g)}(\frac{\xi}{h}) \\ & \times \Big[a(x,\xi/h)\Big(A_-(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h)\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\Big) \\ & + b(x,\xi/h)\Big(A'_-(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\Big)\Big]d\xi, \end{aligned} \tag{6.22}$$

*where the symbols $a$, $b$ and the phases $\theta$, $\zeta$ have the following properties: $a$ and $b$ are symbols of type* $(1,0)$ *and order* $1/6$ *and* $-1/6$, *respectively, both of which are supported in a small conic neighborhood of the $\xi_1$ axis and where $K$ is a classical Fourier integral operator of order* $0$ *in $f$ and order* $-1$ *in $g$, compactly supported on both sides. The phases $\theta$ and $\zeta$ are real, smooth and homogeneous of degree* $1$ *and* $2/3$, *respectively. If we denote $Ai$ the Airy function then $A_\pm$ are defined by $A_\pm(z) = Ai(e^{\mp 2\pi i/3}z)$.*

*Remark* 6.4. If local coordinates are chosen so that $\Omega$ is given by $x_n > 0$, the phases functions $\theta$, $\zeta$ satisfy the eikonal equations

$$\begin{cases} \xi_1^2 - <d\theta, d\theta>_g + \zeta <d\zeta, d\zeta>_g = 0, \\ <d\theta, d\zeta>_g = 0, \\ \zeta(x',0,\xi) = \zeta_0(\xi) = -\xi_1^{-1/3}\xi_n, \end{cases} \tag{6.23}$$

in the region $\zeta \leq 0$. Here $x' = (x_1, .., x_{n-1})$ and $< .,. >_g$ denotes the inner product given by the metric $g$. The phases also satisfy the eikonal equations (6.23) to infinite order at $x_n = 0$ in the region $\zeta > 0$.

*Remark* 6.5. Notice that one can think of $A_-(\zeta)$ as the incoming contribution and of $A_+(\zeta)\frac{A_-(\zeta_0)}{A_+(\zeta_0)}$ as the outgoing one. The $A_\pm$ terms are not temperate and one really exploits the cancellation

$$A_-(\zeta) - A_+(\zeta)\frac{A_-(\zeta_0)}{A_+(\zeta_0)} = e^{i\pi/3}(Ai(\zeta) - A_+(\zeta)\frac{Ai(\zeta_0)}{A_+(\zeta_0)}),$$

since $Ai(\zeta) = e^{i\pi/3}A_+(\zeta) + e^{-i\pi/3}A_-(\zeta)$. The term $Ai(\zeta)$ gives us the direct term (bicharacteristic not hitting the boundary), while the oscillatory one corresponds to the billiard ball map shift corresponding to reflection.

**Parametrix for the solution to the semi-classical Schrödinger equation near the glancing region:** Let now $v(x,t)$ be the solution of the semi-classical Schrödinger equation (6.5) where the initial data $v_0 \in L^2(M)$ is spectrally localized at spatial frequency $1/h$, i.e. $v_0(x) = \Psi(-h^2\Delta_g)v_0(x)$. From the discussion at the beginning of this section we see that it will be enough to consider $v_0$ compactly supported outside some small neighborhood of $\partial M$. Under this assumption $\Psi(-h^2\Delta_g)v_0$ is a well-defined pseudo-differential operator for which the results of Burq, Gérard and Tzvetkov [27, Section 2] apply.

Let $(e_\lambda(x))_{\lambda\geq 0}$ be the eigenbasis of $L^2(M)$ consisting in eigenfunctions of $-\Delta_g$ associated to the eigenvalues $(\lambda^2)$, so that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$. We write

$$\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{6.24}$$

and hence

$$e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)e^{-ith\lambda^2}v_\lambda e_\lambda(x). \tag{6.25}$$

If $\delta$ denotes the Dirac function, then the Fourier transform of $v(x,t)$ writes

$$\hat{v}(x,\frac{\tau}{h}) = h \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)\delta_{\{-\tau=h^2\lambda^2\}}v_\lambda e_\lambda(x). \tag{6.26}$$

For $t \in \mathbb{R}$ we can define (since $\hat{v}$ has compact support away from 0)

$$\begin{aligned} w(x,t) := & \frac{1}{2\pi h}\int_0^{\infty} e^{\frac{it\sigma}{h}}\hat{v}(x,-\frac{\sigma^2}{h})d\sigma \\ = & -\frac{1}{4\pi h}\int_{-\infty}^{0} e^{\frac{it\sqrt{-\tau}}{h}}\frac{1}{\sqrt{-\tau}}\hat{v}(x,\frac{\tau}{h})d\tau \\ = & -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\Psi(h^2\lambda^2)\Big(\frac{1}{2\pi}\int_{-\infty}^{0} e^{i\frac{t\sqrt{-\tau}}{h}}\frac{1}{\sqrt{-\tau}}\delta_{\{-\tau=h^2\lambda^2\}}d\tau\Big)v_\lambda e_\lambda(x) \\ = & -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\frac{1}{h\lambda}\Psi(h^2\lambda^2)e^{it\lambda}v_\lambda e_\lambda(x). \end{aligned} \tag{6.27}$$

Then $w(x,t)$ solves the wave equation

$$\begin{cases} h^2D_t^2w + h^2\Delta_g w = 0, \quad \text{on} \quad M\times\mathbb{R}, \quad w|_{\partial M\times\mathbb{R}} = 0, \\ w(x,0) = f_h(x), \quad D_tw(x,0) = g_h(x), \end{cases} \tag{6.28}$$

where the initial data are given by

$$f_h(x) = -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\frac{1}{h\lambda}\Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{6.29}$$

$$\begin{aligned} g_h(x) = & -\frac{1}{2h}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]}\Psi(h^2\lambda^2)v_\lambda e_\lambda(x) \\ = & -\frac{1}{2h}\Psi(-h^2\Delta_g)v_0(x). \end{aligned} \tag{6.30}$$

From (6.29), (6.30) it follows that

$$h\|g_h\|_{L^2(M)} \simeq \|f_h\|_{L^2(M)} \simeq \|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}, \tag{6.31}$$

where by $\alpha \simeq \beta$ we mean that there is $C > 0$ such that $C^{-1}\alpha < \beta < C\alpha$.

Indeed, in order to prove (6.31) notice that $w$ defined by (6.27) satisfies in fact

$$(hD_t - h\sqrt{-\Delta_g})w = 0$$

and (since $\Delta_g$ and $D_t$ commute) we have

$$f_h = w|_{t=0} = [(\sqrt{-\Delta_g})^{-1}D_tw]|_{t=0} = (\sqrt{-\Delta_g})^{-1}(D_tw|_{t=0}) = (\sqrt{-\Delta_g})^{-1}g_h.$$

Due to the spectral localization and since $g_h = -\frac{1}{2h}\Psi(-h^2\Delta_g)v_0$ we deduce (6.31).

By the $L^2$ continuity of the (classical) Fourier integral operator $K$ introduced in Proposition 6.1 we deduce

$$\|K(f_h,g_h)\|_{L^2(M)} \leq C(\|f_h\|_{L^2(M)} + h\|g_h\|_{L^2(M)}) \simeq \|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)}. \tag{6.32}$$

The solution $v(x,t)$ of (6.5) writes

$$v(x,t) = \frac{1}{2\pi h}\int e^{-\frac{it\sigma^2}{h}}(-2\sigma)\hat{v}(x,-\frac{\sigma^2}{h})d\sigma$$
$$= \frac{1}{2\pi h}\int_0^\infty e^{-i\frac{t\sigma^2}{h}}(-2\sigma)\int_{s\in\mathbb{R}} e^{-\frac{is\sigma}{h}}w(x,s)dsd\sigma. \quad (6.33)$$

The next step is to use Proposition 6.21 in order to obtain a representation of $v(x,t)$ near the glancing region: notice that the glancing part of the stationary wave $\hat{w}(x,\frac{\sigma}{h})$ is given by

$$1_{\{\sigma^2+r(x',0,\eta')\in[-c,c]\}}\hat{w}(x,\frac{\sigma}{h}) = 1_{\{\sigma^2+r(x',0,\eta')\in[-c,c]\}}\hat{v}(x,-\frac{\sigma^2}{h})$$
$$\overset{(\tau=-\sigma^2)}{=} 1_{\{-\tau+r(x',0,\eta')\in[-c,c]\}}\hat{v}(x,\frac{\tau}{h}), \quad (6.34)$$

where $c>0$ is sufficiently small. The equality in (6.34) follows from (6.27) and from the fact that $\hat{v}$ is essentially supported for the second variable in the interval $[-\beta_0,-\alpha_0]$. Consequently we can apply Proposition 6.21 and determine a representation for $v$ near the glancing region (for the Schrödinger equation) as follows

$$v(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)}2\xi_1\widehat{K(f_h,g_h)}(\frac{\xi}{h})$$
$$\times\Big[a(x,\xi/h)\Big(Ai(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h))\frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\Big)$$
$$+ b(x,\xi/h)\Big(Ai'(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\Big)\Big]d\xi, \quad (6.35)$$

where $a$, $b$ and $K$ are those defined in Proposition 6.1 and $f_h$, $g_h$ are given by (6.29), (6.30). The initial data $f_h$, $g_h$ are both supported, like $v_0$, away from $\partial M$, and consequently their $\dot{H}^\sigma(M)$ norms for $\alpha < n/2$ will be comparable to the norms of the non-homogeneous Sobolev space $H^\sigma(\mathbb{R}^n)$, therefor we shall work with the latter norms on the data $f_h$, $g_h$.

*Remark* 6.6. Notice that it is enough to prove semi-classical Strichartz estimates only for the "outgoing" piece corresponding to the oscillatory term $A_+(\zeta)\frac{Ai(\zeta_0)}{A_+(\zeta_0)}$, since the direct solution has already been dealt with (see the remark following (6.20)).

We deduce from (6.32), (6.35) that in order to finish the proof of Theorem 6.1 we need only to show that the operator $A_h$ defined, for $f$ supported away from $\partial M$ and spectrally localized at frequency $1/h$, i.e. such that $f = \Psi(-h^2\Delta_g)f$, by

$$A_hf(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1(a(x,\xi/h)A_+(\zeta(x,\xi/h)) + b(x,\xi/h)A'_+(\zeta(x,\xi/h)))$$
$$\times e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)}\frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\widehat{f}(\frac{\xi}{h})d\xi, \quad (6.36)$$

satisfies

$$\|A_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq C h^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{6.37}$$

*Remark* 6.7. We introduce a cut-off function $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the support of $f$ and equal to 0 near $\partial M$. Since $\chi_1$ is supported away from the boundary it follows from [27, Prop.2.1] (which applies here in its adjoint form) that $\Psi(-h^2\Delta_g)\chi_1 f$ is a pseudo-differential operator and writes (in a patch of local coordinates)

$$\Psi(-h^2\Delta_g)\chi_1 f = d(x, hD_x)\chi_2 f + O_{L^2(M)}(h^\infty), \tag{6.38}$$

where $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on the support of $\chi_1$ and where $d(x, D_x)$ is defined for $x$ in the suitable coordinate patch using the usual pseudo-differential quantization rule,

$$d(x, D_x)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi} d(x,\xi)\hat{f}(\xi)d\xi, \quad d \in C_0^\infty,$$

with symbol $d$ compactly supported for $|\xi|_g^2 :=< \xi, \xi >_g \in [\alpha_0, \beta_0]$, which follows by the condition of the support of $\Psi$. Since the principal part of the Laplace operator $\Delta_g$ is uniformly elliptic, we can introduce a smooth radial function $\psi \in C_0^\infty([\frac{1}{\delta}\alpha_0^{1/2}, \delta\beta_0^{1/2}])$ for some $\delta \geq 1$ such that $\psi(|\xi|)d = d$ everywhere. In what follows we shall prove (6.37) where, instead of $f$ we shall write $\psi(|\xi|)f$, keeping in mind that $f$ is supported away from the boundary and localized at spatial frequency $1/h$.

The proof of Theorem 6.1 will be completed once we prove (6.37). In order to do that, we split the operator $A_h$ into two parts: a main term and a diffractive term. To this end, let $\chi(s)$ be a smooth function satisfying

$$\text{supp}\chi \subset (-\infty, -1], \quad \text{supp}(1-\chi) \subset [-2, \infty).$$

We write this operator as a sum $A_h = M_h + D_h$, by decomposing

$$A_+(\zeta) = (\chi A_+)(\zeta) + ((1-\chi)A_+)(\zeta),$$

and letting the "main term" be defined for $f$ like in Remark 6.7 by

$$M_h f(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1 e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)} \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}(\frac{\xi}{h}) \\ \times (a(x,\xi/h)(\chi A_+)(\zeta(x,\xi/h)) + b(x,\xi/h)(\chi A'_+)(\zeta(x,\xi/h)))d\xi. \tag{6.39}$$

The "diffractive term" is then defined for $f$ like before by

$$D_h f(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1 e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)} \frac{Ai(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}(\frac{\xi}{h}) \\ \times (a(x,\xi/h)((1-\chi)A_+)(\zeta(x,\xi/h)) + b(x,\xi/h)((1-\chi)A'_+)(\zeta(x,\xi/h)))d\xi. \tag{6.40}$$

We analyze these operators separately, following the ideas of [95]:

**The main term $M_h$:** To estimate the "main term" $M_h$ we first use the fact that

$$|\frac{Ai(s)}{A_+(s)}| \leq 2, \quad s \in \mathbb{R}. \tag{6.41}$$

Consequently, since the term $\frac{Ai(\zeta_0)}{A_+(\zeta_0)}$ acts like a multiplier and so does $\xi_1$ which is localized in the interval $[\alpha_0, \beta_0]$ (this follows from (6.13)), the estimates for $M_h$ will follow from showing that the operator

$$f \to \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} (a(x,\xi/h)(\chi A_+)(\zeta(x,\xi/h)) + b(x,\xi/h)(\chi A'_+)(\zeta(x,\xi/h))) \times e^{\frac{i}{h}(\theta(x,\xi)-t\xi_1^2)}\psi(|\xi|)\hat{f}(\frac{\xi}{h})d\xi \tag{6.42}$$

satisfies the same bounds like in (6.37) for $f$ spectrally localized at frequency $1/h$. Following [111, Lemma 4.1], we write $\chi A_+$ and $(\chi A_+)'$ in terms of their Fourier transform to express the phase function of this operator

$$\phi(t,x,\xi) = -t\xi_1^2 + \theta(x,\xi) - \frac{2}{3}(-\zeta)^{3/2}(x,\xi), \tag{6.43}$$

which satisfies the eikonal equation (6.23). We denote its symbol $c_m(x,\xi/h)$, $c_m(x,\xi) \in \mathcal{S}^0_{2/3,1/3}(\mathbb{R}^n \times \mathbb{R}^n)$ and we also denote the operator defined in (6.42) by $W_h^m$, thus

$$W_h^m f(x,t) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(t,x,\xi)} c_m(x,\xi/h)\psi(|\xi|)\hat{f}(\frac{\xi}{h})d\xi.$$

**Proposition 6.2.** *Let $(q,r)$ be an admissible pair with $q > 2$, let $T > 0$ be sufficiently small and for $f = d(x,D_x)\chi_2 f + O_{L^2(\Omega)}(h^\infty)$ like in Remark 6.7 let*

$$W_h f(x,t) := \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} c_m(x,\xi/h)\psi(|\xi|)\hat{f}(\frac{\xi}{h})d\xi.$$

*Then the following estimates hold*

$$\|W_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq C h^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{6.44}$$

In the rest of this section we prove Proposition 6.2. The first step in the proof is a TT* argument. Explicitly,

$$\widehat{W_h^*(F)}(\frac{\xi}{h}) = \int e^{-\frac{i}{h}\phi(s,y,\xi)} F(y,s)\overline{c_m(y,\xi/h)}dyds,$$

and if we set

$$(T_h F)(x,t) = (W_h W_h^* F)(x,t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t,x,\xi)-\phi(s,y,\xi))} c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)F(y,s)d\xi ds dy, \tag{6.45}$$

then inequality (6.44) is equivalent to

$$\|T_hF\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq Ch^{-n(1/2-1/r)}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}, \tag{6.46}$$

where $(q', r')$ satisfies $1/q+1/q' = 1$, $1/r+1/r' = 1$. To see, for instance, that (6.46) implies (6.44), notice that the dual version of (6.44) is

$$\|W_h^*F\|_{L^2(\mathbb{R}^n)} \leq Ch^{-n/2(1/2-1/r)}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))},$$

and we have

$$\|W_h^*F\|^2_{L^2(\mathbb{R}^n)} = \int W_hW_h^*F\bar{F}dtdx \leq \|T_hF\|_{L^q((0,T],L^r(\mathbb{R}^n))}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}.$$

Therefore we only need to prove (6.46). Since the symbols are of type $(2/3, 1/3)$ and not of type $(1, 0)$, before starting the proof of (6.46) for the operator $T_h$ we need to make a further decomposition: let $\rho \in C_0^\infty(\mathbb{R})$ satisfying $\rho(s) = 1$ near 0, $\rho(s) = 0$ if $|s| \geq 1$. Let

$$T_hF = T_h^fF + T_h^sF,$$

where

$$T_h^sF(x,t) = \int K_h^s(t,x,s,y)F(y,s)dsdy, \tag{6.47}$$

$$K_h^s(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\phi(t,x,\xi)-\phi(s,y,\xi))}(1-\rho(h^{-1/3}|t-s|)) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)d\xi, \tag{6.48}$$

while

$$T_h^fF(x,t) = \int K_h^f(t,x,s,y)F(y,s)dsdy, \tag{6.49}$$

$$K_h^f(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\phi(t,x,\xi)-\phi(s,y,\xi))}\rho(h^{-1/3}|t-s|) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)d\xi. \tag{6.50}$$

*Remark* 6.8. The two pieces will be handled differently. The kernel of $T_h^f$ is supported in a suitable small set and it will be estimate by "freezing" the coefficients. To estimate $T_h^s$ we shall use the stationary phase method for type $(1, 0)$ symbols. For type $(2/3, 1/3)$ symbols, these stationary phase arguments break down if $|t-s|$ is smaller than $h^{1/3}$, which motivates the decomposition. We use here the same arguments as in [95].

— The "stationary phase admissible" term $T_h^s$

**Proposition 6.3.** *There is a constant $1 < C_0 < \infty$ such that the kernel $K_h^s$ of $T_h^s$ satisfies*

$$|K_h^s(t,x,s,y)| \leq C_N h^N \quad \forall N, \quad \textit{if} \quad \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0]. \tag{6.51}$$

*Moreover, there is a function $\xi_c(t,x,s,y)$ which is smooth in the variables $(t,s)$, uniformly over $(x,y)$, so that if $C_0^{-1} \leq \frac{|t-s|}{|x-y|} \leq C_0$, then*

$$|K_h^s(t,x,s,y)| \lesssim h^{-n}(1 + \frac{|t-s|}{h})^{-n/2}, \quad \textit{for} \quad |t-s| \geq h^{1/3}. \tag{6.52}$$

*Proof.* We shall use stationary phase lemma to evaluate the kernel $K_h^s$ of $T_h^s$. The critical points occur when $|t-s| \simeq |x-y|$. For some constant $C_0$ and for $|\xi| \in \text{supp}\psi$, $\xi_1$ in a small neighborhood of 1, we have

$$|\nabla_\xi(\phi(t,x,\xi) - \phi(s,y,\xi))| \simeq |t-s| + |x-y| \geq h^{1/3}, \quad \text{if} \quad \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0].$$

Since $c \in S^0_{2/3,1/3}$, an integration by parts leads to (6.51). If $|t-s| \simeq |x-y|$ we introduce a cutoff function $\kappa(\frac{|x-y|}{|t-s|})$, $\kappa \in C_0^\infty(\mathbb{R} \setminus \{0\})$. The phase function can be written as

$$\phi(t,x,\xi) - \phi(s,y,\xi) = (t-s)\Theta(t,x,s,y,\xi) \quad \text{for} \quad |t-s| \simeq |x-y| \geq h^{1/3}.$$

We want to apply the stationary phase method with parameter $|t-s|/h \geq h^{-2/3} \gg 1$ to estimate $K_h^s$. For $x$, $y$, $t$, $s$ fixed we must show that the critical points of $\Theta$ are non-degenerate.

**Lemma 6.1.** *If $T$ is sufficiently small then the phase function $\Theta(t,x,s,y,\xi)$ admits a unique, non-degenerate critical point $\xi_c$. Moreover, for $0 \leq t,s \leq T$, the function $\xi_c(t,x,s,y)$ solving $\nabla_\xi \Theta(t,x,s,y,\xi_c) = 0$ is smooth in $t$ and $s$, with uniform bounds on derivatives as $x$ and $y$ vary and we have*

$$|\partial_{t,s}^\alpha \partial_{x,y}^\gamma \xi_c(t,x,s,y)| \leq C_{\alpha,\gamma} h^{-|\alpha|/3} \quad \textit{if} \quad |x-y| \geq h^{1/3}. \tag{6.53}$$

*Proof.* The phase $\Theta(t,x,s,y,\xi)$ writes

$$\begin{aligned}\Theta(t,x,s,y,\xi) &= -\xi_1^2 + \frac{1}{(t-s)}(\phi(0,x,\xi) - \phi(0,y,\xi)) \\ &= -\xi_1^2 + \frac{1}{(t-s)} \sum_{j=1}^n (x_j - y_j)\partial_{x_j}\phi(0, z_{x,y}, \xi), \end{aligned} \tag{6.54}$$

for some $z_{x,y}$ close to $x$, $y$ (if $T$ is sufficiently small then $|t-s| \simeq |x-y|$ is small), and using the eikonal equations (6.23) we can write

$$\Theta(t,x,s,y,\xi) = - < \nabla_x \phi, \nabla_x \phi >_g (0, z_{x,y}, \xi) + \frac{1}{(t-s)} \sum_{j=1}^n (x_j - y_j)\partial_{x_j}\phi(0, z_{x,y}, \xi).$$

Let us write $<\nabla_x\phi,\nabla_x\phi>_g=\sum_{j,k}g^{j,k}\partial_{x_j}\phi\partial_{x_k}\phi$ and compute explicitly $\nabla_\xi\Theta$. For each $l\in\{1,..,n\}$ we have

$$\partial_{\xi_l}\Theta(t,x,s,y,\xi)=-\sum_{j=1}^n\partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\Big(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi)-\frac{(x_j-y_j)}{(t-s)}\Big),\tag{6.55}$$

thus

$$\nabla_\xi\Theta(t,x,s,y,\xi)=-\nabla^2_{\xi,x}\phi(0,z_{x,y},\xi).\left(\begin{array}{c}2\sum_k g^{1,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi)-\frac{x_1-y_1}{(t-s)}\\ .\\ .\\ 2\sum_k g^{n,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi)-\frac{x_n-y_n}{(t-s)}\end{array}\right),\tag{6.56}$$

where $\nabla^2_{\xi,x}\phi=(\partial^2_{\xi_l,x_j}\phi)_{l,j\in\{1,..,n\}}$ is the matrix $n\times n$ whose elements are the second derivatives of $\phi$ with respect to $\xi$ and $x$. We need the next lemma:

**Lemma 6.2.** *(see [94, Lemma 3.9]) For $\xi$ in a conic neighborhood of the $\xi_1$ axis the mapping*

$$x\to\nabla_\xi\Big(\theta(x,\xi)-\frac{2}{3}(-\zeta)^{3/2}(x,\xi)\Big)$$

*is a diffeomorphisme on the complement of the hypersurface $\zeta=0$, with uniform bounds of the Jacobian of the inverse mapping.*

A direct corollary of Lemma 6.2 is the following:

**Corollary 6.2.** *If $T$ is small enough and $|x-y|\simeq|t-s|\leq 2T$ then*

$$\det(\nabla^2_{\xi,x}\phi)(0,z_{x,y},\xi)\neq 0.\tag{6.57}$$

In what follows we complete the proof of Lemma 6.1. A critical point for $\Theta$ satisfies $\nabla_\xi\Theta(t,x,s,y,\xi)=0$ and from (6.56) and (6.57) this translates into

$$\Big((g^{j,k}(z_{x,y}))_{j,k}\Big)(\nabla_x\phi)^t(0,z_{x,y},\xi)=\frac{(x-y)}{(t-s)}.\tag{6.58}$$

Since $(g^{j,k})_{j,k}$ is invertible and using again (6.57) we can apply the implicit function's theorem to obtain (for $T$ small enough) a critical point $\xi_c=\xi_c(t,x,s,y)$ for $\Theta$. In order to show that $\xi_c$ is non-degenerate we compute

$$\begin{aligned}\partial_{\xi_q}\partial_{\xi_l}\Theta(t,x,s,y,\xi)=&-\sum_{j=1}^n\partial^3_{\xi_q,\xi_l,x_j}\phi(0,z_{x,y},\xi)\\ &\times\Big(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi)-\frac{(x_j-y_j)}{(t-s)}\Big)\\ &+2\sum_{j=1}^n\partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\Big(\sum_{k=1}^n g^{j,k}(z_{x,y})\partial^2_{\xi_q,x_k}\phi(0,z_{x,y},\xi)\Big),\end{aligned}\tag{6.59}$$

consequently at the critical point $\xi = \xi_c$ the hessian matrix $\nabla^2_{\xi,\xi}\Theta$ is given by

$$\nabla^2_{\xi,\xi}\Theta(t,x,s,y,\xi_c) = 2(\nabla^2_{\xi,x}\phi)(g^{ij}(z_{x,y}))_{i,j}(\nabla^2_{\xi,x}\phi)|_{(0,z_{x,y},\xi_c)},$$

consequently for $T$ small enough the critical point $\xi_c$ is non-degenerate by (6.57).

□

On the support of $\kappa$ it follows that the kernel $K^s_h$ writes

$$K^s_h(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}|t-s|\Theta(t,x,s,y,\xi)}\psi^2(|\xi|)(1-\rho(h^{-1/3}|t-s|))c_m(x,\xi/h)\overline{c_m(y,\xi/h)}d\xi, \tag{6.60}$$

where, if $\omega = |t-s|/h$ and $\xi_1 \simeq 1$, the symbol satisfies

$$|\partial^\alpha_{t,s}\partial^k_\omega \sigma_h(t,x,s,y,\omega\xi/|t-s|)| \le C_{\alpha,k}h^{-|\alpha|/3}(|t-s|^{3/2}/h)^{-2k/3},$$

where we set

$$\sigma_h(t,x,s,y,\omega\xi/|t-s|) = (1-\rho(h^{-1/3}|t-s|))c_m(x,\omega\xi/|t-s|)\overline{c_m(y,\omega\xi/|t-s|)}.$$

Indeed, since $c_m \in S^0_{2/3,1/3}$, for $\alpha = 0$ one has

$$\begin{aligned}|\partial^k_\omega\sigma_h| &\le |\xi||t-s|^{-k}|(\partial^k_\xi c)(t,x,\omega\xi/|t-s|)| \\ &\le C_{0,k}|t-s|^{-k}(\omega/|t-s|)^{-2k/3} = C_{0,k}|t-s|^{-k}h^{2k/3}. \end{aligned}\tag{6.61}$$

We conclude using the next lemma with $\omega = \frac{|t-s|}{h}$ and $\delta = |t-s|^{3/2} \ge h^{1/2} \gg h$.

**Lemma 6.3.** *Suppose that $\Theta(z,\xi) \in C^\infty(\mathbb{R}^{2(n+1)} \times \mathbb{R}^n)$ is real, $\nabla_\xi\Theta(z,\xi_c(z)) = 0$, $\nabla_\xi\Theta(z,\xi) \neq 0$ if $\xi \neq \xi_c(z)$, and*

$$|\det \nabla^2_{\xi\xi}\Theta| \ge c_0 > 0, \ \text{if} \ |\xi| \le 1.$$

*Suppose also that*

$$|\partial^\alpha_z\partial^\beta_\xi\Theta(z,\xi)| \le C_{\alpha,\beta}h^{-|\alpha|/3}, \quad \forall\alpha,\beta.$$

*In addition, suppose that the symbol $\sigma_h(z,\xi,\omega)$ vanishes when $|\xi| \ge 1$ and satisfies*

$$|\partial^\alpha_z\partial^\gamma_\xi\partial^k_\omega\sigma_h(z,\xi,\omega)| \le C_{k,\alpha,\gamma}h^{-(|\alpha|+|\gamma|)/3}(\delta/h)^{-2k/3}, \quad \forall k,\alpha,\gamma,$$

*where on the support of $\sigma_h$ we have $\omega \ge h^{-2/3}$ and $\delta > 0$. Then we can write*

$$\int_{\mathbb{R}^n} e^{i\omega\Theta(z,\xi)}\sigma_h(z,\xi,\omega)d\xi = \omega^{-n/2}e^{i\omega\Theta(z,\xi_c(z))}b_h(z,\omega),$$

*where $b_h$ satisfies*

$$|\partial^k_\omega\partial^\alpha_z b_h(z,\omega)| \le C_{k,\alpha}h^{-|\alpha|/3}(\delta/h)^{-2k/3}$$

*and where each of the constants depend only on $c_0$ and the size of finitely many of the constants $C_{\alpha,\beta}$ and $C_{k,\alpha,\gamma}$ above. In particular, the constants are uniform in $\delta$ if $1 \ge \delta \ge h$.*

This Lemma is used in [95, Lemma 2.6] and also in the thesis of Grieser [53] and it follows easily from the proof of standard stationary phase lemma (see [97, pag. 45]). Proposition 6.3 is thus proved. □

For each $t$, $s$, let $T_h^s(t,s)$ be the "frozen" operator defined by

$$T_h^s(t,s)g(x) = \int K_h^s(t,x,s,y)g(y)dy.$$

From Proposition 6.3 we deduce

$$\|T_h^s(t,s)g\|_{L^\infty(\mathbb{R}^n)} \leq C\max(h^{-n},(h|t-s|)^{-n/2})\|g\|_{L^1(\mathbb{R}^n)}. \tag{6.62}$$

We need the following

**Lemma 6.4.** *For $t$, $s$ fixed the frozen operator $T_h^s(t,s)$ is bounded on $L^2(\mathbb{R}^n)$,*

$$\|T_h^s(t,s)g\|_{L^2(\mathbb{R}^n)} \leq C\|g\|_{L^2(\mathbb{R}^n)}. \tag{6.63}$$

*Proof.* By energy conservation it follows that both $M_h(.)(.,t)$ and $D_h(.)(.,t)$ are bounded on $L^2(\mathbb{R}^n)$, and so is the operator $W_h(.,t)$, from which the $L^2$ continuity for both $T_h^s$ and $T_h^f$ follows. □

Interpolation between (6.62) and (6.63) with weights $1-2/r$ and $2/r$ respectively yields

$$\|T_h^s(t,s)g\|_{L^r(\mathbb{R}^n)} \leq Ch^{-n(1-2/r)}(1+\frac{|t-s|}{h})^{-n(1/2-1/r)}\|g\|_{L^{r'}(\mathbb{R}^n)} \tag{6.64}$$

and hence

$$\|T_h^sF\|_{L^q(0,T],L^r(\mathbb{R}^n)} \leq Ch^{-n/2(1-2/r)}\|\int_{1\ll\frac{|t-s|}{h}}^T |t-s|^{-n/2(1-2/r)}\|F(.,s)\|_{L^{r'}(\mathbb{R}^n)}ds\|_{L^{q'}((0,T])}.$$

Since $n(\frac{1}{2}-\frac{1}{r}) = \frac{2}{q} < 1$ the application $|t|^{-2/q} : L^{q'} \to L^q$ is bounded and by Hardy-Littlewood-Sobolev inequality we deduce

$$\|T_h^sF\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq Ch^{-n(1/2-1/r)}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}. \tag{6.65}$$

— The "frozen" term $T_h^f$

To estimate $T_h^f$ it suffices to obtain bounds for its kernel $K_h^f$ with both the variables $(t,x)$ and $(s,y)$ restricted to lie in a cube of $\mathbb{R}^{n+1}$ of sidelength comparable to $h^{1/3}$. Let us decompose $S_T$ into disjoint cubes $Q = Q_x \times Q_t$ of sidelength $h^{1/3}$. We then have

$$\begin{aligned}\|T_h^fF\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} &= \int_0^T \Big(\sum_{Q=Q_x\times Q_t}\|\chi_Q T_h^fF\|^r_{L^r(Q_x)}\Big)^{q/r}dt \\ &= \sum_Q \|\chi_Q T_h^fF\|^q_{L^q([0,T],L^r(\mathbb{R}^n))},\end{aligned} \tag{6.66}$$

where by $\chi_Q$ we denoted the characteristic function of the cube $Q$. In fact, by the definition, the integral kernel $K_h^f(t,x,s,y)$ of $T_h^f$ vanishes if $|t-s| \geq h^{1/3}$. If $|t-s| \leq h^{1/3}$ and $|x-y| \geq C_0 h^{1/3}$, then the phase

$$\phi(t,x,\xi) - \phi(s,y,\xi)$$

has no critical points with respect to $\xi_1$ (on the support of $\psi$), so that

$$|K_h^f(t,x,s,y)| \leq C_N h^N \quad \forall N, \ if \ |x-y| \geq C_0 h^{1/3}.$$

It therefore suffices to estimate $\|\chi_Q T_h^f \chi_{Q^*} F\|_{L^q([0,T],L^r(\mathbb{R}^n))}$, where $Q^*$ is the dilate of $Q$ by some fixed factor independent of $h$. Since $q > 2 > q'$, $r \geq 2 \geq r'$, where $q'$, $r'$ are such that $1/q + 1/q' = 1$, $1/r + 1/r' = 1$, then we shall obtain

$$\begin{aligned}\sum_Q \|\chi_Q T_h^f \chi_{Q^*} F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} &\leq C_1 \sum_Q \|\chi_{Q^*} F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))} \\ &\leq C_2 \|F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))}. \quad (6.67)\end{aligned}$$

In order to prove (6.67) we shall use the following:
**Proposition 6.4.** *Let $b(\xi) \in L^\infty(\mathbb{R}^n)$ be elliptic near $\xi_1 \simeq 1$, $b_h(\xi) := b(\xi/h)$, then for $h \ll |t-s| \leq h^{1/3}$, $h \ll |x-y| \leq h^{1/3}$ the operator defined by*

$$B_h f(x,t) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\phi(t,x,\xi)} \psi(|\xi|) b_h(\xi) \hat{f}(\frac{\xi}{h}) d\xi \qquad (6.68)$$

*satisfies*

$$\|B_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq C h^{-n/2(1/2-1/r)} \|f\|_{L^2(\mathbb{R}^n)}. \qquad (6.69)$$

*Proof.* We use again the TT$^*$ argument. Since $b(\xi)$ acts as an $L^2$ multiplier we can apply the stationary phase theorem in the integral

$$\int e^{\frac{i}{h}(\phi(t,x,\xi)-\phi(s,y,\xi))} \psi(|\xi|) d\xi$$

in order to obtain

$$\|B_h B_h^* F\|_{L^q((0,T],L^r((\mathbb{R}^n))} \lesssim h^{-n(1/2-1/r)} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}.$$

Notice that we haven't used the special properties of the phase function at $t=0$. □

Let now $Q$ be a fixed cube in $\mathbb{R}^{n+1}$ of sidelength $h^{1/3}$. Let

$$b_h(t,x,s,y,\xi) = \rho(h^{-1/3}|t-s|) c_m(x,\xi/h) \overline{c_m(y,\xi/h)},$$

and write

$$b_h(t,x,s,y,\xi) = b_h(0,0,s,y,\xi) + \int_0^t \partial_t b_h(r,0,s,y,\xi)dr \\ + .. + \int_0^t ... \int_0^{x_n} \partial_t..\partial_{x_n} b_h(r,z_1,..,z_n,s,y,\xi)drdz. \quad (6.70)$$

If the symbol $c$ is independent of $t$, $x$ then the estimates (6.44) follow from Proposition 6.4. We use this, for instance, to deduce

$$\|\chi_Q T_h^f \chi_{Q*} F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq C h^{-n/2(1/2-1/r)} \\ \times \left( \left\| \int\int e^{\frac{i}{h}(x\xi-\phi(s,y,\xi))}\psi(|\xi|)b_h(0,0,s,y,\xi)F(y,s)d\xi dsdy \right\|_{L^2(\mathbb{R}^n)} \right. \\ \left. + \dots + \int_0^{h^{1/3}} \int_0^{h^{1/3}} \left\| \int\int e^{\frac{i}{h}(x\xi-\phi(s,y,\xi))}\partial_t..\partial_{x_n}\psi(|\xi|)b_h(r,z,s,y,\xi)F(y,s)d\xi dsdy \right\|_{L^2(\mathbb{R}^n)} drdz \right). \quad (6.71)$$

Each derivative of $b_h(t,x,s,y,\xi)$ loses a factor of $h^{-1/3}$, but this is compensated by the integral over $(r,z)$, so that it suffices to establish uniform estimates for fixed $(r,z)$. By duality, we have to establish the estimate

$$\| \int\int e^{\frac{i}{h}\phi(s,y,\xi)}\psi(|\xi|)b_h(0,0,s,y,\xi)\hat{f}(\frac{\xi}{h})d\xi\|_{L^q((0,T],L^r(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

which follows by using the same argument of freezing the variables $(s,y)$ together with the Proposition 6.4.

### 6.2.3 The diffractive term

In order to estimate the diffractive term we shall proceed again like in [95, Sect.2].

**Lemma 6.5.** *For $x_n \geq 0$ and for $\xi$ in a small conic neighborhood of the positive $\xi_1$ axis, the symbol $q$ of $S_h$ can be written in the form*

$$q(x,\xi) := (a(x,\xi)((1-\chi)A_+)(\zeta(x,\xi)) + b(x,\xi)((1-\chi)A_+)'(\zeta(x,\xi)))\frac{Ai(\zeta_0(\xi))}{A_+(\zeta_0(\xi))} \\ =: p(x,\xi,\zeta(x,\xi)), \quad (6.72)$$

*where, for some $c > 0$*

$$|\partial_\xi^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k p(x,\xi,\zeta(x,\xi))| \leq C_{\alpha,j,\beta,k}\xi_1^{1/6-|\alpha|+2k/3}e^{-cx_n^{3/2}\xi_1-|\zeta|^{3/2}/2}.$$

*Proof.* Since

$$|\partial_\zeta^k((1-\chi)A_+)(\zeta)| \le C_{k,\epsilon} e^{(2/3+\epsilon)|\zeta|^{3/2}}, \quad \forall \epsilon > 0,$$

and the symbols $a$ and $b$ belong to $S^{1/6}_{1,0}$, the above fact will follow by showing that in the region $\zeta(x,\xi) \ge -2$,

$$\frac{Ai}{A_+}(\zeta_0(\xi)) = \tilde{p}(x,\xi',\zeta(x,\xi)),$$

where if $\xi' = (\xi_1,..,\xi_{n-1})$

$$|\partial_{\xi'}^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k \tilde{p}(x,\xi',\zeta)| \le C_{\alpha,j,\beta,k,\epsilon} \xi_1^{-|\alpha|+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta|^{3/2}}. \tag{6.73}$$

At $x_n = 0$, one has $\zeta = \zeta_0$, $\partial_{x_n}\zeta < 0$. It follows that for some $c > 0$

$$\zeta_0(x,\xi) \ge \zeta(x,\xi) + cx_n\xi_1^{2/3}.$$

By the asymptotic behavior of the Airy function we have, in the region $\zeta(x,\xi) \ge -2$

$$|\Big(\frac{Ai}{A_+}\Big)^{(k)}(\zeta_0)| \le C_{k,\epsilon} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta(x,\xi)|^{3/2}}. \tag{6.74}$$

We introduce a new variable $\tau(x,\xi) = \xi_1^{1/3}\zeta(x,\xi)$. At $x_n = 0$ one has $\tau = -\xi_n$, so that we can write $\xi_n = \sigma(x,\xi',\tau)$, where $\sigma$ is homogeneous of degree 1 in $(\xi',\tau)$. We set

$$\tilde{p}(x,\xi',\zeta) = \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\xi^{1/3}\zeta)).$$

The estimates (6.73) will follow by showing that

$$|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \partial_{x_n}^k \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))| \le C_{\alpha,j,\beta,k,\epsilon}\xi_1^{-|\alpha|-j+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\tau|^{3/2}\xi_1^{-1/2}}. \tag{6.75}$$

For $k = 0$, the estimates (6.75) follow from (6.74), together with the fact that

$$|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \frac{Ai}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))| \le C_{\alpha,\beta,j}(x_n\xi_1^{2/3} + \xi_1^{-1/3}|\tau|)\xi_1^{-|\alpha|-j},$$

which, in turn, holds by homogeneity, together with the fact that $\sigma(x,\xi',\tau) = 0$ if $x_n = \tau = 0$. If $k > 0$, the estimate (6.75) follows by observing that the effect of differentiating in $x_n$ is similar to multiplying by a symbol of order 2/3. This concludes the proof of Lemma 6.5. □

**Lemma 6.6.** *The Schwartz kernel of the diffractive term $D_h$ writes in the form*

$$\int e^{i(\theta(x,\xi)-ht\xi_1^2)}\psi(h|\xi|)q(x,\xi)d\xi$$
$$= \int e^{i(\theta(x,\xi)-ht\xi_1^2+\sigma\xi_1^{-2/3}\zeta(x,\xi)+\sigma^3/3\xi_1^2-<y,\xi>)}\psi(h|\xi|)c_d(x,\xi,\sigma)d\sigma d\xi, \tag{6.76}$$

*where $< .,. >$ denotes the scalar product and where*

$$|\partial_\xi^\alpha \partial_\sigma^j \partial_{x'}^\beta \partial_{x_n}^k c_d(x,\xi,\sigma)| \le C_{\alpha,j,\beta,k,N}\xi_1^{-1/2-|\alpha|-2j/3+2k/3} e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2}, \quad \forall N.$$

*Proof.* The symbol $c_d$ of the Schwartz kernel of $D_h$ writes as a product of two symbols

$$c_d(x,\xi,\sigma)=c_1(x,\xi,\sigma\xi_1^{-2/3})c_2(x,\xi,\zeta(x,\xi)),$$

where

$$c_1(x,\xi,\sigma\xi_1^{-2/3})=\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}\sigma)(a(x,\xi)+\sigma\xi_1^{-2/3}b(x,\xi))\in S^{-1/2}_{2/3,1/3}(\mathbb{R}^n_x,\mathbb{R}^{n+1}_{\xi,\sigma})$$

comes from the Fourier transform of $A_+$ (here $\Psi_+$ is a symbol of order 0) and where $c_2$ satisfies for all $N\geq 0$ (for $\sigma^2\xi_1^{-4/3}+\zeta(x,\xi)=0$)

$$\begin{aligned}&|\partial^\alpha_{\xi'}\partial^j_\sigma\partial^\beta_{x'}\partial^k_{x_n}c_2(x,\xi',-(\sigma^2\xi_1^{-4/3}))|\\&\qquad\leq C_{\alpha,j,\beta,k,N}\xi_1^{-2j/3}|\sigma\xi_1^{-2/3}|^j\xi_1^{-|\alpha|+2k/3}e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2},\quad(6.77)\end{aligned}$$

which follows from (6.73). We use the exponential factor $e^{-cx_n^{3/2}\xi_1}$ to deduce from (6.77)

$$|x_n^j\partial^k_{x_n}c_2(x,\xi',-(\sigma^2\xi_1^{-4/3}))|\leq C_{j,k,N}(x_n\xi_1^{2/3})^je^{-c(x_n\xi_1^{2/3})^{3/2}}\xi_1^{2/3(k-j)}(1+\xi_1^{-4/3}\sigma^2)^{-N/2},\quad\forall N$$

□

From now on we proceed as for the main term and we reduce the problem to considering the operator

$$W_h^df(x,t)=\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}c_d(x,\xi/h,\sigma)\psi(|\xi|)\hat f(\frac{\xi}{h})d\xi,$$

where $x_n^j\partial^k_{x_n}c_d\in S^{2(k-j)/3}_{2/3,1/3}(\mathbb{R}^{n-1}_{x'}\times\mathbb{R}^n_\xi)$ uniformly over $x_n$ and where we set

$$\tilde\phi(t,x,\xi,\sigma):=-t\xi_1^2+\theta(x,\xi)+\sigma\xi_1^{1/3}\zeta(x,\xi)+\xi_1\sigma^3/3,\quad(6.78)$$

obtained after the changes of variables $\sigma\to\sigma\xi_1$, $\xi\to\xi/h$ in (6.76). Using the freezing arguments behind the proof of the estimates for $T_h^f$ and Minkovski inequality we have

$$\begin{aligned}\|W_h^df\|_{L^q((0,T],L^r(\mathbb{R}^n))}&\leq\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}c_d(x',0,\xi/h,\sigma)\psi(|\xi|)\hat f(\frac{\xi}{h})d\sigma d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^n))}\\&+h^{-2/3}\int_0^{h^{2/3}}\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}h^{2/3}\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f(\frac{\xi}{h})d\sigma d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}dr\\&+h^{2/3}\int_{r>h^{2/3}}\frac{dr}{r^2}\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f(\frac{\xi}{h})d\sigma d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}.\end{aligned}$$

Since $c_d(x',0,\xi,\sigma)$ and $h^{2/3}(1+h^{-4/3}r^2)\partial_{x_n}c_d(x',r,\xi,\sigma)$ are symbols of order 0 and type $(2/3,1/3)$ with uniform estimates over $r$, the estimates for the diffractive term also follow from Proposition 6.2. Indeed, the term in the second line loses a factor $h^{-2/3}$, but this is compensated by the integral over $r \leq h^{2/3}$. The term in the third line can be bounded from above by

$$h^{2/3}\int_{r>h^{2/3}}\frac{dr}{r^2}\times\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde{\phi}(t,x,\xi,\sigma)}(h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h,\sigma))\psi(|\xi|)\hat{f}(\frac{\xi}{h})d\sigma d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^n))}$$
$$\leq\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde{\phi}(t,x,\xi,\sigma)}(h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h))\psi(|\xi|)\hat{f}(\frac{\xi}{h})d\sigma d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^n))}.$$

We conclude using the same arguments as in the proof of Proposition 6.2, where now $W_h$ is replaced by operators with symbols $c_d(x',0,\xi,\sigma)$, $h^{2/3}\partial_{x_n}c_d(x',r,\xi,\sigma)$ and $h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi,\sigma)$ respectively. Notice however that for this term we can't apply directly Lemma 6.1 since the expansion of the Airy function giving the phase function (6.43) is available only for $\zeta \leq -1$. Writing the phase function of (6.76) in the form $\tilde{\phi}(t,x,\xi,\sigma) - <y,\xi>$, we notice that at $t=0$ this phase is homogeneous of degree 1 in $\xi$ and the proof of the non-degeneracy of the critical points in the TT$^*$ argument of Lemma 6.1 reduces to checking that the Jacobian $J$ of the mapping

$$(\xi,\sigma)\to(\nabla_x(\theta(x,\xi)+\sigma\zeta(x,\xi)),\zeta(x,\xi)+\sigma^2) \tag{6.79}$$

does not vanish at the critical point of the phase of (6.76). Hence we will obtain a phase function $\breve{\phi}(t,x,\xi)$ which will satisfy $\nabla^2_{x,\xi}\breve{\phi}(0,x,\xi)\neq 0$ and this will hold also for small $|t|\leq T$ and we can use the same argument as in Lemma 6.1. To prove that the Jacobian of the application (6.79) doesn't vanish we use [94, Lemma A.2]. Precisely, at this (critical) point $\sigma=\zeta(x,\xi)=0$, $y=0$ and $\nabla_{x'}\zeta(x,\xi)=0$. Since $\partial_{x_n}\zeta(x,\xi)\neq 0$ and $\partial_{\xi_n}\zeta(x,\xi)\neq 0$ there, the result follows by the nonvanishing of $|\nabla_{x'}\nabla_{\xi'}\theta(x,\xi)|$. In fact we have

$$\det\begin{pmatrix}\nabla_{x'}\nabla_{\xi'}\theta & \nabla_{\xi'}\partial_{x_n}\theta & \nabla_{\xi'}\zeta\\ \partial_{\xi_n}\nabla_{x'}\theta & \partial_{\xi_n}\partial_{x_n}\theta & \partial_{\xi_n}\zeta\\ \nabla_{x'}\zeta & \partial_{x_n}\zeta & 2\sigma\end{pmatrix}|_{\sigma^2=-\zeta=0}\neq 0.$$

## 6.3 Strichartz estimates for the classical Schrödinger equation outside a strictly convex obstacle in $\mathbb{R}^n$

In this section we prove Theorem 6.2 under the Assumptions 6.2. In what follows we shall work with the Laplace operator with constant coefficients $\Delta_D=\sum_{j=1}^n\partial_j^2$ acting on $L^2(\Omega)$ to avoid technicalities, where $\Omega$ is the exterior in $\mathbb{R}^n$ of a strictly convex domain $\Theta$. In the proof of Theorem 6.2 we distinguish two main steps: we start by performing a time rescaling which transforms the equation (6.9) into a semi-classical problem: due to the finite speed of propagation (proved by Lebeau [73]), we can use the (local) semi-classical result of Theorem 6.1 together with the smoothing effect (following Staffilani and Tataru [98] and Burq [23]) to obtain classical Strichartz estimates near the boundary. Outside a fixed neighborhood of $\partial\Omega$ we use a method suggested by Staffilani and Tataru [98] which consists in considering the Schrödinger flow as a solution of a problem in the whole space $\mathbb{R}^n$, for which the Strichartz estimates are known.

We start by proving that using Theorem 6.1 on a compact manifold with strictly concave boundary we can deduce sharp Strichartz estimates for the semi-classical Schrödinger flow on $\Omega$. Precisely, the following holds

**Proposition 6.5.** *Given $(q,r)$ satisfying the scaling condition (6.3) with $q>2$ there exists a constant $C>0$ such that the (classical) Schrödinger flow on $\Omega\times\mathbb{R}$ with Dirichlet boundary condition and spectrally localized initial data $\Psi(-h^2\Delta_D)u_0$, where $\Psi\in C_0^\infty(\mathbb{R}\setminus 0)$, satisfies*

$$\|e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0\|_{L^q(\mathbb{R})L^r(\Omega)}\leq C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \tag{6.80}$$

*Remark* 6.9. We first proceed with the proof of Proposition 6.5 and then we show how it can be used to prove Theorem 6.2. For the proof of Proposition 6.5 we use a similar method as the one given in our recent paper [64] in collaboration with F.Planchon.

*Proof.* Let $\tilde{\Psi}\in C_0^\infty(\mathbb{R}\setminus\{0\})$ be such that $\tilde{\Psi}=1$ on the support of $\Psi$, hence

$$\tilde{\Psi}(-h^2\Delta_D)\Psi(-h^2\Delta_D)=\Psi(-h^2\Delta_D).$$

Following [23], [64], we split $e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0(x)$ as a sum of two terms

$$\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0+\tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0,$$

where $\chi\in C_0^\infty(\mathbb{R}^n)$ equals 1 in a neighborhood of $\partial\Omega$.

— Study of $\tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$ :
Set $w_h(x,t)=(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(x)$. Then $w_h$ satisfies

$$\begin{cases} i\partial_t w_h+\Delta_D w_h=-[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0,\\ w_h|_{t=0}=(1-\chi)\Psi(-h^2\Delta_D)u_0. \end{cases} \tag{6.81}$$

Since $\chi$ is equal to 1 near the boundary $\partial\Omega$, the solution to (6.81) solves also a problem in the whole space $\mathbb{R}^n$. Consequently, the Duhamel formula writes

$$w_h(t,x)=e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0-\int_0^t e^{i(t-s)\Delta}[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)ds, \tag{6.82}$$

where by $\Delta$ we denoted the free Laplacian on $\mathbb{R}^n$ and thereforee the contribution of $e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0$ satisfies the usual Strichartz estimates. For the second term in the right hand side of (8.90) we use the next lemma, due to Christ and Kiselev [35]:
**Lemma 6.7.** *(Christ and Kiselev) Consider a bounded operator*

$$T:L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)$$

*given by a locally integrable kernel $K(t,s)$ with values in bounded operators from $B_1$ to $B_2$, where $B_1$ and $B_2$ are Banach spaces. Suppose that $q' < q$. Then the operator*

$$\tilde{T}f(t) = \int_{s<t} K(t,s)f(s)ds$$

*is bounded from $L^{q'}(\mathbb{R}, B_1)$ to $L^q(\mathbb{R}, B_2)$ and*

$$\|\tilde{T}\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)} \leq C(1-2^{-(1/q-1/q')})^{-1}\|T\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)}.$$

This lemma allows (since $q > 2$) to replace the study of the second term in the right hand side of (8.90) by that of

$$\int_0^\infty e^{i(t-s)\Delta}[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)(s)ds =: U_0U_0^* f(x,t),$$

where $U_0 = e^{it\Delta}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ and $U_0^*$ is bounded from $L^2(\mathbb{R}, H^{-1/2}_{\text{comp}})$ to $L^2(\mathbb{R}^n)$ and where we set $f := [\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$ which belongs to $L^2H^{-1/2}_{\text{comp}}(\Omega)$ by [26, Prop.2.7]. The estimates for $w_h$ follow like in [26] and we find

$$\begin{aligned}\|w_h\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|(1-\chi)\Psi(-h^2\Delta_D)u_0\|_{L^2(\mathbb{R}^n)} \\ + \|[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{-1/2}_{\text{comp}}(\Omega))}.\end{aligned} \tag{6.83}$$

The last term in (6.83) can be estimated using [26, Prop.2.7] by

$$C\|\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R},H^{1/2}_{\text{comp}}(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \tag{6.84}$$

Finally, we conclude this part using [63, Thm.1.1] which gives

$$\|\Psi(-h^2\Delta_D)w_h\|_{L^r(\Omega)} \leq \|w_h\|_{L^r(\Omega)}. \tag{6.85}$$

— Study of $\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$:
Let $\varphi \in C_0^\infty((-1,2))$ equal to 1 on $[0,1]$. For $l \in \mathbb{Z}$ let

$$v_{h,l} = \varphi(t/h-l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \tag{6.86}$$

which is a solution to

$$\begin{cases} i\partial_t v_{h,l} + \Delta_D v_{h,l} = \Big(\varphi(t/h-l)[\Delta_D,\chi] + i\frac{\varphi'(t/h-l)}{h}\chi\Big)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \\ v_{h,l}|_{t<hl-h} = 0, \quad v_{h,l}|_{t>hl+2h} = 0. \end{cases} \tag{6.87}$$

We denote by $V_{h,l}$ the right-hand side of (9.17), so that

$$V_{h,l} = \Big(\varphi(t/h-l)[\Delta_D,\chi] + i\frac{\varphi'(t/h-l)}{h}\chi\Big)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0. \tag{6.88}$$

Let $Q \subset \mathbb{R}^n$ be an open cube sufficiently large such that $\partial\Omega$ is contained in the interior of $Q$. We denote by $S$ the punctured torus obtained from removing the obstacle $\Theta$ (recall that $\Omega = \mathbb{R}^n \setminus \Theta$) in the compact manifold obtained from $Q$ with periodic boundary conditions on $\partial Q$. Notice that defined in this way $S$ coincides with the Sinaï billiard. Let $\Delta_S := \sum_{j=1}^n \partial_j^2$ denote the Laplace operator on the compact domain $S$.
On $S$, we may define a spectral localization operator using eigenvalues $\lambda_k$ and eigenvectors $e_k$ of $\Delta_S$: if $f = \sum_k c_k e_k$, then

$$\Psi(-h^2\Delta_S)f = \sum_k \Psi(-h^2\lambda_k^2)c_k e_k. \tag{6.89}$$

*Remark* 6.10. Notice that in a neighborhood of the boundary, the domains of $\Delta_S$ and $\Delta_D$ coincide, thus if $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ is supported near $\partial\Omega$ then $\Delta_S\tilde{\chi} = \Delta_D\tilde{\chi}$.

In what follows let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of $\chi$ and be supported in a neighborhood of $\partial\Omega$ such that on its support the operator $-\Delta_D$ coincide with $-\Delta_S$. From their respective definition, $v_{h,l} = \tilde{\chi}v_{h,l}$, $V_{h,l} = \tilde{\chi}V_{h,l}$, consequently $v_{h,l}$ will also solve the following equation on the compact domain $S$

$$\begin{cases} i\partial_t v_{h,l} + \Delta_S v_{h,l} = V_{h,l}, \\ v_{h,l}|_{t<h(l-1/2)\pi} = 0, \quad v_{h,l}|_{t>h(l+1)\pi} = 0. \end{cases} \tag{6.90}$$

Writing the Duhamel formula for the last equation (9.21) on $S$, applying $\tilde{\Psi}(-h^2\Delta_D)$ and using that $\tilde{\chi}v_{h,l} = v_{h,l}$, $\tilde{\chi}V_{h,l} = V_{h,l}$ and writing

$$\begin{aligned} \tilde{\Psi}(-h^2\Delta_D)\tilde{\chi} = \chi_1\tilde{\Psi}(-h^2\Delta_S)\tilde{\chi} + (1-\chi_1)\tilde{\Psi}(-h^2\Delta_D)\tilde{\chi} \\ + \chi_1(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S))\tilde{\chi} \end{aligned} \tag{6.91}$$

for some $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ equal to one on the support of $\tilde{\chi}$, yields

$$\begin{aligned} \tilde{\Psi}(-h^2\Delta_D)v_{h,l}(x,t) = \chi_1 \int_{hl-l}^t e^{i(t-s)\Delta_S}\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)ds + \\ + (1-\chi_1)\int_{hl-l}^t \tilde{\Psi}(-h^2\Delta_D)e^{i(t-s)\Delta_S}V_{h,l}(x,s)ds \\ + \chi_1(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S))v_{h,l}. \end{aligned} \tag{6.92}$$

Denote by $v_{h,l,m}$ the first term of (6.92), by $v_{h,l,f}$ the second one and by $v_{h,l,s}$ the last one. We deal we them separately. To estimate the $L_t^q L^r(\Omega)$ norm of $v_{h,l,f}$ we notice that it is supported away from the boundary, therefore the estimates will follow as in the previous part of this section. Indeed, notice that since $v_{h,l}$ solves also the equation (9.17) on $\Omega$ we can use the Duhamel formula on $\Omega$ so that in the integral defining $v_{h,l,f}$ to have $\Delta_D$ instead of $\Delta_S$. We then estimate the $L_t^q L^r(\Omega)$ norm of $v_{h,l,f}$ applying the Minkovski inequality and using the sharp Strichartz estimates for

$(1-\chi_1)\tilde{\Psi}(-h^2\Delta_D)e^{i(t-s)\Delta_D}V_{h,l}$ deduced in the first part of the proof of Proposition 6.5 and obtain, denoting $I_l^h=[hl-h,hl+2h]$,

$$\|v_{h,l,f}\|_{L^q(I_l^h,L^r(\Omega))}\leq C\int_{I_l^h}\|V_{h,l}(x,s)\|_{L^2(\Omega)}ds. \tag{6.93}$$

For the last term $v_{h,l,s}$ we use the next lemma that will be proved in the Appendix:
**Lemma 6.8.** *Let $\chi_1\in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$. Then we have*

$$\|v_{h,l,s}\|_{L^q(I_l^h,L^r(\Omega))}\leq C_Nh^N\|V_{h,l}(x,s)\|_{L^2(I_l^h,H_0^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))},\quad \forall N\in\mathbb{N}. \tag{6.94}$$

To estimate the main contribution $v_{h,l,m}$ we use the Minkovski inequality which yields

$$\begin{aligned}\|v_{h,l,m}\|_{L^q(I_l^h,L^r(\Omega))}&=\|v_{h,l,m}\|_{L^q(I_l^h,L^r(S))}\\&\leq C\int_{I_l^h}\|e^{i(t-s)\Delta_S}\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)\|_{L^q(I_l^h,L^r(S))}ds.\end{aligned} \tag{6.95}$$

Applying Theorem 6.1 for the linear semi-classical Schrödinger flow on $S$, the term to integrate in (6.95) is bounded by $C\|\tilde{\Psi}(-h^2\Delta_S)V_{h,l}(x,s)\|_{L^2(S)}$. Using [**?**, Thm.1.1] and the fact that $\tilde{\chi}V_{h,l}=V_{h,l}$ (so that taking the norm over $\Omega$ or $S$ makes no difference) we obtain

$$\|v_{h,l,m}\|_{L^q(I_l^h,L^r(\Omega))}\leq C\int_{I_l^h}\|V_{h,l}(x,s)\|_{L^2(\Omega)}ds. \tag{6.96}$$

After applying the Cauchy-Schwartz inequality in (6.93), (6.96) it remains to estimate the $L^2(I_l^h,H^\sigma(\Omega))$ norm of $V_{h,l}$, where $\sigma\in\{0,n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}\}$. We do this using the precise form (6.88) and obtain

$$\begin{aligned}&\|V_{h,l}\|_{L^2(I_l^h,H^\sigma(\Omega))}\\&\quad\leq C\|\varphi(t/h-l)[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H^\sigma(\Omega))}\\&\quad\quad+Ch^{-1}\|\varphi'(t/h-l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H^\sigma(\Omega))}.\end{aligned} \tag{6.97}$$

Since the operator $[\Delta_D,\chi]\Psi(-h^2\Delta_D)$ is bounded from $H^{\sigma+1}$ to $H^\sigma$, we deduce from (6.92), (6.93), (6.97), (6.98) and Lemma 9.3 that

$$\begin{aligned}&\|\tilde{\Psi}(-h^2\Delta_D)v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))}\\&\quad\leq Ch^{1/2}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^1(\Omega))}\\&\quad+Ch^{-1/2}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,L^2(\Omega))}\\&\quad+C_Nh^{N+1/2}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^{n(\frac{1}{2}-\frac{1}{r})+\frac{1}{2}}(\Omega))}\\&\quad+C_Nh^{N-1/2}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(I_l^h,H_0^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))},\end{aligned} \tag{6.98}$$

where $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ is chosen equal to 1 on the supports of $\varphi$. Since $q \geq 2$ we estimate

$$
\begin{aligned}
\|\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^q(\mathbb{R},L^r(\Omega))} &\leq C\sum_{l=-\infty}^{\infty}\|\tilde{\Psi}(-h^2\Delta_D)v_{h,l}\|^q_{L^q(I_l^h,L^r(\Omega))}\\
&\leq Ch^{q/2}\Big(\sum_{l=-\infty}^{\infty}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^1(\Omega))}\Big)^{q/2}\\
&+ Ch^{-q/2}\Big(\sum_{l=-\infty}^{\infty}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,L^2(\Omega))}\Big)^{q/2}\\
&+ C_Nh^{q(N+1/2)}\Big(\sum_{l=-\infty}^{\infty}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^{n(\frac{1}{2}-\frac{1}{r})+\frac{1}{2}}(\Omega))}\Big)^{q/2}\\
&+ C_Nh^{q(N-1/2)}\Big(\sum_{l=-\infty}^{\infty}\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^2_{L^2(I_l^h,H_0^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}\Big)^{q/2}.
\end{aligned}
\tag{6.99}
$$

The almost orthogonality of the supports of $\tilde{\varphi}(.-l)$ in time allows to estimate the term in the second line of (6.99) by

$$Ch^{q/2}\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^1(\Omega))}, \tag{6.100}$$

the one in the third line by

$$Ch^{-q/2}\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},L^2(\Omega))}, \tag{6.101}$$

the term in the fourth line by

$$C_Nh^{q(N+1/2)}\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^{n(\frac{1}{2}-\frac{1}{r})+\frac{1}{2}}(\Omega))}, \tag{6.102}$$

and the one in the last line of (6.99) by

$$C_Nh^{q(N-1/2)}\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|^q_{L^2(\mathbb{R},H_0^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}. \tag{6.103}$$

We need the following smoothing effect on a non trapping domain:
**Proposition 6.6.** *( [26, Prop.2.7]) Assume that $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, where $\mathcal{O} \neq \emptyset$ is a compact non-trapping obstacle. Then for every $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, $\sigma \in [-1/2, 1]$, one has*

$$\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R},H_0^{\sigma+1/2}(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}. \tag{6.104}$$

*Remark* 6.11. In [26], Proposition 6.6 is proved for $\sigma \in [0,1]$, but for spectrally localized data the result also follows using the estimates (2.15) of [26, Prop.2.7].

We apply Proposition 6.6 with $\sigma = 1/2$ in (6.100), with $\sigma = -1/2$ in (6.101) and with $\sigma = n(\frac{1}{2} - \frac{1}{r}) = \frac{2}{q} \in [0,1]$ in (6.102). In (6.103) we use that $n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{2} \leq \frac{1}{2}$ in order to estimate the $L^2(\mathbb{R}, H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))$ norm by the $L^2(\mathbb{R}, H^{\frac{1}{2}}(\Omega))$ norm and use Proposition 6.6 with $\sigma = 0$. This yields

$$\|\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}, \quad (6.105)$$

where we used the spectral localization $\Psi$ to estimate $\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}$ by $h^{-\sigma}\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}$. This achieves the proof of Proposition 6.5.

□

In the rest of this section we show how Proposition 6.5 implies Theorem 6.2. We need the next lemma proved in [63]:

**Lemma 6.9.** *(see [63, Thm.1.1]) Let $\Psi_0 \in C_0^\infty(\mathbb{R})$, $\Psi \in C_0^\infty((1/2, 2))$ satisfy*

$$\Psi_0(\lambda) + \sum_{j\geq 1} \Psi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}.$$

*Then for all $r \in [2, \infty)$ we have*

$$\|f\|_{L^r(\Omega)} \leq C_r\Big(\|\Psi_0(-\Delta_D)f\|_{L^r(\Omega)} + (\sum_{j=1}^{\infty} \|\Psi(-2^{-2j}\Delta_D)f\|^2_{L^r(\Omega)})^{1/2}\Big). \quad (6.106)$$

Applying Lemma 6.9 to $f = e^{it\Delta_D}u_0$ and taking the $L^q$ norm in time yields

$$\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R},L^r(\Omega))} \leq \|\|e^{it\Delta_D}\Psi_0(-\Delta_D)u_0\|_{L^r(\Omega)} + (\sum_{j\geq 1} \|e^{it\Delta_D}\Psi(-2^{-2j}\Delta_D)u_0\|^2_{L^r(\Omega)})^{1/2}\|_{L^q(\mathbb{R})}$$

which, by Minkowski inequality, leads to $\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}$. The proof of Theorem 6.2 is complete.

## 6.4 Applications

In this section we sketch the proofs of Theorem 6.3 and Theorem 6.4.

We start with Theorem 6.3. From Theorem 6.2 we have an estimate of the linear flow of the Schrödinger equation

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R},L^{30/11}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \quad (6.107)$$

One may shift regularity by 1 and obtain

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R},W^{1,30/11}(\Omega))} \leq C\|u_0\|_{H^1_0(\Omega)}. \quad (6.108)$$

Hence for small $T > 0$ the left hand side in (6.107), (6.108) will be small; for such $T$ let $X_T := L^5((0,T], W^{1,30/11}(\Omega))$. One may then set up the usual fixed point argument in $X_T$, as if $u \in X_T$ then $u^5 \in L^1([0,T], H^1(\Omega))$.

Let us proceed with Theorem 6.4. From the work of Planchon and Vega [87], one has a global in time control on the solution $u$, at the level of $\dot{H}^{\frac{1}{4}}$ regularity:

$$u \in L^4((0,+\infty), L^4(\Omega)).$$

By interpolation with either mass or energy conservation, combined with the local existence theory, one may bootstrap this global in time control into

$$u \in L^{p-1}((0,+\infty), L^\infty(\Omega)),$$

from which scattering in $H_0^1(\Omega)$ follows immediately.

## 6.5 Appendix

### 6.5.1 Finite speed of propagation for the semi-classical equation

In this section we recall several properties of the semi-classical Schrödinger flow (for further discussions and proofs we refer the reader to [73]). Let $M$ be a compact manifold with smooth boundary $\partial M$.

*Definition* 6.2. We say that a symbol $q(y,\eta) \in S^m_{\rho,\delta}$ is of type $(\rho,\delta)$ and of order $m$ if

$$\forall \alpha, \beta \quad \exists C_{\alpha,\beta} > 0 \quad |\partial_y^\beta \partial_\eta^\alpha q(y,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{m-\rho|\alpha|+\delta|\beta|}.$$

For $q \in S^m_{1,0}$ we let $Op_h(q) = Q(y, hD, h)$ be the $h$-pseudo differential operator defined by

$$Op_h(q)f(y) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(y-\tilde{y})\eta} q(y,\eta,h) f(\tilde{y}) d\tilde{y}.$$

We set $y = (x,t) \in M$ and denote $\eta = (\xi, \tau)$ the dual variable of $y$. Near a point $x_0 \in \partial M$ we can choose a system of local coordinates such that $M$ is given by $M = \{x = (x', x_n) | x_n > 0\}$. We define the tangential operators

$$Op_{h,\text{tang}}(q)f(y) = \frac{1}{(2\pi h)^{n-1}} \int e^{\frac{i}{h}(y'-\tilde{y}')\eta'} q(y,\eta',h) f(\tilde{x}', x_n, \tilde{t}) d\tilde{y}' d\eta',$$

where $y = (x', x_n, t)$, $y' = (x', t)$, $\tilde{y}' = (\tilde{x}', \tilde{t})$, $\eta = (\xi', \xi_n, \tau)$, $\eta' = (\xi', \tau)$ and where the symbol $q(y, \eta', h) \in S^m_{1,0,\text{tang}}$ i.e. such that

$$\forall \alpha, \beta \quad \exists C_{\alpha,\beta} > 0 \quad |\partial_y^\alpha \partial_{\eta'}^\beta q(y,\eta',h)| \le C_{\alpha,\beta}(1+|\eta'|)^{m-|\beta|}.$$

In what follows we let $(M, g)$ be a compact Riemannian manifold with strictly concave boundary satisfying the Assumptions 6.1. Let also $v_0 \in L^2(M)$ be compactly supported outside a small neighborhood of the boundary, $\Psi \in C_0^\infty((\alpha_0, \beta_0))$ and let $v(x,t) = e^{iht\Delta_g}\Psi(-h^2\Delta_g)v_0$ denote the linear semi-classical Schrödinger flow with initial data at time $t = 0$ equal to $\Psi(-h^2\Delta_g)v_0$ and such that $\|\Psi(-h^2\Delta_g)v_0\|_{L^2(M)} \lesssim 1$.

Let $\pi : T^*(\bar{M} \times \mathbb{R}) \to T^*(\partial M \times \mathbb{R}) \cup T^*(M \times \mathbb{R})$ be the canonical projection defined, for $y = (x, t)$, $\eta = (\xi, \tau)$ by

$$\pi|_{T^*(M\times\mathbb{R})} = Id, \quad \pi(y, \eta) = (y, \eta|_{T^*(\partial M\times\mathbb{R})}), \quad for\ (y, \eta) \in T^*(\bar{M} \times \mathbb{R})|_{\partial M\times\mathbb{R}}.$$

We introduce the characteristic set

$$\Sigma_b := \pi\{(y, \eta)|\eta = (\xi, \tau), \tau = |\xi|_g^2, \alpha_0 \leq \tau \leq \beta_0\},$$

where $|\xi|_g^2 =< \xi, \xi >_g =: \xi_n^2 + r(x, \xi')$ denotes the inner product given by the metric $g$ and where, due to the strict concavity of the boundary we have $\partial_{x_n} r(x, \eta')|_{\partial M} < 0$.

*Definition* 6.3. We say that a point $\rho_0 = (y_0, \eta_0) \in T_b^*(\partial M\times\mathbb{R}) := T^*(\partial M\times\mathbb{R})\cup T^*(M\times\mathbb{R})$ doesn't belong to the $b$-wave front set $WF_b(v)$ of $v$ if there exists a $h$-pseudo differential operator of symbol $q(y, \eta, h)$ (respectively $q(y, \eta', h)$ if $\rho_0 \in T^*(\partial M \times \mathbb{R})$) with compact support in $(y, \eta)$, elliptic at $\rho_0$, and a smooth function $\phi \in C_0^\infty$ equal to 1 near $y_0$, such that for every $\sigma \geq 0$ the following holds

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|Op_h(q)\phi v\|_{H^\sigma(M\times\mathbb{R})} \leq C_N h^N.$$

We shall write $\rho_0 \notin WF_b(v)$.

**Proposition 6.7.** *(Elliptic regularity [73, Thm.3.1]) Let $q(y, \eta)$ a symbol such that $q = 0$ on $\Sigma_b$. Then for every $\sigma \geq 0$ we have*

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|Op_h(q)v\|_{H^\sigma(M)} \leq C_N h^N.$$

Proposition 6.7 is proved by Lebeau [73] for eigenfunctions of the Laplace operator, but the same arguments apply in this setting. From Proposition 6.7 and [73, Sections 2,3] it follows:

**Corollary 6.3.** *There exists a constant $D > 0$ such that*

$$WF_b(v) \subset \Sigma_b \cap \{\tau \in [\alpha_0, \beta_0], |\xi|_g \leq D\}.$$

**Proposition 6.8.** *Let $y_0 \notin pr_y(WF_b(v))$, where by $pr_y$ we mean the projection on the variable $y = (x, t)$. Then there exists $\phi \in C_0^\infty$, $\phi = 1$ near $y_0$ such that for every $\sigma \geq 0$ we have*

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|\phi v\|_{H^\sigma(M)} \leq C_N h^N.$$

*Proof.* Let $y_0 \notin pr_y(WF_b(v))$. It follows that $(y_0, \eta) \notin WF_b(v)$ for every $\eta \neq 0$. Fix $\eta = \eta_0$, then by Definition 6.3 there exists a symbols $q_0(y, \eta, h)$ with compact support in $(y, \eta)$ near $(y_0, \eta_0)$ and elliptic at $(y_0, \eta_0)$, and there exists $\phi_0 \in C_0^\infty$ equal to 1 in a neighborhood $U_0$ of $y_0$ such that for every $\sigma \geq 0$

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|Op_h(q_0)\phi v\|_{H^\sigma(M\times\mathbb{R})} \leq C_N h^N.$$

Let $V_0 \times W_0$ be an open neighborhood of $(y_0, \eta_0)$ such that $q_0$ is elliptic on $V_0 \times W_0$. Then $(U_0 \cap V_0) \times W_0$ is a neighborhood of $(y_0, \eta_0)$ where $\phi_0 = 1$ and where for every $\sigma \geq 0$

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|\phi v\|_{H^\sigma(U_0\cap V_0)} \leq C_N h^N.$$

From Corollary 6.3 we deduce that it is enough to consider only the case $\eta_0 \in pr_\eta(\Sigma_b)$, where by $pr_\eta$ we denote the projection on the variable $\eta = (\xi, \tau)$. Since this set is compact there exist $\eta^\alpha$, $\alpha \in \{1, .., N\}$ for some fixed $N \geq 1$ and for each $\eta^\alpha$ there exist symbols $q_\alpha$ elliptic on some neighborhoods $V_\alpha \times W_\alpha$ of $(y_0, \eta^\alpha)$ and smooth functions $\phi_\alpha \in C_0^\infty$ equal to 1 on some neighborhoods $U_\alpha$ of $y_0$, such that $pr_\eta(\Sigma_b) \subset \cup_{j=1}^N W_\alpha$. Let $\phi \in C_0^\infty$ be equal to 1 in an open neighborhood of $y_0$ strictly included in the intersection $\cap_{\alpha=1}^N (U_\alpha \cap V_\alpha)$ (which has nonempty interior) and supported in the compact set $\cap_{\alpha=1}^N \text{supp}(\phi_\alpha) \cap \bar{V}_\alpha$. Then $\phi$ satisfies Proposition6.8. □

Since the $b$-wave front set $WF_b$ is the union of broken bicharacteristics ([73]), Proposition 6.8 can be restated as

**Proposition 6.9.** *(Propagation of singularities) Let $y_0 = (x_0, t_0)$ and assume that there exists a neighborhood $V_0$ of $y_0$ such that there is no broken bicharacteristic starting at time $t = 0$ from the support supp$(v_0)$ of $v_0$ and arriving at $t = t_0$ in $V_0$. Then there exists $\phi \in C_0^\infty$ equal to 1 near $y_0$ such that for every $\sigma \geq 0$ we have*

$$\forall N \geq 0 \quad \exists C_N > 0 \quad \|\phi v\|_{H^\sigma(M)} \leq C_N h^N.$$

**Proposition 6.10.** *[21, Lemma B.7] Let $v(x,t) = e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0$ like before, $v_0 \in L^2(M)$ and let $Q$ be a $h$-pseudo-differential operator of order 0, $t_0 > 0$ and $\tilde{\psi} \in C_0^\infty((-2t_0, -t_0))$. Let $w$ denote the solution to*

$$\begin{cases} (ih\partial_t + h^2\Delta_g)w = ih\tilde{\psi}(t)Q(v), & \text{on} \quad M \times \mathbb{R}, \\ w|_{\partial M} = 0, \quad w|_{t<-2t_0} = 0. \end{cases} \tag{6.109}$$

*If $\rho_0 \in WF_b(w)$ then the broken bicharacteristic starting from $\rho_0$ has a nonempty intersection with $WF_b(v) \cap \{t \in \text{supp}(\tilde{\psi})\}$.*

### 6.5.2 Proof of Lemma 9.3

In this section $(M, \Delta_M)$ denotes either $(S, \Delta_S)$ or $(\Omega, \Delta_D)$, respectively. This notation will be used to refer both domains at the same time. Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be such that $\Delta_D\tilde{\chi} = \Delta_S\tilde{\chi}$.

Let $\varphi_0 \in C^\infty(\mathbb{R})$ be supported in the interval $[-4,4]$ and $\varphi \in C^\infty(\mathbb{R})$ be supported in $[-4,-1]\cup[1,4]$ such that for all $\xi \in \mathbb{R}$

$$\varphi_0(\xi) + \sum_{k\geq 1} \varphi(2^{-k}\xi) = 1.$$

If $\hat{\Psi}$ denotes the Fourier transform of $\Psi$, we write it using the preceding sum

$$\hat{\Psi}(\xi) = \hat{\Psi}(\xi)\Big(\varphi_0(\xi) + \sum_{k\geq 1} \varphi(2^{-k}\xi)\Big)$$

and denote $\phi_k \in \mathcal{S}(\mathbb{R})$ the functions such that $\hat{\phi}_0(\xi) = \hat{\Psi}(\xi)\varphi_0(\xi)$, $\hat{\phi}_k(\xi) = \hat{\Psi}(\xi)\varphi(2^{-k}\xi)$. We denoted by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions. Hence we have

$$\Psi(\lambda) = \sum_{k\in\mathbb{N}} \phi_k(\lambda), \quad \text{where} \quad \|\hat{\phi}_k\|_{L^\infty} = \|\hat{\Psi}(\xi)\varphi(2^{-k}\xi)\|_{L^\infty} \leq C_N 2^{-kN}, \quad \forall N \in \mathbb{N}. \tag{6.110}$$

For $k \in \mathbb{N}$ write

$$\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l} = \frac{1}{2\pi}\int_{\operatorname{supp}(\hat{\phi}_k)} e^{i\xi h\sqrt{-\Delta_M}}\tilde{\chi}v_{h,l}\hat{\phi}_k(\xi)d\xi. \tag{6.111}$$

On the support of $\hat{\phi}_k(\xi)$, $|\xi| \simeq 2^k$ and for $k \leq \frac{1}{2}\log_2(1/h)$ for example we see, by the finite speed of propagation of the wave operator, that on a time interval of size $2^k h \leq h^{1/2}$ we remain in a fixed neighborhood of the boundary of $\Omega$ where $\Delta_D$ coincides with $\Delta_S$, therefore we can introduce $\chi_1$ equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$ (independent of $k$, $h$) such that for every $k \leq \frac{1}{2}\log_2(1/h)$

$$\chi_1\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}v_{h,l} = \chi_1\phi_k(h\sqrt{-\Delta_\Omega})\tilde{\chi}v_{h,l}. \tag{6.112}$$

Since $v_{h,l,s} = \chi_1(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S))v_{h,l}$ and $v_{h,l} = \tilde{\chi}v_{h,l}$, we obtain, using (6.112)

$$v_{h,l,s} = \chi_1\Big(\sum_{k\geq\frac{1}{4}\log_2(1/h)} (\phi_k(h\sqrt{-\Delta_\Omega}) - \phi_k(h\sqrt{-\Delta_S}))\Big)\tilde{\chi}v_{h,l}. \tag{6.113}$$

In order to estimate the $L^q(I_l^h, L^r(\Omega))$ norm of $v_{h,l,s}$ it will be enough to estimate separately the norms of $\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}$ for $k \geq \frac{1}{4}\log_2(1/h)$ where $(M,\Delta_M) \in \{(\Omega,\Delta_D),(S,\Delta_S)\}$. Using the Cauchy-Schwartz inequality and the Sobolev embeddings gives

$$\begin{aligned}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))} &\leq Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,L^r(\Omega))}\\ &\leq Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))}\\ &\leq C_N h^{1/q}2^{-kN}\|\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))}, \quad \forall N \in \mathbb{N},\end{aligned} \tag{6.114}$$

where in the last line we used (6.110). We estimate the last term in (6.114) writing the Duhamel formula for $v_{h,l}$ only on $\Omega$ using the equation (9.17), since in this case the smoothing effect yields (see [98], [26] or the dual estimates of (6.104) in Proposition 6.6):

$$\|\tilde{\chi} v_{h,l}\|_{L^\infty(I_l^h, H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \leq C \|V_{h,l}\|_{L^2(I_l^h, H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}. \tag{6.115}$$

Since we consider here only large values $k \geq \frac{1}{4}\log_2(1/h)$, each $2^{-k}$ is bounded by $h^{1/4}$, therefore, after summing over $k$ we obtain

$$\|v_{h,l,s}\|_{L^q(I_l^h, L^r(\Omega))} \leq C_N h^{1/q+N/4} \|V_{h,l}\|_{L^2(I_l^h, H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}, \quad \forall N \in \mathbb{N}. \tag{6.116}$$

# 7 Square function and heat flow estimates on domains

This paper was written in collaboration with Fabrice Planchon.

The first purpose of this note is to provide a proof of the usual square function estimate on $L^p(\Omega)$. It turns out to follow directly from a generic Miklhin multiplier theorem obtained by Alexopoulos, which requires very little on the underlying space, besides Gaussian bounds on the heat kernel. We also provide a simple proof of a weaker version of the square function estimate, which is enough in most instances involving dispersive PDEs. Moreover, we obtain, by a relatively simple integration by parts, several useful $L^p(\Omega; H)$ bounds for the heat flow and its derivatives with values in a given Hilbert space $H$.

## 7.1 Introduction

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Let $\Delta_D$ denote the Laplace operator on $\Omega$ with Dirichlet boundary conditions, acting on $L^2(\Omega)$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$.

The first result reads as follows:

**Theorem 7.1.** *Let $f \in C^\infty(\Omega)$ and $\Psi \in C_0^\infty(\mathbb{R}^*)$ such that*

$$\sum_{j\in\mathbb{Z}} \Psi(2^{-2j}\lambda) = 1, \quad \lambda \in \mathbb{R}. \tag{7.1}$$

*Then for all $p \in (1,\infty)$ we have*

$$\|f\|_{L^p(\Omega)} \approx C_p \Big\| \Big( \sum_{j\in\mathbb{Z}} |\Psi(-2^{-2j}\Delta_D) f|^2 \Big)^{1/2} \Big\|_{L^p(\Omega)}, \tag{7.2}$$

*where the operator $\Psi(-2^{-2j}\Delta_D)$ is defined by* (7.30) *below.*

Readers who are familiar with functional spaces'theory will have recognized the equivalence $\dot{F}_p^{0,2} \approx L^p$, where the Triebel-Lizorkin space is defined using the right hand-side of (7.2) as a norm. In other words, $L^p(\Omega)$ and the Triebel-Lizorkin space $\dot{F}_p^{0,2}(\Omega)$ coincide. Such an equivalence (and much more !) is proven in [107, 108, 109], though one has to reconstruct it from several different sections (functional spaces are defined differently, only the inhomogeneous ones are treated, among other things). As such, the casual user with mostly a PDE background might find it difficult to reconstruct the argument for his own sake without digesting the whole theory. It turns out that the proof of (7.2) follows directly from the classical argument (in $\mathbb{R}^n$) involving Rademacher functions, provided that an appropriate Mikhlin-Hörmander multiplier theorem is available. We will provide details below.

A weaker version of Theorem 7.1 is often used in the context of dispersive PDEs:

**Theorem 7.2.** *Let $f \in C^\infty(\Omega)$, then for all $p \in [2, \infty)$ we have*

$$\|f\|_{L^p(\Omega)} \leq C_p |\Big(\sum_{j\in\mathbb{Z}} \|\Psi(-2^{-2j}\Delta_D)f\|^2_{L^p(\Omega)}\Big)^{1/2}. \tag{7.3}$$

The second part of the present note aims at giving a self-contained proof of (7.3) , with "acceptable" black boxes, namely complex interpolation and spectral calculus. In fact, if one accepts to replace the spectral localization by the heat flow, the proof can be made entirely self-contained, relying only on integration by parts. Our strategy to prove Theorem 7.2 is indeed to reduce matters to an estimate involving the heat flow, by proving almost orthogonality between spectral projectors and heat flow localization; this only requires basic parabolic estimates in $L^p(\Omega)$, together with a little help from spectral calculus.

*Remark* 7.1. For compact manifolds without boundaries, one may find a direct proof of (7.3) (with $\Delta_D$ replaced by the Laplace-Beltrami operator) in [27], which proceeds by reduction to the $\mathbb{R}^n$ case using standard pseudo-differential calculus. Our elementary approach provides an alternative direct proof. However, the true square function bound (7.2) holds on such manifolds, as one has a Mikhlin-Hörmander theorem from [90].

*Remark* 7.2. One can also adapt all proofs to the case of Neumann boundary conditions, provided special care is taken of the zero frequency (note that on an exterior domain, a decay condition at infinity solves the issue). The Gaussian bound which is required later holds in the Neumann case, see [41, 40].

*Remark* 7.3. As mentioned before, Theorem 7.2 is useful, among other things, when dealing with $L^p$ estimates for wave or dispersive evolution equations. For such equations, one naturally considers initial data in Sobolev spaces, and spectral localization conveniently reduces matters to data in $L^2$, and helps with finite speed of propagation arguments. One however wants to sum eventually over all frequencies in $l^2$, if possible without loss. Recent examples on domains may be found in [61] or [87], as well as in [74].

We now state estimates involving directly the heat flow, which will be proved by direct arguments. It should be noted that for nonlinear applications, it is quite convenient to have bounds on derivatives of spectral multipliers, and such bounds do not follow immediately from the multiplier theorem from [3]. We consider the linear heat equation on $\Omega$ with Dirichlet boundary conditions and initial data $f$

$$\partial_t u - \Delta_D u = 0, \text{ on } \Omega \times \mathbb{R}_+; \quad u|_{t=0} = f \in C^\infty(\Omega); \quad u|_{\partial\Omega} = 0. \tag{7.4}$$

We denote the solution $u(t,x) = S(t)f(x)$, where we set $S(t) = e^{t\Delta_D}$. For the sake of simplicity $\Delta_D$ has constant coefficients, but the same method applies in the case when the coefficients belong to a bounded set of $C^\infty$ and the principal part is uniformly elliptic (one may lower the regularity requirements on both the coefficients and the boundary, and a nice feature of the proofs which follow is that counting derivatives is relatively straightforward).

Let us define two operators which are suitable heat flow versions of $\Psi(-2^{-2j}\Delta_D)$:

$$Q_t = \sqrt{t}\nabla S(t) \text{ and } \mathbf{Q}_t \stackrel{\text{déf}}{=} t\partial_t S(t). \tag{7.5}$$

**Theorem 7.3.** *Let $1 < p < +\infty$, then we have*

$$\|f\|_{L^p(\Omega)} \approx c_{,p} \Big\| \Big( \int_0^\infty |Q_t f|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^p(\Omega)}, \tag{7.6}$$

*which implies, for $p \in [2, +\infty)$,*

$$\|f\|_{L^p(\Omega)} \leq C_p \Big( \int_0^\infty \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \Big)^{1/2}, \tag{7.7}$$

*and $Q_t$ may be replaced by $\mathbf{Q}_t$ in both statements.*

Notice that there is no difficulty to define $Q_t f$ or $\mathbf{Q}_t f$ as distributional derivatives for $f \in L^p(\Omega)$, while simply defining $\Psi(-2^{-2j}\Delta_D)$ on $L^p(\Omega)$ is already a non trivial task. The purpose of the next Proposition is to prove that both operators are in fact bounded on $L^p(\Omega)$.

**Proposition 7.1.** *Let $1 < p < +\infty$. The operators $Q_t$, $\mathbf{Q}_t$ are bounded on $L^p(\Omega)$, uniformly in $t \geq 0$. Moreover $\mathbf{Q}_t$ is bounded on $L^1(\Omega)$ and $L^\infty(\Omega)$.*

For practical applications, one may need a vector valued version of Proposition 7.2. Let us consider now $u = (u_l)_{l \in \{1,..,N\}}$ for $N \geq 2$, where each $u_l$ solves (7.4) with Dirichlet condition and initial data $f_l$. Let $H$ be the Hilbert space with norm $\|u\|_H^2 = \sum_l |u_l|^2$, and $L^p(\Omega; H)$ the Hilbert valued Lebesgue space. Then we have

**Proposition 7.2.** *Let $1 < p < +\infty$. The operators $Q_t$, $\mathbf{Q}_t$ are bounded on $L^p(\Omega; H)$, uniformly in $t \geq 0$ and $N$. Moreover $\mathbf{Q}_t$ is bounded on $L^1(\Omega; H)$ and $L^\infty(\Omega; H)$.*

*Remark* 7.4. One may therefore extend the finite dimensional case to any separable Hilbert space. The typical setting would be to consider the solution $u$ to the heat equation with initial data $f(x, \theta) \in L^2_\theta = H$. Notice that the Hilbert valued bound does not follow from the previous scalar bound; however the argument is essentially the same, replacing $| \cdot |$ norms by Hilbert norms.

*Remark* 7.5. A straightforward consequence of Propositions 7.2, 7.1 is that the Riesz transforms $\partial_j(-\Delta_D)^{-\frac{1}{2}}$ are continuous on Besov spaces defined by the RHS of (7.7); these spaces are equivalent to the ones defined by the RHS of (7.3), see Remark 7.10 later on.

Alternatively, one can derive all the (scalar, at least) results on the heat flow from adapting to the domain case the theory which ultimately led to the proof of the Kato conjecture ([10, 9]). Such a possible development is pointed out by P. Auscher in [8] (chap. 7, p. 66) and was originally our starting point; eventually we were led to the elementary approach we present here, but we provide a sketch of an alternate proof in the next remark, which was kindly outlined to us by Pascal Auscher.

*Remark* 7.6. The main drawback from (7.7) is the presence of $\nabla S(t)$ on the right hand-side: one is leaving the functional calculus of $\Delta_D$, and in fact for domains with Lipschitz boundaries the operator $\nabla S(t)$ may not even be bounded. As such, a suitable alternative is to replace $\nabla S(t)$ by $\sqrt{\partial_t} S(t)$. Then the square function estimate may be obtained following [8] as follows:

- prove that the associated square function in time is bounded by the $L^p$ norm, for all $1 < p \leq 2$, essentially following step 3 in chapter 6, page 55 in [8]. This requires very little on the semi-group, and Gaussian bounds on $S(t)$ and $\partial_t S(t)$ ([42]) are more than enough to apply the weak $(1, 1)$ criterion from [8] (Theorem 1.1, chapter 1). Moreover, the argument can be extended to domains with Lipschitz boundaries, assuming the Laplacian is defined through the associated Dirichlet form;
- by duality, we get the square function bound for $p > 2$ (step 5, page 56 in [8]);
- from now on one proceeds as in the remaining part of our paper to obtain the bound with spectral localization, and almost orthogonality (7.17) is even easier because we stay in the functional calculus. One has, however, to be careful if one is willing to extend this last step to Lipschitz boundaries, as this would most likely require additional estimates on the resolvent to deal with the $\Delta_j$.

## 7.2 From a Mikhlin multiplier theorem to the square function

The following "Fourier multiplier" theorem is obtained in [3] under very weak hypothesis on the underlying manifold (see also [4] for a specific application to Markov chains, and [105] for a version closer to the sharp Hörmander's multiplier theorem, under suitable additional hypothesis, all of which are verified on domains). For $m \in L^\infty(\mathbb{R}^+)$, one usually defines the operator $m(-\Delta_D)$ on $L^2(\Omega)$ through the spectral measure $dE_\lambda$:

$$m(-\Delta_D) = \int_0^{+\infty} dE_\lambda, \tag{7.8}$$

and $m(-\Delta_D)$ is bounded on $L^2$.

*Remark* 7.7. One may alternatively use the Dynkin-Helffer-Sjöstrand formula as in the Appendix, and both definitions are known to coincide on $L^2(\Omega)$. However, the Dynkin-Helffer-Sjöstrand formula seems to be restricted to defining $m(-\Delta_D)$ for functions $m$ which exhibit slightly more decay than required in the next theorem, at least if one proceeds as exposed in the Appendix.

**Theorem 7.4** ([3]). *Let $m \in C^N(\mathbb{R}^+)$, $N \in \mathbb{N}$ and $N \geq n/2 + 1$, such that*

$$\sup_{\xi, k \leq N} |\xi \partial_\xi^k m(\xi)| < +\infty. \tag{7.9}$$

*Then the operator defined by* (7.8) *extends to a continuous operator on $L^p(\Omega)$, and sends $L^1(\Omega)$ to weak $L^1(\Omega)$.*

In order to use the argument of [3], we need the Gaussian upper bound on the heat kernel, which is provided in our case by [41]. Once we have Theorem 7.4, all we need to do to prove Theorem 7.1 is to follow Stein's classical proof from [99] [1], and we recall it briefly for the convenience of the reader. Let us introduce the Rademacher functions, which are defined as follows:

— the function $r_0(t)$ is defined by $r_0(t) = 1$ on $[0, 1/2]$ and $r_0(t) = -1$ on $(1/2, 1)$, and then extended to $\mathbb{R}$ by periodicity;

— for $m \in \mathbb{N} \setminus \{0\}$, $r_m(t) = r_0(2^m t)$.

Their importance is outlined by the following inequalities (see the Appendix in [99]),

$$c_p \| \sum_m a_m r_m(t) \|_{L^p_t} \leq (\sum_m |a_m|^2)^{\frac{1}{2}} \leq C_p \| \sum_m a_m r_m(t) \|_{L^p_t}. \tag{7.10}$$

Now, define

$$m^{\pm}(t, \xi) = \sum_{j=0}^{+\infty} r_j(t) \Psi_{\pm j}(\xi),$$

where $\Psi_j$ was defined in the introduction. A straightforward computation proves that the bound (7.9) holds for $m^{\pm}(t, \xi)$. Therefore,

$$\|m^{\pm}(t, -\Delta_D) f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)};$$

integrating in time over $[0, 1]$, exchanging space and time norms, and using (7.10),

$$\|m^{\pm}(t, -\Delta_D) f\|_{L^p(\Omega) L^2(0,1)} \approx \| \big( \sum_{j=0}^{\pm\infty} |\Psi(-2^{-2j} \Delta_D) f|^2 \big)^{\frac{1}{2}} \|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.$$

This proves one side of the equivalence in (7.2): the other side follows from duality, once we see the above estimate as an estimate from $L^p(\Omega)$ to $L^p(\Omega; l^2)$, which maps $f$ to $(\Psi(-2^{-2j} \Delta_D) f)_{j \in \mathbb{Z}}$.

## 7.3 Heat flow estimates

In order to prove Proposition 7.3 we need the following lemma.

**Lemma 7.1.** *For all* $1 \leq p \leq +\infty$*, we have*

$$\|S(t) f\|_{L^p(\Omega)} \to_{t \to \infty} 0, \tag{7.11}$$

$$\sup_{t \geq 0} \|S(t) f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}. \tag{7.12}$$

*Moreover,*

$$\| \sup_{t \geq 0} |S(t) f| \|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}. \tag{7.13}$$

1. we thank Hart Smith for bringing this to our attention

*Proof:* The estimate (7.12) clearly follows from (7.13), which in turn is a direct consequence of the Gaussian nature of the Dirichlet heat kernel, see [41]. The same Gaussian estimate implies (7.11). However we do not need such a strong fact to prove (7.12), which will follow from the next computation as well (see (7.14)) when $1 < p < +\infty$. Estimate (7.11) can also be obtained through elementary arguments. We defer such a proof to the end of the section.

### 7.3.1 Proof of Theorem 7.3

If $p = 2$ the proof is nothing more than the energy inequality, combined with (7.11). In fact, for $p = 2$, we have equality in (7.6) with $C_2 = 2$. We now take $p = 2m$ where $m \geq 2$. Multiplying equation (7.4) by $\bar{u}|u|^{p-1}$ and taking the integral over $\Omega$ and $[0, T]$, $T > 0$ yields, taking advantage of the Dirichlet boundary condition,

$$\frac{1}{p}\int_0^T \partial_t \|u\|^p_{L^p(\Omega)} dt + \int_0^T \int_\Omega |\nabla u|^2 |u|^{p-2} dx dt + \\ + \frac{(p-2)}{2}\int_0^T \int_\Omega (\nabla(|u|^2))^2 |u|^{p-4} dx dt = 0, \quad (7.14)$$

from which we can estimate either $\|u\|^p_{L^p(\Omega)}(T) \leq \|f\|^p_{L^p(\Omega)}$ (which is (7.12)) or

$$\|f\|^p_{L^p(\Omega)} \leq \|u\|^p_{L^p(\Omega)}(T) + p(p-1)\int_0^T \int_\Omega |\nabla u|^2 |u|^{p-2} dx dt.$$

Letting $T$ go to infinity and using (7.11) from Lemma 7.1 and Hölder inequality we find

$$\|f\|^p_{L^p(\Omega)} \leq p(p-1)\Big(\int_\Omega \Big(\int_0^\infty |\nabla u|^2\, dt\Big)^{\frac{p}{2}} dx\Big)^{\frac{2}{p}} \Big(\int_\Omega (\sup_t |u|^{p-2})^{\frac{p}{p-2}} dx\Big)^{\frac{p-2}{p}}.$$

The proof follows using again Lemma 7.1, as

$$\|f\|^p_{L^p(\Omega)} \leq C_p \|\Big(\int_0^\infty |\nabla u|^2 dt\Big)^{\frac{1}{2}}\|_{L^p(\Omega)} \Big(\|\sup_{t\geq 0} |u|\|_{L^p(\Omega)}\Big)^{p-2}.$$

Note that we may prove the weaker part, (7.7), without assuming the maximal in time bound, by reversing the order of integration in our argument. This would keep the argument for heat square functions essentially self-contained, without any need for Gaussian bounds on the heat kernel.

*Remark* 7.8. We do not claim novelty here: our argument follows closely (a dual version of) the proof of a classical square function bound for the Poisson kernel in the whole space, see [99].

We have proved one side of the equivalence in (7.6) involving the $Q_t$ square function, in the range $2 \leq p < +\infty$; we now prove the other side, by duality. Let $\phi \in L^q(\Omega)$, with $1/q = 1 - 2/p$, and consider

$$I = \int_\Omega \Big( \int_0^{+\infty} |\nabla u|^2 \, dt \Big) \phi(x) \, dx.$$

Without any loss of generality, we may assume $\phi \geq 0$. On the other hand, let $v = |\nabla u|^2$, then

$$\partial_t v - \Delta v = -2|\nabla^2 u|^2,$$

and one checks easily that $\partial_n v = 0$ on $\partial\Omega$. Let $S_n(t)$ be the solution to the heat equation on $\Omega$ with Neumann boundary condition, by comparing $v$ and $S_n(t/2)v(t/2)$ (formally, take the difference, multiply by the positive part and integrate by parts) we have

$$0 \leq v \leq S_n(t/2)v(t/2) = S_n(t/2)|\nabla u(t/2)|^2,$$

and therefore

$$I \leq 2 \int_\Omega \int_0^{+\infty} |\nabla u|^2 \, S_n(t)\phi \, dxdt. \tag{7.15}$$

Now, we also have

$$\partial_t u^2 - \Delta_D u^2 = -2|\nabla u|^2,$$

and therefore

$$I \leq - \int_\Omega \int_0^{+\infty} (\partial_t - \Delta)(u^2) S_n(t)\phi \, dxdt.$$

From $(\partial_t - \Delta)(u^2 S_n(t)\phi) = -2\nabla(u^2) \cdot \nabla S_n(t)\phi$, we get

$$I \leq \int_\Omega 4 \sup_t |u| \Big( \int_0^{+\infty} |Q_t u|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \Big( \int_0^{+\infty} |\nabla S_n(t)\phi|^2 \, dt \Big)^{\frac{1}{2}}.$$

The bound we already proved with $Q_t$ can easily be reproduced with $S(t)$ replaced by $S_n(t)$, and therefore, provided $q \leq 2$, we may use the dual bound on the square function of $\phi \in L^q(\Omega)$ and conclude by Hölder, using (7.13) on the first factor. The condition on $q$ translates into $p \geq 4$, and the remaining $2 < p < 4$ are handled by interpolation.

*Remark* 7.9. Actually, we may directly bound $S_n(t)$ by a Gaussian in (7.15), extend $\phi$ by 0 outside $\Omega$, and use the heat square function bounds in $\mathbb{R}^n$. This provides a direct argument, irrespective of the value of $p$.

It remains to prove the equivalence between the $Q_t$ square function and the $\mathbf{Q}_t$ square function. For this, we repeat the duality argument but we replace $|\nabla u|^2$ by $t|\partial_t u|^2$. Notice that $\partial_t u$ is also a solution to the heat equation with Dirichlet boundary condition, and if $w = |\partial_t u|^2$,

$$(\partial_t - \Delta)w = -2|\nabla \partial_t u|^2.$$

Therefore, comparing $w$ and $S(t/2)w(t/2)$,

$$0 \leq |\partial_t u|^2 \leq S(t/2)|\partial_t u(t/2)|^2,$$

and

$$J = \int_\Omega \int_0^{+\infty} |\partial_t u|^2 t\phi\, dxdt \leq 2 \int_\Omega \int_0^{+\infty} |\partial_t u|^2 tS(t)\phi\, dxdt.$$

Now,

$$\begin{aligned} J &\leq 2 \int_\Omega \int_0^{+\infty} t\partial_t u \Delta u S(t)\phi\, dxdt \\ &\leq -\int_\Omega \int_0^{+\infty} t\partial_t |\nabla u|^2 S(t)\phi\, dxdt - 2\int_\Omega \int_0^{+\infty} t\partial_t u \nabla u \nabla S(t)\phi\, dxdt \\ &\leq \int_\Omega \int_0^{+\infty} |\nabla u|^2 (1 + t\partial_t) S(t)\phi\, dxdt + 2\int_\Omega \int_0^{+\infty} |\mathbf{Q}_t u Q_t u Q_t \phi| \frac{dt}{t}\, dx \end{aligned}$$

from which we can easily conclude by Hölder (using Lemma 7.2 to bound $t\partial_t S(t)\phi$). Duality takes care of the reverse bound, and this concludes the proof of Theorem 7.3, except for the equivalence between the $Q_t$ and $\mathbf{Q}_t$ Besov norms in (7.7); we defer this to the end of the next subsection.

Notice that, at this point, we proved Theorem 7.2, but with the $\Psi$ operator replaced by the gradient heat kernel and the discrete parameter $2^{-2j}$ by the continuous parameter $t$. The rest of this section is devoted to proving the equivalence between the Besov norms which are defined by the heat kernel or the spectral localization.

**Lemma 7.2.** *Let $1 \leq p \leq +\infty$. We have the following equivalence between dyadic and continuous versions of the Besov norm:*

$$\frac{3}{4} \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)} \leq \int_0^\infty \|Q_t f\|^2_{L^p(\Omega)} \frac{dt}{t} \leq 3 \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)}.$$

This follows at once from factoring the semi-group: for $2^{-2j} \leq t \leq 2^{-2(j-1)}$, write $S(t) = S(t - 2^{-2j})S(2^{-2j})$ and use (7.12). We now turn to the direct proof of Theorem 7.2 from the heat flow version. Let $\Psi \in C_0^\infty(\mathbb{R}^*)$ satisfying (7.1) and denote $\Delta_j f \overset{\text{déf}}{=} \Psi(2^{-2j}\Delta_D) f$, where $\Psi(2^{-2j}\Delta_D) f$ is given by the Dynkin-Helffer-Sjöstrand formula (see the Appendix, (7.30)). From Proposition 7.3 and Lemma 7.2 we have

$$\|f\|_{L^p(\Omega)} \leq 3C_p \Big( \sum_{k \in \mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)} \Big)^{1/2} \tag{7.16}$$

and we will show that (7.16) implies (7.2): it suffices to prove the following almost orthogonality property between localization operators $\Delta_j$ and $Q_{2^{-2k}}$:

$$\forall k, j \in \mathbb{Z}, \qquad \|Q_{2^{-2k}} \Delta_j f\|_{L^p(\Omega)} \lesssim 2^{-|j-k|} \|\Delta_j f\|_{L^p(\Omega)}. \tag{7.17}$$

Then, from $(2^{-|j-k|})_k \in l^1$ and $(\|\Delta_j f\|_{L^p(\Omega)})_j \in l^2$ we estimate

$$\sum_{k\in\mathbb{Z}} \|Q_{2^{-2k}} f\|^2_{L^p(\Omega)} = \sum_{k\in\mathbb{Z}} \| \sum_{j\in\mathbb{Z}} Q_{2^{-2k}} \Delta_j f\|^2_{L^p(\Omega)} \tag{7.18}$$

as an $l^1 * l^2$ convolution and conclude using Lemma 7.2. It remains to show (7.17):

— for $k < j$ we write

$$Q_{2^{-2k}} \Delta_j f = 2^{3/2} 2^{-2(j-k)} \Big(2^{-(2k+1)/2} \nabla S(2^{-(2k+1)})\Big) \Big(2^{-(2k+1)} \Delta_D S(2^{-(2k+1)})\Big) \breve{\Psi}(-2^{-2j}\Delta_D)\Psi(-2^{-2j}\Delta_D) f,$$

where we set $\breve{\Psi}(\lambda) \overset{\text{déf}}{=} \frac{1}{\lambda}\tilde{\Psi}(\lambda)$, and $\tilde{\Psi} \in C_0^\infty$, $\tilde{\Psi} = 1$ on $\mathrm{supp}\Psi$. By Lemma 7.2, the operators $Q_{2^{-(2k+1)}} = 2^{-(2k+1)/2} \nabla S(2^{-(2k+1)})$ and $\mathbf{Q}_{2^{-(2k+1)}} = 2^{-(2k+1)} \Delta_D S(2^{-(2k+1)})$ are bounded on $L^p(\Omega)$ and we obtain (7.17) using Corollary 7.1 for $\breve{\Psi}$.

— for $k \geq j$ we set $\Psi_1(\xi) = \tilde{\Psi}(\xi)\exp(\xi)$, $\Psi_2(\xi) = \Psi(\xi)$, and we use again Lemma 7.6 to write (slightly abusing the notation as $2^{-2k} - 2^{-2j} < 0$)

$$S(2^{-2k} - 2^{-2j})\Delta_j f = S(2^{-2k})\Psi_1(-2^{-2j}\Delta_D)\Psi_2(-2^{-2j}\Delta_D) f. \tag{7.19}$$

Then

$$Q_{2^{-2k}} \Delta_j f = 2^{-(k-j)} \Big(2^{-j} \nabla S(2^{-2j})\Big) \Big(S(2^{-2k} - 2^{-2j})\Delta_j f\Big),$$

and using again Lemma 7.2 we see that the operator $2^{-j}\nabla S(2^{-2j})$ is bounded while the remaining operator (7.19) is bounded by Corollary 7.1. This ends the proof.

*Remark* 7.10. One may prove a similar bound with $Q_{2^{-2k}}$ and $\Delta_j$ reversed, either directly or by duality. Hence Besov norms based on $\Delta_j$ or $Q_{2^{-2k}}$ are equivalent.

### 7.3.2 Proof of Proposition 7.2

For $\mathbf{Q}_t$, boundedness on all $L^p$ spaces, including $p = 1, +\infty$, follows once again from a Gaussian upper bound on $\partial_t S(t)$ (see [42] or [44]). However the subsequent Gaussian bound on the gradient $\nabla_x S(t)$ in [42] is a direct consequence of the Li-Yau inequality, which holds only inside convex domains. We were unable to find a reference which would provide the desired bound for $Q_t$ in the context of the exterior domain. Therefore we provide an elementary detailed proof for $Q_t$. Furthermore, we only deal with $1 < p < 2$ or powers of two, $p = 2^m$, $m \in \mathbb{N}^*$: complex interpolation takes care of remaining values of $p$, though one could adapt the following argument to generic values $p > 2$, at the expense of lengthier computations.

Set $v(x,t) = (v_1, .., v_n)(x,t) := Q_t f = t^{1/2}\nabla u(x,t)$ and assume without loss of generality that $v_j$ are real: we multiply the equation satisfied by $v$ by $v|v|^{p-2}$, where $|v|^2 = \sum_{j=1}^n v_j^2$, and integrate over $\Omega$,

$$\partial_t\left(\frac{1}{p}\|v\|^p_{L^p(\Omega)}\right)-\sum_{j=1}^n\int_{\partial\Omega}((\overrightarrow{\nu}\cdot\nabla)v_j)\cdot v_j|v|^{p-2}d\sigma+ \\ +\int_\Omega|\nabla v|^2|v|^{p-2}\,dx+\frac{(p-2)}{2}\int_\Omega\nabla(|v|^2)|v|^{p-4}\,dx=\frac{1}{2t}\|v\|^p_{L^p(\Omega)},\quad(7.20)$$

where $\overrightarrow{\nu}$ is the outgoing unit normal vector to $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$. We claim that the second term in the left hand side vanishes: in fact we write

$$\sum_{j=1}^n\int_{\partial\Omega}(\overrightarrow{\nu}\cdot\nabla v_j)\cdot v_j|v|^{p-2}d\sigma= \\ =\frac{t^{p/2}}{2}\int_{\partial\Omega}\partial_\nu(|\partial_\nu u|^2+|\nabla_{tang}u|^2)(|\partial_\nu u|^2+|\nabla_{tang}u|^2)^{(p-2)/2}d\sigma,\quad(7.21)$$

and from $u|_{\partial\Omega}=0$ the time and tangential derivative $(\partial_t,\nabla_{\text{tang}})u|_{\partial\Omega}$ vanishes; furthermore, using the equation, $\partial_\nu^2 u=0$ on $\partial\Omega$.

*Remark* 7.11. Notice that while this term does not vanish with Neumann boundary conditions, it will be a lower order term (like $|\nabla u|^2$ on $\partial\Omega$) which can be controled by the trace theorem.

Now, if $1<p<2$, multiply by $\|v\|^{2-p}_{L^p(\Omega)}$ and integrate over $[0,T]$,

$$\|v\|^2_{L^p(\Omega)}(T)\lesssim\int_0^T\|Q_tf\|^2_{L^p(\Omega)}\frac{dt}{t}\lesssim\|f\|_p^2,$$

where the last inequality is the dual of (7.6). Hence we are done with $1<p<2$.

*Remark* 7.12. We ignored the issue of $v$ vanishing in the third term in (7.20). This is easily fixed by replacing $|v|^{p-2}$ by $(\sqrt{\varepsilon+|v|^2})^{p-2}$ and proceeding with the exact same computation. Then let $\varepsilon$ go to 0 after dropping the positive term on the left handside of (7.20).

Now let $p=2^m$ with $m\geq 1$: we proceed directly by integrating (7.20) over $[0,T]$, to get

$$\frac{1}{p}\|v\|^p_{L^p(\Omega)}(T)+\int_0^T\int_\Omega|\nabla v|^2|v|^{p-2}\,dxdt+ \\ +\frac{(p-2)}{2}\int_0^T\int_\Omega|\nabla(|v|^2)|^2|v|^{p-4}\,dxdt=\int_0^T\frac{1}{2t}\|v\|^p_{L^p(\Omega)}\,dt.\quad(7.22)$$

On the other hand (recall (7.14)),

$$\frac{1}{p}\|u\|^p_{L^p(\Omega)}(T)+(p-1)\int_0^T\int_\Omega|\nabla u|^2|u|^{p-2}\,dxdt=\frac{1}{p}\|f\|^p_{L^p(\Omega)}.\quad(7.23)$$

If $p = 2$ the estimates are trivial since from (7.22), (7.23),

$$\frac{1}{2}\|v\|^2_{L^2(\Omega)}(T) \le \int_0^T \frac{1}{2t}\|v\|^2_{L^2(\Omega)}dt = \frac{1}{2}\int_0^T \|\nabla u\|^2_{L^2(\Omega)}\,dt \le \frac{1}{4}\|f\|^2_{L^2(\Omega)}.$$

Now, let $p \ge 4$; for convenience, denote by $J$ the second integral in the left hand-side of (7.22) (notice that the third integral is bounded from above by $J$), hence

$$J = \int_0^T \int_\Omega |\nabla^2 u|^2 |\nabla u|^{p-2} t^{\frac{p}{2}}\,dxdt = \int_0^T \int_\Omega (\sum_{i,j} |\partial^2_{i,j} u|^2)(\sum_j |\partial_j u|^2)^{\frac{(p-2)}{2}} t^{\frac{p}{2}}\,dxdt,$$

and set

$$I_k = \int_0^T \int_\Omega |\nabla u|^{2k}|u|^{p-2k}t^{k-1}\,dxdt \text{ where } 2 \le 2k \le p. \tag{7.24}$$

For our purposes, it suffices to estimate the right hand-side of (7.22), which rewrites

$$\frac{1}{2}\int_0^T t^{\frac{p}{2}-1}\|\nabla u\|^p_{L^p(\Omega)}\,dt = \frac{1}{2}I_{\frac{p}{2}}. \tag{7.25}$$

Integrate by parts the inner (space) integral in $I_k$, the boundary term vanishes and collecting terms,

$$\int_\Omega \nabla u \nabla u |\nabla u|^{2(k-1)}|u|^{p-2k}\,dx \quad \le \quad \frac{(2k-1)}{(p-2k+1)}\int_\Omega |\nabla^2 u||\nabla u|^{2k-2}|u|^{p-2k+1}dx. \tag{7.26}$$

By Cauchy-Schwarz the integral in the right hand side of (7.26) is bounded by

$$\Big(\int_\Omega |\nabla^2 u|^2|\nabla u|^{p-2}dx\Big)^{1/2}\Big(\int_\Omega |\nabla u|^{4k-4-(p-2)}|u|^{2p+2-4k}dx\Big)^{1/2},$$

therefore for $k \ge \frac{p}{4}+1$ we have

$$I_k \lesssim \frac{(2k-1)}{(p-2k+1)}J^{\frac{1}{2}}I^{\frac{1}{2}}_{2k-\frac{p}{2}-1}.$$

We aim at controlling $I_m$ by $J^{1-\eta}I_1^\eta$, for some $\eta > 0$ which depends on $m$ (notice that when $p = 4$, which is $m = 2$, we are already done, using $k = 2$ !). Set $k = \frac{p}{2} - (2^j - 1)$ with $j \le m-2$,

$$I_{2^{m-1}-(2^j-1)} \le \frac{(2^m - (2^{j+1}-1))}{(2^{j+1}-1)}J^{\frac{1}{2}}I^{\frac{1}{2}}_{2^{m-1}-(2^{j+1}-1)},$$

and iterating $m-2$ times, we finally control $I_{\frac{p}{2}}$ by $J^{1-\eta}I_1^\eta$, which proves that $Q_t$ is bounded on $L^p(\Omega)$.

We now proceed to obtain boundedness of $\mathbf{Q}_t$ on $L^p(\Omega)$ from the $Q_t$ bound; this is worse than using the Gaussian properties of its kernel, as the constants blow up when $p \to 1, +\infty$. It is, however, quite simple. By duality $Q_t^\star$ is bounded on $L^p(\Omega)$, and

$$\mathbf{Q}_t = t\partial_t S(t) = tS(\frac{t}{2})\Delta S(\frac{t}{2}) = 2\sqrt{\frac{t}{2}}S(\frac{t}{2})\nabla\cdot\sqrt{\frac{t}{2}}\nabla S(\frac{t}{2}) = 2Q^\star_{\frac{t}{2}}Q_{\frac{t}{2}},$$

and we are done with Lemma 7.2.

From the previous decomposition, we also obtain

$$\|\mathbf{Q}_t f\|_{L^p(\Omega)} \lesssim \|Q_t f\|_{L^p(\Omega)},$$

which implies that any Besov norm defined with $\mathbf{Q}_t$ is bounded by the corresponding norm for $Q_t$. The reverse bound is true as well, though slightly more involved. We provide the proof for completeness. Consider $f, h \in C_0^\infty(\Omega)$ and $\langle f, g\rangle = \int_\Omega fg$. Then

$$\begin{aligned}
\langle f, g\rangle &= -\int_0^{+\infty} \langle \partial_t S(t)f, h\rangle\, dt = -2\int_0^{+\infty} \langle \partial_t S(t)f, S(t)h\rangle\, dt \\
&= 2\int_{t<s} \langle \partial_t S(t)f, \partial_s S(s)h\rangle\, dtds = 4\int_0^{+\infty} \langle \nabla S(s)\partial_t S(t)f, \nabla S(s)h\rangle\, dtds \\
&\lesssim \int_s \left\| \int_0^s \nabla S(t)\partial_s S(s)f\, dt\right\|_p \|\nabla S(s)h\|_{p'}\, ds \lesssim \int_s \sqrt{s}\|\partial_s S(s)f\|_p \|\nabla S(s)h\|_{p'}\, ds
\end{aligned}$$

where we used our bound on $\sqrt{t}\nabla S(t)$ at fixed $t$. Then

$$\langle f, h\rangle \lesssim \int_s \|\mathbf{Q}_s f\|_p \|Q_s h\|_{p'} \frac{ds}{s}$$

from which we are done by Hölder.

### 7.3.3 Proof of Proposition 7.1

Let us consider now the vector valued case $u = (u_l)_{l\in\{1,..,N\}}$ for $N \geq 2$, where each $u_l$ solves (7.4) with Dirichlet condition and initial data $f_l$. For the sake of simplicity we consider only real valued $u_l$, and write

$$|u|^2 = \sum_{l=1}^N u_l^2, \;\; |\nabla u_l|^2 = \sum_{j=1}^n (\partial_j u_l)^2, \;\; |\nabla u|^2 = \sum_{j=1}^n \sum_{l=1}^N (\partial_j u_l)^2$$

Notice that $n$ is the spatial dimension and is fixed through the argument: hence all constants may depend implicitely on $n$, while $N$ is the dimension of $H$. For $p = 1, +\infty$, the boundedness of $\mathbf{Q}_t$ follows from the Gaussian character of the time derivative heat kernel, which is diagonal on $H$.

We proceed with $Q_t$. Multiplying the equation satisfied by $u_l$ by $u_l|u|^{p-2}$, integrating over $\Omega$ and summing up we immediately get (7.14). We now proceed to obtain bounds for $v(x,t) = (v_l(x,t))_l$, where $v_l(x,t) = t^{1/2}\nabla u_l(x,t)$. Multiplying the equation satisfied by $v_l$

by $v_l|v|^{p-2}$ where $|v|^2 = t|\nabla u|^2$, summing up over $l$ and taking the integral over $\Omega$ yields

$$\frac{1}{p}\|v\|^p_{L^p(\Omega)}(T) + \sum_{l=1}^N \sum_{j=1}^n \int_0^T \int_\Omega |\nabla(\partial_j u_l)|^2 |\nabla u|^{p-2}\, dxdt+$$
$$+ \frac{(p-2)}{4}\int_0^T \int_\Omega |\nabla|\nabla u|^2|^2 |\nabla u|^{p-4} t^{p/2}\, dxdt = \int_0^T \|\nabla u\|^p_{L^p(\Omega)} t^{p/2-1}\, dt = \frac{1}{2} I_{\frac{p}{2}}, \quad (7.27)$$

where $|\nabla(\partial_j u_l)|^2 = \sum_{i=1}^n (\partial^2_{i,j} u_l)^2$, $|\nabla u|^2 = \sum_{l=1}^N \sum_{j=1}^n (\partial_j u_l)^2$. Notice again that the boundary term vanishes. Denote the last two integrals in the left hand side by $J_1$, $J_2$. Like before, we perform integrations by parts in $I_k$ defined in (7.24) to obtain

$$\int_\Omega |\nabla u|^{2k} |u|^{p-2k}\, dx = -\sum_{l=1}^N \int_\Omega u_l \Delta u_l |\nabla u|^{2(k-1)} |u|^{p-2k}\, dx-$$
$$-(k-1)\sum_{l=1}^N \int_\Omega u_l \nabla u_l \nabla(|\nabla u|^2) |\nabla u|^{2(k-2)} |u|^{p-2k}\, dx-$$
$$-(p-2k)\sum_{i=1}^n \int_\Omega (\sum_{l=1}^N \partial_i u_l u_l)^2 |u|^{p-2k-2}\, dx. \quad (7.28)$$

For $k \geq \frac{p}{4}+1$ we estimate the first term in the right hand side of (7.28) by

$$\int_\Omega (\sum_{l=1}^N u_l^2)^{1/2} (\sum_{l=1}^N (\Delta u_l)^2)^{1/2} |\nabla u|^{2(k-1)} |u|^{p-2k}\, dx \leq$$
$$(\sum_{l=1}^N \sum_{j=1}^n \int_\Omega \int_\Omega |\nabla(\partial_j u_l)|^2 |\nabla u|^{p-2}\, dx)^{1/2} (\int_\Omega |\nabla u|^{4k-p-2} |u|^{2p-4k+2}\, dx)^{1/2},$$

and the second term in the right hand side of (7.28) by

$$(k-1)(\int_\Omega \sum_{i=1}^n (\partial_i(|\nabla u|^2))^2 |\nabla u|^{p-4}\, dx)^{1/2} (\int_\Omega |\nabla u|^{4k-p-2} |u|^{2p-4k+2}\, dx)^{1/2},$$

where we used that

$$\sum_{l=1}^N u_l \nabla u_l \nabla(|\nabla u|^2) \leq \sum_{i=1}^n (\sum_{l=1}^N u_l^2)^{1/2} (\sum_{l=1}^N (\partial_i u_l)^2)^{1/2} |\partial_i(|\nabla u|^2)| \lesssim |u||\nabla u||\nabla(|\nabla u|^2)|.$$

Since the last term in (7.28) is negative, while the quantity we want to estimate is positive we obtain from the last inequalities

$$\int_0^T \int_\Omega |\nabla u|^{2k} |u|^{p-2k} t^{k-1}\, dxdt \lesssim (J_1^{1/2} + J_2^{1/2}) I_{2k-\frac{p}{2}-1} \lesssim (J_1+J_2)^{1/2} I_{2k-\frac{p}{2}-1}. \quad (7.29)$$

From now on we proceed exactly like in the scalar case iterating sufficiently many times to obtain the desired result, since we control $I_{p/2}$ which is the RHS term of (7.28) using (7.29).

### 7.3.4 A simple argument for (7.11)

We now return to the first estimate in Lemma 7.1: while we only deal with $p = 2$, there is nothing specific to the $L^2$ case in what follows. Let $\chi$ be a smooth cut-off near the boundary $\partial\Omega$. Then $v = (1-\chi)u$ solves the heat equation in the whole space, with source term $[\chi, \Delta]u$:

$$(1-\chi)u = S_0(t)(1-\chi)u_0 + \int_0^t S_0(t-s)[\chi,\Delta]u(s)\,ds,$$

where $S_0$ is the free heat semi-group. We have, taking advantage of the localization near the boundary,

$$\|[\chi,\Delta]u\|_{L^2_t(L^{\frac{2n}{n+2}})} \lesssim C(\chi,\chi')\|\nabla u\|_{L^2_t(L^2)} < +\infty,$$

by the energy inequality (7.14). The integral equation on $(1-\chi)u$ features $S_0$ for which we have trivial Gaussian estimates, and both the homogeneous and inhomogeneous terms are $C_t(L^2)$ and go to zero as time goes to $+\infty$. On the other hand, by Poincaré inequality (or Sobolev),

$$\int_0^t \|\chi u\|_2^2\,ds \lesssim \int_0^t \|\nabla u\|_2^2\,ds,$$

which ensures that $\|\chi u\|_2$ goes to zero as well at $t = +\infty$.

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## 7.4 Appendix: functional calculus

We start by recalling the Dynkin-Helffer-Sjöstrand formula ([47, 55]) and refer to the appendix of [83] for a nice presentation of the use of almost-analytic extensions in the context of functional calculus. In what follows we will also rely on Davies'presentation ([43]) from which we will use a couple of useful lemma.

*Definition* 7.1. (see [83, Lemma A.1]) Let $\Psi \in C_0^\infty(\mathbb{R})$, possibly complex valued. We assume that there exists $\tilde\Psi \in C_0^\infty(\mathbb{C})$ such that $|\bar\partial\tilde\Psi(z)| \le C|\mathrm{Im} z|$ and $\tilde\Psi|_{\mathbb{R}} = \Psi$. Then we have (as a bounded operator in $L^2(\Omega)$)

$$\Psi(-h^2\Delta_D) = \frac{i}{2\pi}\int_{\mathbb{C}} \bar\partial\tilde\Psi(z)(z+h^2\Delta_D)^{-1}d\bar z\wedge dz. \tag{7.30}$$

The next result ensures the existence of $\tilde{\Psi}$ in the previous definition ( see [83, Lemma A.2] and [106], where it is linked with Hadamard's problem of finding a smooth function with prescribed derivatives at a given point):

**Lemma 7.3.** *If $\Psi$ belongs to $C_0^\infty(\mathbb{R})$ there exists $\tilde{\Psi} \in C_0^\infty(\mathbb{C})$ such that $\tilde{\Psi}|_{\mathbb{R}} = \Psi$ and*

$$|\bar{\partial}\tilde{\Psi}(z)| \leq C_{N,\Psi}|\mathrm{Im}z|^N, \quad \forall z \in \mathbb{C}, \quad \forall N \in \mathbb{N}. \tag{7.31}$$

*Moreover, if $\Psi$ belongs to a bounded subset of $C_0^\infty(\mathbb{R})$ (elements of $\mathcal{B}$ are supported in a given compact subset of $\mathbb{R}$ with uniform bounds), then the mapping $\mathcal{B} \ni \Psi \to \tilde{\Psi} \in C_0^\infty(\mathbb{C})$ is continuous and $C_{N,\Psi}$ can be chosen uniformly w.r.t $\Psi \in \mathcal{B}$.*

*Remark* 7.13. Estimate (7.31) simply means that $\bar{\partial}\tilde{\Psi}(z)$ vanishes at any order on the real axis. Precisely, if $z = x + iy$

$$\partial_y^N \tilde{\Psi}|_{\mathbb{R}} = (i\partial_x)^N \tilde{\Psi}|_{\mathbb{R}} = (i\partial_x)^N \Psi|_{\mathbb{R}}.$$

In particular if $\langle x\rangle = (1 + x^2)^{1/2}$ then for any given $N \geq 0$, a useful example of an almost analytic extension of $\Psi \in C_0^\infty(\mathbb{R})$ is given by

$$\tilde{\Psi}(x + iy) = \Big(\sum_{m=0}^{N} \partial^m \Psi(x)(iy)^m/m!\Big)\tau(\frac{y}{\langle x\rangle}),$$

where $\tau$ is a non-negative $C^\infty$ function such that $\tau(s) = 1$ if $|s| \leq 1$ and $\tau(s) = 0$ if $|s| \geq 2$. For later purposes, we also set

$$\|\Psi\|_N \overset{\text{déf}}{=} \sum_{m=0}^{N} \int_{\mathbb{R}} |\partial^m \Psi(x)|\langle x\rangle^{m-1} dx.$$

Our next lemma lets us deal with Lebesgue spaces.

**Lemma 7.4.** *Let $z \notin \mathbb{R}$ and $|\mathrm{Im}z| \lesssim |\mathrm{Re}z|$, then $\Delta_D$ satisfies*

$$\|(z - \Delta_D)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \leq \frac{c}{|\mathrm{Im}z|}\left(\frac{|z|}{|\mathrm{Im}z|}\right)^\alpha, \quad \forall z \notin \mathbb{R} \tag{7.32}$$

*for $1 \leq p \leq +\infty$, with a constant $c = c(p) > 0$ and $\alpha = \alpha(n, p) > n|\frac{1}{2} - \frac{1}{p}|$.*

*Remark that, for all $h \in (0, 1]$, the operator $h^2\Delta_D$ satisfies (7.32) with the same constants $c$ and $\alpha$ (this is nothing but scale invariance).*

For $p = 2$ the proof of Lemma 7.4 is trivial by multiplying the resolvent equation $-\Delta_D u + zu = f$ by $\bar{u}$ and we get $\alpha = 0$; however for $p \neq 2$ it requires a non trivial argument which we postpone to the end of this Appendix.

**Corollary 7.1.** *For $N \geq \alpha + 1$ the integral* (7.30) *is norm convergent and* $\forall h \in (0, 1]$

$$\|\Psi(-h^2\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \leq c\|\Psi\|_{N+1}, \tag{7.33}$$

*for some constant c independent of h.*

*Remark* 7.14. Notice how the Mikhlin multiplier condition (7.9) on $\Psi$ does not imply boundedness of $\|\Psi\|_{N+1}$: we need extra decay at infinity.

*Proof:* By scale invariance it is enough to prove (7.33) for $h = 1$. The integrand in (7.30) is norm continuous for $z \notin \mathbb{R}$. If we set

$$U \stackrel{\text{déf}}{=} \{z = x + iy | \langle x\rangle < |y| < 2\langle x\rangle\}, \quad V \stackrel{\text{déf}}{=} \{z = x + iy | 0 < |y| < 2\langle x\rangle\},$$

then the norm of the integrand is dominated by

$$c\sum_{m=0}^{N} |\partial^m \Psi(x)| \frac{2^m}{m!} \langle x\rangle^{m-2} \|\partial\tau\|_{L^\infty([1,2])} 1_U(x + iy) +$$

$$+c|\partial^{N+1}\Psi(x)| \frac{2^N}{N!} |y|^N \Big(\frac{\langle x\rangle}{|y|}\Big)^\alpha \|\tau\|_{L^\infty([0,2])} 1_V(x + iy).$$

Integrating with respect to $y$ for $N \geq \alpha + 1$ yields the bound

$$\|\Psi(-\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \lesssim \int_{\mathbb{R}} \Big(\sum_{m=0}^{N} |\partial^m\Psi(x)|\langle x\rangle^{m-1} +$$

$$+ |\partial^{N+1}\Psi(x)|\langle x\rangle^N\Big) dx = \|\Psi\|_{N+1}.$$

One may then prove that the operator $\Psi(-\Delta_D)$, acting on $L^p(\Omega)$, is independent of $N \geq 1 + n/2$ and of the cut-off function $\tau$ in the definition of $\tilde{\Psi}$, see [43].

We now recall two lemma which will be useful when composing operators.

**Lemma 7.5.** *([43, Lemma 2.2.5]) If $\Psi \in C_0^\infty(\mathbb{R})$ has support disjoint from the spectrum of $-h^2\Delta_D$ then $\Psi(-h^2\Delta_D) = 0$.*

**Lemma 7.6.** *([43, Lemma 2.2.5]) If $\Psi_1$, $\Psi_2 \in C_0^\infty(\mathbb{R})$, then $(\Psi_1\Psi_2)(-h^2\Delta_D) = \Psi_1(-h^2\Delta_D)\Psi_2(-h^2\Delta_D)$.*

For the remaining part of the Appendix we prove the resolvent estimate (7.32) from Lemma 7.4. If $\mathrm{Re}z > 0$, this is nothing but a standard elliptic estimate. The trouble comes with $\mathrm{Re}z < 0$ and getting close to the spectrum. In $\mathbb{R}^n$, one may evaluate directly the convolution operator by proving its kernel to be in $L^1$: this follows from

$$|z + |\xi|^2|^2 = \sin^2\frac{(\pi-\theta)}{2}(|z| + |\xi|^2)^2 + \cos^2\frac{(\pi-\theta)}{2}(|\xi|^2 - |z|)^2, \quad \text{with} \quad z = |z|e^{i\theta},$$

and a direct computation of $L^2$ norms of $\partial^\alpha(z+|\xi|^2)^{-1}$. By reflection, one then extends this estimate to the half-space case, with both Dirichlet and Neumann boundary conditions. By localizing $L^p$ estimates close to the boundary and flattening, one may then obtain the desired estimate (7.32); such an approach is carried out in [2] in a greater generality (systems of Laplace equations, mixed boundary conditions), at the expense of fixing the angle $\theta$ and not tracking explicit dependances on $|z|$ and $\theta$. While (relatively) elementary, such a proof is, out of necessity, filled with lenghty calculations and most certainly does not provide the sharpest constant. It is worth noting, however, that it relies on standard elliptic techniques.

To keep in line with the parabolic approach, we present a short proof, relying on the holomorphic nature of $S(w)$ in the half-plane $\mathrm{Re}w > 0$. Remark that by our $L^p$ bound on $S(t)$, $t \in \mathbb{R}_+$, the trivial $L^2$ bound on $S(w)$, $\mathrm{Re}w \geq 0$, and Stein's parameter version of complex interpolation, one may easily derive that $S(w)$ is holomorphic in a sector around the positive real axis; but its angle will narrow with large or small $p$. However the argument may be refined and $S(w)$ was proved to be holomorphic in the whole right half-plane in [85], using in a crucial way the Gaussian nature of the heat kernel on domains ([41]). This was extented to more general settings in [31], where an explicit bound is stated:

$$\|S(w)\|_{L^p \to L^p} \leq C_\varepsilon \left(\frac{|w|}{|\mathrm{Re}w|}\right)^{n\left|\frac{1}{2}-\frac{1}{p}\right|+\varepsilon}. \tag{7.34}$$

Then (7.32) is a direct consequence of the following standard computation: recall the following formula, which is simply a Laplace transform,

$$(z-\Delta_D)^{-1} = \int_L e^{w\Delta_D - wz}\,dw, \tag{7.35}$$

where $L$ can be chosen to be a half ray from the origin. Set $z = re^{i\theta}$, $w = \rho e^{i\phi}$, then

$$(z-\Delta_D)^{-1} = \int_0^{+\infty} e^{\rho\exp(i\phi)\Delta_D - r\rho\exp i(\theta+\phi)}\,d\rho.$$

Now, if $\mathrm{Re}z > 0$, we may take $\phi = 0$ and use estimates for the semi-group $S(\rho)$. We would like to extend the range to the $\mathrm{Re}z < 0$ region, up to a thin sector around the negative real axis ($|\pi - \theta| < \epsilon$); getting close to the spectrum is required if we want to define $\Psi(-\Delta_D)$ with $\Psi \in C_0^\infty(]0,+\infty[)$. One picks $\phi$ such that $2|\theta+\phi| < \pi$, which ensures a decaying exponential in 7.35, provided we bound $S(w)$ in $L^p$. But the condition on $\phi$ yields $|\phi| < \pi/2$, and the bound amounts to the holomorphy of $S(w)$. The constant in (7.34) translates into a $(|z|/|\mathrm{Im}z|)^\alpha$ factor, while integration over $\rho$ provides the remaining $1/|\mathrm{Im}z|$ in (7.32). This concludes the proof.

# 8 Precise smoothing effect in the exterior of balls

## 8.1 Introduction

We are interested in this article in investigating the smoothing effect properties of the solutions of the Schrödinger equation. Since the work by Craig, Kapeller and Strausss [39], Kato [68], Constantin and Saut [37] establishing the smoothing property, many works have dealt with the understanding of this effect. In particular the work by Doi [46] and Burq [25, 24] shows that it is closely related to the infinite speed of propagation for the solutions of Schrödinger equation. Roughly speaking, if one considers a wave packet with wave length $\lambda$, it is known that it propagates with speed $\lambda$ and the wave will stay in any bounded domain only for a time of order $1/\lambda$. As a consequence, taking the $L^2$ in time norm will lead to an improvement of $1/\lambda^{1/2}$ with respect to taking an $L^\infty$ norm, leading to a gain of $1/2$ derivatives. This heuristic argument can be transformed into a proof of the smoothing effect either by direct calculations (in the case of the free Schrödinger equation) or by means of resolvent estimates (see [13] for the case of a perturbation by a potential or [26] for the boundary value problem). In view of this simple heuristics, it is natural to ask whether one can refine (and improve) such smoothing type estimates if one considers smaller space domains (whose size will shrink as the wave length increases). A very natural context in which one can test this heuristics is the case of the exterior of a convex body (or more generally the exterior of several convex bodies), in which case natural candidates for the $\lambda$ dependent domains are $\lambda^{-\alpha}$ neighborhoods of the boundary. This is the main aim of this paper. To keep the paper at a rather basic technical level, we choose to consider only balls, for which direct calculations (with Bessel functions) can be performed. Our first result reads as follows:

**Theorem 8.1.** *Let $\Omega = \mathbb{R}^3 \setminus B(0,1)$, $T > 0$, $0 \leq \alpha < \frac{2}{3}$ and $\lambda \geq 1$. Let $\psi$ and $\chi \in C_0^\infty(\mathbb{R}^*)$ be smooth functions with compact support, $\psi = 1$ near 1, $\chi = 1$ near 0. Set $\chi_\lambda(|x|) := \chi(\lambda^\alpha(|x|-1))$, where $x$ denotes the variable on $\Omega$. Then one has*

1. *For $s \in [-1,1]$ and $v(t) = \int_0^t e^{i(t-\tau)\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda g d\tau$*

$$\|\chi_\lambda \psi(\frac{-\Delta_D}{\lambda^2})v\|_{L^2_T H^{s+1}_D(\Omega)} \leq C\lambda^{-\frac{\alpha}{2}}\|\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda g\|_{L^2_T H^s_D(\Omega)}, \tag{8.1}$$

2. *For $s \in [0,1]$*

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{s+\frac{1}{2}}(\Omega)} \leq C\lambda^{s-\frac{\alpha}{4}}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.2}$$

*Here the constants $C$ do not depend on $T$, i.e. the estimates are* global *in time.*

Using this result, we can deduce new Strichartz type estimates for the solution of the linear Schrödinger equation in the exterior, $\Omega$, of a smooth bounded obstacle $\Theta \subset \mathbb{R}^3$,

$$(i\partial_t + \Delta_{D,N})u = 0, \qquad u(0,x) = u_0(x), \qquad u|_{\partial\Omega} = 0 \text{ or } \partial_n u\,|_{\partial\Omega} = 0, \tag{8.3}$$

that we denote respectively by $e^{it\Delta_D}u_0$ and $e^{it\Delta_N}u_0$.

*Definition* 8.1. Let $q, r \geq 2$, $(q, r, d) \neq (2, \infty, 2)$, $2 \leq p \leq \infty$. A pair $(q, r)$ is called admissible in dimension $d$ if $q$, $r$ satisfy the scaling admissible condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{8.4}$$

**Theorem 8.2.** *(Strichartz estimates) Let $\Theta = B(0,1) \subset \mathbb{R}^3$ and $\Omega = \mathbb{R}^3 \setminus \Theta$, $T > 0$ and $(p, q)$ an admissible pair in dimension 3. Let $\epsilon > 0$ be an arbitrarily small constant. Then there exists a constant $C > 0$ such that, for all $u_0 \in H^{\frac{4}{5p}+\epsilon}_{D,N}(\Omega)$ the following holds*

$$\|e^{it\Delta_{D,N}} u_0\|_{L^p([-T,T],L^q(\Omega))} \leq C\|u_0\|_{H^{\frac{4}{5p}+\epsilon}_{D,N}(\Omega)}. \tag{8.5}$$

*Moreover, a similar result holds true for a class of trapping obstacles (Ikawa's example), i.e. for the case where $\Theta$ is a finite union of balls in $\mathbb{R}^3$.*

As a consequence of Theorem 8.2 we deduce new global well-posedness results for the non-linear Schrödinger equation in the exterior of several convex obstacles, improving previous results by Burq, Gerard and Tzvetkov [26]. Consider the nonlinear Schrödinger equation on $\Omega$ subject to Dirichlet boundary condition

$$(i\partial_t + \Delta_D)u = F(u) \quad \text{in} \quad \mathbb{R} \times \Omega, \quad u(0, x) = u_0(x), \quad \text{on} \quad \Omega, \quad u|_{\partial\Omega} = 0. \tag{8.6}$$

The nonlinear interaction $F$ is supposed to be of the form $F = \partial V/\partial \bar{z}$, with $F(0) = 0$, where the "potential" $V$ is real valued and satisfies $V(|z|) = V(z)$, $\forall z \in \mathbb{C}$. Moreover we suppose that $V$ is of class $C^3$, $|D^k_{z,\bar{z}}V(z)| \leq C_k(1 + |z|)^{4-k}$ for $k \in \{0, 1, 2, 3\}$ and that $V(z) \geq -(1 + |z|)^\beta$, for some $\beta < 2 + \frac{4}{d}$, $d = 3$ (the last assumption avoid blow-up in the focussing case).

Some phenomena in physics turn out to be modeled by exterior problems and one may expect rich dynamics under various boundary conditions. A first step in this direction is to establish well defined dynamics in the natural spaces determined by the conservation lows associated to (8.6). If $u(t, .) \in H^1_0(\Omega) \bigcap H^2(\Omega)$ is a solution of (8.6) then it satisfies the conservation lows

$$\frac{d}{dt}\int_\Omega |u(t, x)|^2 dx = 0; \quad \frac{d}{dt}\Big(\int_\Omega |\nabla u(t, x)|^2 dx + \int_\Omega V(u(t, x))dx\Big) = 0 \tag{8.7}$$

and therefore for a large class of potentials $V$ the quantity $\|u(t, .)\|_{H^1_0(\Omega)}$ remains finite along the trajectory starting from $u_0 \in H^1_0(\Omega) \bigcap H^2(\Omega)$. This fact makes the study of (8.6) in the energy space $H^1_0(\Omega)$ of particular interest. It is also of interest to study (8.6) in $L^2(\Omega)$: the main issue in the analysis is that the regularities of $H^1$ and $L^2$ are a priori too poor to be achieved by the classical methods for establishing local existence and uniqueness for (8.6). We state the result concerning finite energy solutions, which will be a consequence of Theorem 8.2.

**Theorem 8.3.** *(Global existence theorem) Let* $\Omega = \mathbb{R}^3 \setminus B(0,1)$*. For any* $u_0 \in H_0^1(\Omega)$ *the initial boundary value problem* (8.6) *has a unique global solution* $u \in C(\mathbb{R}, H_0^1(\Omega))$ *satisfying the conservation laws* (8.7)*. Moreover, for any* $T > 0$ *the flow map* $u_0 \to u$ *is Lipschitz continuous from any bounded set of* $H_0^1(\Omega)$ *to* $C(\mathbb{R}, H_0^1(\Omega))$*.*

*Remark* 8.1. In [26], Strichartz type estimates with loss of $\frac{1}{p}$ derivative have been obtained for the Schrödinger equation in the exterior of a non-trapping obstacle $\Theta \subset \mathbb{R}^d$ which allowed to prove the same global existence result in dimension 3 provided $\|u_0\|_{H_0^1(\Omega)}$ is sufficiently small.

The Cauchy problem associated to (8.6) has been extensively studied in the case $\Omega = \mathbb{R}^d$, $d \geq 2$, by Bourgain [18], Cazenave [32], Sulem et Sulem [101], Ginibre and Velo [49, 50, 51], Kato [68] and the theory of existence of finite energy solutions to (8.6) for potentials $V$ with polynomial growth has been much developed. Roughly speaking the argument for establishing finite energy solutions of (8.6) consists in combining $H^1$ local well-posedness with conservation laws (8.7) which provide a control on the $H^1$ norm.

The article is written as follows: in Section 8.2 we obtain bounds for the $L^2$ norms for the outgoing solution of the Helmholtz equation that will be used in Section 8.3 in order to prove Theorem 4.1. In Section 8.4 we use a strategy inspired from [98], [26] to handle the case when the solution is supported outside a neighborhood of $\partial\Omega$; in Section 8.5 we achieve the proof of Theorem 8.2. The last section is dedicated to the applications of these results; precisely, we give the proofs of Theorems 8.2 and 8.3 in the case where the obstacle consists of a union of balls. In the Appendix we recall some properties of the Hankel functions.

## 8.2 Precise smoothing effect

### 8.2.1 Preliminaries

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a smooth domain . For $s \geq 0$, $p \in [1, \infty]$ we denote by $W^{s,p}(\Omega)$ the Sobolev spaces on $\Omega$. We write $L^p$ and $H^s$ instead of $W^{0,p}$ and $W^{s,2}$. By $\Delta_D$ (resp. $\Delta = \Delta_N$) we denote the Dirichlet Laplacian (resp. Neumann Laplacian) on $\Omega$, with domain $H^2(\Omega) \cap H_0^1(\Omega)$ (resp. $\{u \in H^2(\Omega) | \partial_n u|_{\partial\Omega} = 0\}$). We next define the dual space of $H_0^1(\Omega)$, $H^{-1}(\Omega)$, which is a subspace of $D'(\Omega)$. We construct $H^{-s}(\Omega)$ via interpolation and due to [14, Cor.4.5.2] we have the duality between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$ for $s \in [0, 1]$.

**Proposition 8.1.** *Let* $d \geq 2$ *and* $\Omega \subset \mathbb{R}^d$ *be a smooth domain. The following continuous embeddings hold:*

1. $H_0^1(\Omega) \subset L^q(\Omega)$, $\quad 2 \leq q \leq \frac{2d}{d-2}$ *(*$p < \infty$ *if* $d = 2$*),*
2. $H_D^s(\Omega) \subset L^q(\Omega)$, $\quad \frac{1}{2} - \frac{1}{q} = \frac{s}{d}$, $s \in [0, 1)$,
3. $H_D^{s+1}(\Omega) \subset W^{1,q}(\Omega)$, $\quad \frac{1}{2} - \frac{1}{q} = \frac{s}{d}$, $s \in [0, 1)$,

*4.* $W^{s,p}(\Omega) \subset L^\infty(\Omega), \quad s > \frac{d}{p}, \, p \geq 1.$

*Proof.* The proof follows from the Sobolev embeddings on $\mathbb{R}^d$ and the use of extension operators. □

Let $\psi, \chi \in C_0^\infty(\mathbb{R}^*)$ be smooth functions such that $\chi = 1$ in a neighborhood of 0 and for all $\tau \in \mathbb{R}_+$, $\sum_{k\geq 0} \psi(2^{-2k}\tau) = 1$. For $\lambda > 0$ let $\chi_\lambda(|x|) := \chi(\lambda^\alpha(|x| - 1))$, where $x$ denotes variable on $\Omega$. We introduce spectral localizations which commute with the linear evolution. Since the spectrum of $-\Delta_D$ is confined to the positive real axis it is convenient to introduce $\lambda^2$ as a spectral parameter. We will consider the linear problem (9.4) with initial data localized at frequency $\lambda$, $u|_{t=0} = \psi(-\frac{\Delta_D}{\lambda^2})u_0$.

*Remark* 8.2. In order to prove Theorem 8.2 it will be enough to prove (8.5) with the initial data of the form $\psi(-\frac{\Delta_D}{\lambda^2})u_0$, since than we have for some $\epsilon$ arbitrarily small

$$\|e^{it\Delta_D}u_0\|_{L^p([-T,T],L^q(\Omega))} \approx \|(e^{it\Delta_D}\psi(-\frac{\Delta_D}{2^{2j}})u_0)_j\|_{L^p([-T,T],L^q(\Omega))l_j^2} \leq \tag{8.8}$$

$$\leq \|(e^{it\Delta_D}\psi(-\frac{\Delta_D}{2^{2j}})u_0)_j\|_{l_j^2 L^p([-T,T],L^q(\Omega))} \leq \Big(\sum_j 2^{2j(\frac{4}{5p}+\epsilon)}\|\psi(-\frac{\Delta_D}{2^{2j}})u_0\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \approx \|u_0\|_{H_D^{\frac{4}{5p}+\epsilon}(\Omega)}.$$

### 8.2.2 Estimates in a small neighborhood of the boundary

In what follows let $d = 3$. We study first the outgoing solution to the equation

$$(\Delta_D + \lambda^2)w = \chi_\lambda f, \quad w|_{\partial\Omega} = 0. \tag{8.9}$$

In what follows we will establish high frequencies bounds for the $L^2$ norm of $w$ in a small neighborhood $\Omega_\lambda = \{|x| \lesssim \lambda^{-\alpha}\}$ of size $\lambda^{-\alpha}$ of the boundary. We notice that a ray with transversal, equal-angle reflection spends in the neighborhood $\Omega_\lambda$ a time $\simeq \lambda^{-\alpha}$. If the ray is diffractive then the time spent in $\Omega_\lambda$ equals $\lambda^{-\frac{\alpha}{2}}$.

We analyze the outgoing solution of (8.9) outside the unit ball of $\mathbb{R}^3$. The first step is to introduce polar coordinates and to write the expansion in spherical harmonics of the solution to

$$\begin{cases} (\Delta_D + \lambda^2)\tilde{w} = 0 \quad \text{on} \quad \Omega = \{x \in \mathbb{R}^3 | |x| > 1\}, \\ \tilde{w}|_{\partial\Omega} = \tilde{f} \quad \text{on} \quad \mathbb{S}^2, \\ r(\partial_r w - i\lambda w) \to_{r\to\infty} 0. \end{cases} \tag{8.10}$$

In this coordinates the Laplace operator on $\Omega$ writes

$$\Delta_D = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^2}, \tag{8.11}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace operator on the sphere $\mathbb{S}^2$. Thus the solution $\tilde{w}$ of (8.10) satisfies

$$r^2\partial_r^2\tilde{w} + 2r\partial_r\tilde{w} + (\lambda^2 r^2 + \Delta_{\mathbb{S}^2})\tilde{w} = 0, \quad r > 1. \tag{8.12}$$

In particular, if $\{e_j\}$ is an orthonormal basis of $L^2(\mathbb{S}^2)$ consisting of eigenfunctions of $\Delta_{\mathbb{S}^2}$, with eigenvalues $-\mu_j^2$ and if $\omega$ denotes the variable on the sphere $\mathbb{S}^2$, we can write

$$\tilde{w}(r\omega) = \sum_j \tilde{w}_j(r) e_j(\omega), \quad r \geq 1, \tag{8.13}$$

where the functions $\tilde{w}_j(r)$ satisfy

$$r^2 \tilde{w}_j''(r) + 2r\tilde{w}_j'(r) + (\lambda^2 r^2 - \mu_j^2)\tilde{w}_j(r) = 0, \quad r > 1. \tag{8.14}$$

This is a modified Bessel equation, and the solution satisfying the radiation condition $r(\partial_r w - i\lambda w) \to_{r\to\infty} 0$ is of the form

$$\tilde{w}_j(r) = a_j r^{-\frac{1}{2}} H_{\nu_j}(\lambda r), \tag{8.15}$$

where $H_{\nu_j}(z)$ denote the Hankel function. Recall that the Hankel function is given by

$$H_\nu(z) = (\frac{2}{\pi z})^{1/2} \frac{e^{i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-s} s^{\nu-1/2} (1 - \frac{s}{2iz})^{\nu-1/2} ds, \tag{8.16}$$

where

$$\Gamma(\nu+1/2) = \int_0^\infty e^{-s} s^{\nu-1/2} ds,$$

and it is valid for $Re\nu > \frac{1}{2}$ and $-\pi/2 < \arg z < \pi$. Also, in (8.15) $\nu_j$ is given by

$$\nu_j = (\mu_j^2 + \frac{1}{4})^{1/2} \tag{8.17}$$

and the coefficients $a_j$ are determined by the boundary condition $\tilde{w}_j(1) =< \tilde{f}, e_j >$, so

$$a_j = \frac{< \tilde{f}, e_j >}{H_{\nu_j}(\lambda)}. \tag{8.18}$$

Let us introduce the self-adjoint operator

$$A = (-\Delta_{\mathbb{S}^2} + \frac{1}{4})^{1/2}, \quad Ae_j = \nu_j e_j. \tag{8.19}$$

Then the solution of (8.10) writes, formally,

$$\tilde{w}(r\omega) = r^{-1/2} \frac{H_A(\lambda r)}{H_A(\lambda)} \tilde{f}(\omega), \quad \omega \in \mathbb{S}^2. \tag{8.20}$$

**Proposition 8.2.** *(see [104, Chp.3]) The spectrum of $A$ is $spec(A) = \{m + \frac{1}{2} | m \in \mathbb{N}\}$.*

*Remark* 8.3. (see [104, Chp.3]) The Hankel function $H_{m+1/2}(z)$ and Bessel functions of order $m+\frac{1}{2}$ are all elementary functions of $z$. We have

$$H_{m+1/2}(z) = (\frac{2z}{\pi})^{1/2} q_m(z), \quad q_m(z) = -i(-1)^m (\frac{1}{z}\frac{d}{dz})^m (\frac{e^{iz}}{z}). \tag{8.21}$$

We deduce that

$$r^{-1/2}\frac{H_{m+1/2}(\lambda r)}{H_{m+1/2}(\lambda)} = r^{-m-1}e^{i\lambda(r-1)}\frac{p_m(\lambda r)}{p_m(\lambda)}, \quad p_m(z) = i^{-m-1}\sum_{k=0}^{m}(i/2)^k\frac{(m+k)!}{k!(m-k)!}z^{m-k}. \tag{8.22}$$

We now look for a solution to (8.9). If we write

$$f(r\omega) = \sum_j f_j(r)e_j(\omega), \quad w(r\omega) = \sum_j w_j(r)e_j(\omega),$$

then the functions $w_j(r)$ satisfy

$$r^2 w_j''(r) + 2rw_j'(r) + (\lambda^2 r^2 - \mu_j^2)w_j(r) = r^2 f_j(r), \quad r > 1 \tag{8.23}$$

together with the vanishing condition at $r = 1$ and Sommerfield radiation condition when $r \to \infty$. Applying the variation of constants method together with the outgoing assumption and the formula (8.20), we obtain

$$w_j(r) = \int_0^\infty G_{\nu_j}(r,s,\lambda)\chi_\lambda(s)f_j(s)s^2 ds, \quad \nu_j = (\mu_j^2 + \frac{1}{4})^{1/2}, \tag{8.24}$$

where $G_\nu(r,s,\lambda)$ is the Green kernel for the differential operator

$$L_\nu = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (\lambda^2 - \frac{\mu^2}{r^2}), \quad \nu = (\mu^2 + \frac{1}{4})^{1/2}. \tag{8.25}$$

The fact that $L_\nu$ is self-adjoint implies that $G_\nu(r,s,\lambda) = G_\nu(s,r,\lambda)$ and from the radiation condition we obtain (for $\lambda$ real)

$$G_\nu(r,s,\lambda) = \begin{cases} \frac{\pi}{2i}(rs)^{-1/2}\Big(J_\nu(s\lambda) - \frac{J_\nu(\lambda)}{H_\nu(\lambda)}H_\nu(s\lambda)\Big)H_\nu(r\lambda), & r \geq s, \\ \frac{\pi}{2i}(rs)^{-1/2}\Big(J_\nu(r\lambda) - \frac{J_\nu(\lambda)}{H_\nu(\lambda)}H_\nu(r\lambda)\Big)H_\nu(s\lambda), & r \leq s. \end{cases} \tag{8.26}$$

In what follows we will look for estimates of the $L^2$ norm of $w_j$ on the interval $[1, 1+\lambda^{-\alpha}]$. This problem has to be divided in several classes, according whether $\nu_j/\lambda$ is less than, nearly equal or greater than 1. We distinguish also the simple case of determining bounds when the argument is much larger than the order. Let us explain the meaning of this: in fact, applying the operator $L_{\nu_j}$ to $rw_j(r)$ instead of $w_j(r)$, we eliminate the term involving $w_j'(r)$ in (8.23). On the characteristic set we have

$$\rho_j^2 + \frac{\mu_j^2}{r^2} = \lambda^2, \quad \nu_j = (\mu_j^2 + \frac{1}{4})^{1/2}, \tag{8.27}$$

where $\rho_j$ denotes the dual variable of $r$. Let $\theta_j$ be defined by $\tan\theta_j = \frac{\rho_j}{\mu_j}$, then for $\lambda$ big enough and $r$ in a small neighborhood of 1 we can estimate

$$\tan^2\theta_j + 1 \simeq \frac{\lambda^2}{\nu_j^2}.$$

1. When the quotient $\frac{\nu_j}{\lambda}$ is smaller than a constant $1-\epsilon_0$ where $\epsilon_0$ is fixed, strictly positive, this corresponds to an angle $\theta_j$ between some fixed direction $\theta_0$ and $\pi/2$, with $\tan\theta_0 = \epsilon_0$, and thus to a ray hitting the obstacle transversally. In this case we show that, since a unit speed bicharacteristic spends in the $\lambda$-depending neighborhood $\Omega_\lambda$ a time $\lambda^{-\alpha}$, we have the following
   **Proposition 8.3.** *Let $\epsilon_0 > 0$ be fixed, small. Then there exists a constant $C = C(\epsilon_0)$ such that*
   $$\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\lambda^{-(1+\alpha)}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])}$$
   *uniformly for $j$ such that $\{\nu_j/\lambda \leq 1-\epsilon_0\}$.*
2. When the quotient $\frac{\nu_j}{\lambda}$ is close enough to 1 the angles $\theta_j$ become very small and this is the case of a diffractive ray, which spends in $\Omega_\lambda$ a time proportional to $\lambda^{-\alpha/2}$. In this case $\tan\theta_j \simeq \sqrt{1-\frac{\nu_j^2}{\lambda^2}}$ and we show the following
   **Proposition 8.4.** *If $1-\frac{\nu_j}{\lambda} \simeq \lambda^{-\beta}$ for some $\beta > 0$ then*
   $$\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\lambda^{-1-\alpha/2-(\alpha-\beta)}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])} \quad \text{if} \quad \beta \leq \alpha$$
   *and*
   $$\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\lambda^{-(1+\alpha/2)}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])} \quad \text{if} \quad \beta \geq \alpha.$$
3. In the elliptic case $\frac{\lambda}{\nu_j} \ll 1$ or $\frac{\lambda}{\nu_j} \in [\epsilon_0, 1-\epsilon_0]$ for some small $\epsilon_0 > 0$ there is nothing to do since away from the characteristic variety (8.27) we have nice bounds of the solution of the solution $w_j(r)$ of (8.23).

*Proof.* (*of Proposition 8.3*) The solution $w_j(r)$ given in (8.24) writes

$$w_j(r) = \frac{\pi}{8i}r^{-1/2}\Big(\int_0^r \chi_\lambda(s)f_j(s)s^{3/2}(\bar{H}_{\nu_j}(\lambda s) - \frac{\bar{H}_{\nu_j}(\lambda)}{H_{\nu_j}(\lambda)}H_{\nu_j}(\lambda s))dsH_{\nu_j}(\lambda r)$$
$$+\int_r^\infty \chi_\lambda(s)f_j(s)s^{3/2}H_{\nu_j}(\lambda s)ds(\bar{H}_{\nu_j}(\lambda r) - \frac{\bar{H}_{\nu_j}(\lambda)}{H_{\nu_j}(\lambda)}H_{\nu_j}(\lambda r))\Big), \quad (8.28)$$

thus in order to obtain estimates for $\|w_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])}$ we have to determine bounds for $\|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])}$ since we have for some constant $C > 0$

$$\|w_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])} \leq \|f_j(r)\|_{L^2([1,1+\lambda^{-\alpha}])}\|H_{\nu_j}(\lambda r)\|^2_{L^2([1,1+\lambda^{-\alpha}])}. \quad (8.29)$$

We consider separately two regimes:

1. For $\frac{\nu_j}{\lambda} \ll 1$ we use (8.111) to obtain immediately
$$\|H_{\nu_j}(\lambda r)\|^2_{L^2([1,1+\lambda^{-\alpha}])} \lesssim \lambda^{-1-\alpha}. \tag{8.30}$$

2. For $0 < \epsilon_0 \le c_j := \frac{\nu_j}{\lambda} \le 1-\epsilon_0$ for some fixed $\epsilon_0 > 0$ we use the expansions (8.112), (8.113): set
$$\cos\tau = \frac{c_j}{r}, \quad dr = c_j \frac{\sin\tau}{\cos^2\tau} d\tau, \quad \tau \in [\arccos c_j, \arccos\frac{c_j}{1+\lambda^{-\alpha}}] =: [\tau_{j,0}, \tau_{j,1}].$$
Then we have
$$\begin{aligned}\|H_{\nu_j}(\lambda r)\|^2_{L^2([1,1+\lambda^{-\alpha}])} &= \frac{2}{\pi\lambda}\int_{\tau_{j,0}}^{\tau_{j,1}} \frac{1}{\cos\tau} d\tau = \frac{1}{\pi\lambda}\ln\Big(\frac{1+\sin\tau}{1-\sin\tau}\Big)|_{\tau_{j,0}}^{\tau_{j,1}} \\ &\simeq \frac{1}{\pi\lambda}\Big((1+\lambda^{-\alpha})^2 \frac{(1+\sqrt{1-c_j^2/(1+\lambda^{-\alpha})^2})^2}{(1+\sqrt{1-c_j^2})^2} - 1\Big) \\ &\simeq \frac{1}{\pi\lambda}\frac{\lambda^{-\alpha}}{(1-c_j^2+\lambda^{-\alpha})^{1/2} + (1-c_j^2)^{1/2}},\end{aligned} \tag{8.31}$$
where $1-c_j^2 \in [2\epsilon_0 - \epsilon_0^2, 1-\epsilon_0^2]$ with $0 < \epsilon_0 < 1$ fixed. Notice, however, that if one takes $c_j \simeq \lambda^{-\beta}$ for some $\beta > 0$, then $\tan\theta_j \simeq (1-c_j^2)^{1/2} \simeq \lambda^{-\beta/2}$. If $\beta \le \alpha$ we can estimate (8.31) by $\lambda^{-(1-\alpha/2-(\alpha-\beta)}$, otherwise we have the bound $\lambda^{-(1+\alpha/2)}$.

□

*Proof.* (*of Proposition 8.4*) Here we use Proposition 8.9. Let $1 - \nu_j/\lambda = \tau\lambda^{-\beta}$ for $\tau$ in a neighborhood of 1. For $s \in [1, 1+\lambda^{-\alpha}]$ write $z(s)\nu_j = s\lambda$, thus $z(s) = \frac{s}{1-\tau\lambda^{-\beta}}$. Since
$$\begin{aligned}\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \le C\frac{\lambda^{-\alpha/2}}{|H_{\nu_j}(\lambda)|}\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])}\|H_{\nu_j}(\lambda r)\|_{L^2([1,1+\lambda^{-\alpha}])} \\ \times \|J_{\nu_j}(\lambda s)Y_{\nu_j}(\lambda) - J_{\nu_j}(\lambda)Y_{\nu_j}(\lambda s)\|_{L^2(1,1+\lambda^{-\alpha})},\end{aligned} \tag{8.32}$$
we shall compute separately each factor in (8.32) (modulo small terms). We have
$$\begin{aligned}\|H_{\nu_j}(\lambda r)\|^2_{L^2(1,1+\lambda^{-\alpha})} &= \int_1^{1+\lambda^{-\alpha}} |J_{\nu_j}(\lambda r)|^2 + |Y_{\nu_j}(\lambda r)|^2 dr \\ &\simeq \frac{2}{\pi}\nu_j^{-1}\int_1^{1+\lambda^{-\alpha}} \frac{1}{(z^2(s)-1)^{1/2} ds} \simeq \frac{2}{\pi}\nu_j^{-1}\frac{\lambda^{-\alpha}}{\lambda^{-\alpha}+2\tau\lambda^{-\beta}} \\ &\simeq \begin{cases}\lambda^{-1-(\alpha-\beta)}, & \text{if} \quad 0 \le \beta \le \alpha, \\ \lambda^{-1}, & \text{if}, \quad \beta > \alpha\end{cases}\end{aligned} \tag{8.33}$$
and
$$|H_{\nu_j}(\lambda)|^2 = |J_{\nu_j}(\lambda)|^2 + |Y_{\nu_j}(\lambda)|^2 \simeq \frac{2}{\pi}\nu_j^{-1}\frac{1}{(z^2(1)-1)^{1/2}} \simeq \lambda^{-1+\beta/2}. \tag{8.34}$$

The factor in the second line in (8.32) is estimated as follows:

$$
\begin{aligned}
&\|J_{\nu_j}(\lambda s)Y_{\nu_j}(\lambda) - J_{\nu_j}(\lambda)Y_{\nu_j}(\lambda s)\|_{L^2(1,1+\lambda^{-\alpha})} \\
&\qquad \leq \frac{2}{\pi}\nu_j^{-1}\frac{1}{(z^2(1)-1)^{1/4}}\Big(\int_1^{1+\lambda^{-\alpha}} \frac{1}{(z^2(s)-1)^{1/2}}ds\Big)^{1/2} \\
&\qquad \simeq \begin{cases} \lambda^{-1-\alpha/2+\beta/4}, & \text{if} \quad 0 \leq \beta \leq \alpha, \\ \lambda^{-1+\beta/4}, & \text{if}, \quad \beta > \alpha. \end{cases}
\end{aligned} \tag{8.35}
$$

From (8.32), (8.33), (8.34), (8.35) we deduce

$$
\|w_j\|_{L^2([1,1+\lambda^{-\alpha}])} \leq C\|f_j\|_{L^2([1,1+\lambda^{-\alpha}])} \times \begin{cases} \lambda^{-1-\alpha/2-(\alpha-\beta)}, & \text{if} \quad 0 \leq \beta \leq \alpha, \\ \lambda^{-1-\alpha/2}, & \text{if}, \quad \beta > \alpha. \end{cases} \tag{8.36}
$$

□

**Neumann problem:** *As far as the Neumann problem is concerned, we must solve the problem*

$$
\Big(\partial_r^2 + \frac{2}{r}\partial_r + (\lambda^2 - \frac{\nu^2}{r^2})\Big)w_j(r) = \chi(\lambda^\alpha(r-1))f_j(r), \quad \frac{\partial w_j}{\partial r}(1) = 0, \tag{8.37}
$$

*which gives, after performing similar computations*

$$
\begin{aligned}
w_j^N(r) = &\frac{\pi}{8i}r^{-1/2}H_{\nu_j}(\lambda r)\Big(\frac{\bar{H}'^2_{\nu_j}(\lambda)}{|H'_{\nu_j}(\lambda)|^2}\int_1^\infty \chi_\lambda(s)f_j(s)s^{3/2}H_{\nu_j}(\lambda s)ds \\
&- \int_1^r f_j(s)s^{3/2}\bar{H}_{\nu_j}(\lambda s)ds\Big) - \frac{\pi}{8i}r^{-1/2}\bar{H}_{\nu_j}(\lambda r)\int_r^\infty \chi_\lambda(s)f_j(s)s^{3/2}H_{\nu_j}(\lambda s)ds.
\end{aligned} \tag{8.38}
$$

*Propositions 8.3, 8.4 hold therefor true for the Neumann case too.*

## 8.3 Proof of Theorem 8.1

### 8.3.1 Reduction to the Helmholtz equation

**Proposition 8.5.** *Consider the Helmholtz equation*

$$
\begin{cases} (\Delta_D + \lambda^2)w = \chi_\lambda f \quad on \quad \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \tag{8.39}
$$

*with the Sommerfield "radiation condition" (where $r = |x|$)*

$$
r(\partial_r w - i\lambda w) \to_{r\to\infty} 0. \tag{8.40}
$$

*In order to prove Theorem 8.1 it is enough to establish estimates for the $L^2$ norms of the Helmholtz equation* (8.39).

*Proof.* Consider the inhomogeneous Schrödinger equation

$$(i\partial_t + \Delta_D)v(t,x) = \chi_\lambda g(t,x), \quad v|_{t=0} = 0. \tag{8.41}$$

We denote by $v^\pm$ the solutions to the equations

$$(i\partial_t + \Delta_D)v^\pm(t,x) = \chi_\lambda g(t,x)\mathbf{1}_{\{\pm t>0\}}, \quad v^\pm|_{t=0} = 0. \tag{8.42}$$

For $\epsilon > 0$ and $\pm t > 0$ we define $v^\pm_\epsilon = e^{\mp\epsilon t}v^\pm$ which satisfy

$$\begin{cases} (i\partial_t + \Delta_D \pm i\epsilon)v^\pm_\epsilon = 1_{\{\pm t>0\}}e^{-\epsilon t}\chi_\lambda g_\epsilon, \\ v^\epsilon_\pm|_{t=0} = 0, \quad v^\epsilon_\pm|_{\partial\Omega} = 0. \end{cases} \tag{8.43}$$

After performing the Fourier transform $\mathcal{F}$ with respect to the time variable $t$, the equation (8.43) becomes

$$\begin{cases} (\Delta_D - \tau \pm i\epsilon)\hat{v}^\pm_\epsilon(\tau,x) = \chi_\lambda \mathcal{F}(1_{\{\pm t>0\}}e^{-\epsilon t}\chi_\lambda g_\epsilon)(\tau,x), \\ \hat{v}^\pm_\epsilon|_{\partial\Omega} = 0, \end{cases} \tag{8.44}$$

where $\tau$ denotes the dual variable of $t$ and we deduce

$$\chi_\lambda \hat{v}^+_\epsilon(\tau,x) = \chi_\lambda(\Delta_D - \tau + i\epsilon)^{-1}\chi_\lambda \mathcal{F}(1_{\{\pm t>0\}}e^{-\epsilon t}\chi_\lambda g_\epsilon)(\tau,x). \tag{8.45}$$

Since $-\Delta_D$ is a positive self-adjoint operator, the resolvent $(-\Delta_D - z)^{-1}$ is analytic in $\mathbb{C}\setminus\mathbb{R}_+$. Since the spectrum of $-\Delta_D$ is confined to the positive real axis it is convenient to introduce $\lambda^2 \in \mathbb{R}_+$ as a spectral parameter. Notice, however, that there are two manners to approach $\lambda^2 > 0$ in $\mathbb{C}\setminus\mathbb{R}_+$, choosing the positive imaginary part, which corresponds to considering $\lambda^2 + i\epsilon$, or the negative imaginary part, which corresponds to $\lambda^2 - i\epsilon$. The "physical" choice corresponds to the limiting absorption principal and consists of taking $\lambda^2 + i\epsilon$. In some sense, the limiting absorption principal allows to recover the "sense of time". If one replaces $\lambda^2$ by $\lambda^2 \pm i\epsilon$, $\epsilon > 0$, then $(\lambda^2 \pm i\epsilon)$ belongs to the resolvent of the Laplace operator on $\Omega$ with Dirichlet boundary conditions on $\partial\Omega$ and we can let $\epsilon$ tend to 0 in (8.45) (since the operator $\chi_\lambda(\Delta_D - \tau + i\epsilon)^{-1}\chi_\lambda$ has a limit as $\epsilon \to 0$) and so we can express the Fourier transform of the unique solutions $v^\pm$ of (8.42) as

$$\chi_\lambda \hat{v}^\pm(\tau,x) = \lim_{\epsilon\to 0}\chi_\lambda(\Delta_D - \tau \pm i\epsilon)^{-1}\chi_\lambda \mathcal{F}(1_{\{\pm t>0\}}e^{-\epsilon t}\chi_\lambda g_\epsilon)(\tau,x). \tag{8.46}$$

*Remark* 8.4. Notice that here we used the fact that for $-\frac{\tau}{\lambda^2}$ away from a neighborhood of 1 we have the bounds

$$\|\chi_\lambda \hat{v}^+\|_{L^2} \le \frac{C\|\chi_\lambda \hat{g}\|_{L^2}}{|\tau| + \lambda^2}.$$

We conclude using that if $w_\epsilon = \hat{v}^+_\epsilon$, $f_\epsilon = \mathcal{F}(1_{\{t>0\}}e^{-\epsilon t}\chi_\lambda g_\epsilon)$ the following holds

**Proposition 8.6.** *([104]) As $\epsilon \to 0$, $w_\epsilon$ converges to the unique solution of* (8.39)-(8.40), *where $f = \lim_{\epsilon\to 0} f_\epsilon$.*

It remains to notice that if $\mathcal{H}$ is any Hilbert space, than the Fourier transform defines an isometry on $L^2(\mathbb{R},\mathcal{H})$. □

### 8.3.2 Smoothing effect

In this section we prove Theorem 8.1:

1. (8.1) **case $s = 0$:**

Let $\lambda > 0$, $w$ and $g$ be such that (8.9) holds. We multiply (8.9) by $\chi_\lambda \bar{w}$ and we integrate on $\Omega$

$$\int \chi_\lambda |\nabla w|^2 dx = \lambda^2 \int \chi_\lambda |w|^2 dx - 2\lambda^\alpha \int < \nabla w, \nabla \chi(\lambda^\alpha(|x|-1)) > \bar{w} dx - \int \chi_\lambda^2 f \bar{w} dx, \quad (8.47)$$

thus, using that $0 \leq \chi_\lambda \leq 1$, $\chi_\lambda \leq \chi_\lambda^2$, $\nabla \chi(\lambda^\alpha(|x|-1)) \lesssim \chi_\lambda$, we have for all $\delta > 0$

$$\int |\chi_\lambda \nabla w|^2 dx \lesssim \lambda^2 \int |\chi_\lambda w|^2 dx + \delta \int |\chi_\lambda f|^2 dx + \frac{1}{4\delta} \int |\chi_\lambda w|^2 dx + \lambda^\alpha \int \bar{w} |\nabla w| \chi_\lambda^2 dx. \quad (8.48)$$

Since for all $\delta_1 > 0$ one has

$$\int \bar{w} |\nabla w| \chi_\lambda^2 dx \leq (\int |\chi_\lambda \nabla w|^2 dx)^{1/2} (\int |\chi_\lambda w|^2 dx)^{1/2} \leq \delta_1 \|\chi_\lambda \nabla w\|^2_{L^2(\Omega)} + \frac{1}{4\delta_1} \|\chi_\lambda w\|^2_{L^2(\Omega)}, \quad (8.49)$$

taking $\delta = \lambda^{-\alpha}$, $\delta_1 = \lambda^{-\alpha}/2$ together with $\int |\chi_\lambda w|^2 \lesssim \lambda^{-2-\alpha} \int |\chi_\lambda f|^2$ (according to the computations made in the preceding section) we deduce

$$\int |\chi_\lambda \nabla w|^2 dx \lesssim \lambda^{-\alpha} \int |\chi_\lambda f|^2 dx. \quad (8.50)$$

Thus we have obtained

$$\|\chi_\lambda w\|_{H^1_D(\Omega)} \lesssim \lambda^{-\frac{\alpha}{2}} \|\chi_\lambda f\|_{L^2(\Omega)}. \quad (8.51)$$

Dualizing (8.51) we find

$$\|\chi_\lambda w\|_{L^2(\Omega)} \lesssim \lambda^{-\frac{\alpha}{2}} \|\chi_\lambda f\|_{H^{-1}_D(\Omega)}, \quad (8.52)$$

which will yield (8.1) for $s = -1$.

(8.1) **case $s = 1$:**

Let again $w$ and $f$ be such that (8.9) holds and let $\tilde{\chi}_\lambda$ be a smooth cutoff function equal to 1 on the support of $\chi_\lambda$. Write

$$\|\chi_\lambda w\|_{H^2_D(\Omega)} \approx \|\chi_\lambda w\|_{H^1_D(\Omega)} + \|\Delta_D(\chi_\lambda w)\|_{L^2(\Omega)}. \quad (8.53)$$

Since $\|\chi_\lambda w\|_{H^1_D(\Omega)}$ can be estimated by means of (8.51), we only need to obtain bounds for $\|\Delta_D(\chi_\lambda w)\|_{L^2(\Omega)}$. We write

$$\Delta_D(\chi_\lambda w) = \chi_\lambda \Delta_D w + [\Delta_D, \chi_\lambda]\tilde{\chi}_\lambda w. \tag{8.54}$$

The commutator $[\Delta_D, \chi_\lambda]$ is bounded from $H^1_D(\Omega)$ to $L^2(\Omega)$ with norm less then $C\lambda^\alpha$ and we have chosen $\alpha < 1$ ($C$ does not depend on $\lambda$), while the first term in the right hand side of (8.54) is in $H^1_D(\Omega)$ since $\Delta_D w = \chi_\lambda f - \lambda^2 w$ and satisfies (8.9) with $\chi_\lambda f$ replaced by $\Delta_D(\chi_\lambda f)$. Therefore, we can apply (8.51) in order to deduce the inequality (8.1) for the Fourier transforms in time of $u$ and $f$ which appear in the proposition. Since we have obtained the result for $s = -1$ and $s = 1$ we can use an interpolation argument to get it for $s \in [-1, 1]$.

2. (8.2) **case** $s = 0$**:**
If we denote by $A_\lambda$ the operator which to a given $u_0 \in L^2(\Omega)$ associates $\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0$, we need to prove that $A_\lambda$ is bounded from $L^2(\Omega)$ to $L^2_T H^{\frac{1}{2}}_D(\Omega)$ with the norm less than $C\lambda^{-\frac{\alpha}{4}}$ for some constant $C$ independent of $\lambda$, which in turn is equivalent to the continuity of the adjoint operator,

$$A^*_\lambda(f) = \int_0^T \psi(\frac{-\Delta_D}{\lambda^2})e^{-i\tau\Delta_D}\chi_\lambda f(\tau)d\tau, \tag{8.55}$$

from $L^2_T H^{-\frac{1}{2}}_D(\Omega)$ to $L^2(\Omega)$ with the norm bounded by $C\lambda^{-\frac{\alpha}{4}}$, which is equivalent to showing that the operator $A_\lambda A^*_\lambda$ defined by

$$(A_\lambda A^*_\lambda f)(t) = \int_0^T \chi_\lambda e^{it\Delta_D}\psi^2(\frac{-\Delta_D}{\lambda^2})e^{-i\tau\Delta_D}\chi_\lambda f(\tau)d\tau \tag{8.56}$$

is continuous from $L^2_T H^{-\frac{1}{2}}_D(\Omega)$ to $L^2_T H^{\frac{1}{2}}_D(\Omega)$ and its norm is bounded from above by $C\lambda^{-\frac{\alpha}{2}}$. We write $(A_\lambda A^*_\lambda f)(t)$ as a sum

$$\begin{aligned}(A_\lambda A^*_\lambda f)(t) = \int_0^t \chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})e^{i(t-\tau)\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda f(\tau)d\tau \\ + \int_t^T \chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})e^{i(t-\tau)\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})\chi_\lambda f(\tau)d\tau.\end{aligned} \tag{8.57}$$

Hence, in order to conclude it is sufficient to apply (8.1) with $s = -\frac{1}{2}$ together with time inversion, since the second term on the right hand side of (8.57) will solve the same problem with initial data $u|_{t=T} = u_0$.

(8.2) **case** $s = 1$**:**
**Lemma 8.1.** *We have*

$$\|\Delta_D\Big(\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big)\|_{L^2_T H^{-\frac{1}{2}}_D(\Omega)} \lesssim \lambda^{1-\frac{\alpha}{4}}\|\chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.58}$$

**Corollary 8.1.** *Lemma 8.1 yields*

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{\frac{3}{2}}(\Omega)} \lesssim \lambda^{1-\frac{\alpha}{4}}\|\chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.59}$$

Corollary 8.1 and an interpolation argument now yield for $\theta \in [0,1]$

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{\frac{3\theta}{2}}(\Omega)} \lesssim \lambda^{\frac{3\theta}{2}-\frac{1}{2}-\frac{\alpha}{4}}\|\chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}, \tag{8.60}$$

achieving the proof of Theorem 4.1.

*Proof.* (*of Lemma 8.1*) Write

$$\begin{aligned}(-\Delta_D+1)\Big(\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big) &= \chi_\lambda(-\Delta_D+1)\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big)\\ &\quad - [\Delta_D,\chi_\lambda]\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big).\end{aligned} \tag{8.61}$$

— For the first term in the right hand side of (8.61) we show that the operator

$$B_\lambda := \Big(H^1_D(\Omega) \ni u_0 \to \chi_\lambda(-\Delta_D+1)\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big) \in L^2_T H_D^{-\frac{1}{2}}(\Omega)\Big) \tag{8.62}$$

is continuous and its norm from $H^1_D(\Omega)$ to $L^2_T H_D^{-\frac{1}{2}}(\Omega)$ is bounded from above by $C\lambda^{-\frac{\alpha}{4}}$ for some constant $C$ independent of $\lambda$, or equivalently that

$$\begin{aligned}\Big(B_\lambda(-\Delta_D+1)^{-1}B^*_\lambda f\Big)(t) &= \int_0^T \chi_\lambda(-\Delta_D+1)\psi^2(\frac{-\Delta_D}{\lambda^2})e^{i(t-\tau)\Delta_D}\chi_\lambda f(\tau)d\tau\\ &= (-\Delta_D+1)(A_\lambda A^*_\lambda f)(t) + [\Delta_D,\chi_\lambda]\int_0^T e^{i(t-\tau)\Delta_D}\psi^2(\frac{-\Delta_D}{\lambda^2})\chi_\lambda f(\tau)d\tau\end{aligned} \tag{8.63}$$

is bounded from $L^2_T H_D^{\frac{1}{2}}(\Omega)$ to $L^2_T H_D^{-\frac{1}{2}}(\Omega)$ by $C\lambda^{-\frac{\alpha}{2}}$. Here $A_\lambda$ is the operator introduced in the proof of the case $s=0$. For the first term in the right hand side of (8.63) we apply (8.1) with $s=\frac{1}{2}$ and we obtain a bound $C\lambda^{-\frac{\alpha}{2}}$, while for the second term we make use of (8.2) with $s=-\frac{1}{2}$ and of the fact that $[\Delta_D,\chi_\lambda]$ is bounded from $H_D^{\frac{1}{2}}(\Omega)$ to $H_D^{-\frac{1}{2}}(\Omega)$ with a norm bounded by $C\lambda^\alpha$ in order to find a bound from $L^2_T H_D^{\frac{1}{2}}(\Omega)$ to $L^2_T H_D^{-\frac{1}{2}}(\Omega)$ of at most $C\lambda^{\frac{\alpha}{2}-1} \le C\lambda^{-\frac{\alpha}{2}}$.

— The second term in the right hand side of (8.61) gives

$$\begin{aligned}&\|[\Delta_D,\chi_\lambda]e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)}\\ &\qquad\lesssim \lambda^{2\alpha}\|\chi''_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)}\\ &\qquad\quad + 2\lambda^\alpha\|\chi'_\lambda\nabla\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big)\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)}.\end{aligned} \tag{8.64}$$

For evaluating the first term in the last sum we use again (8.2) with $s = 0$

$$\lambda^{2\alpha}\|\chi_\lambda'' e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H^{\frac{1}{2}}_D(\Omega)} \lesssim \lambda^{2\alpha-\frac{\alpha}{4}}\|\tilde{\chi}_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (8.65)$$

where $\breve{\chi}_\lambda$ is a smooth cutoff function such that $\tilde{\chi}_\lambda$ is equal to 1 on the support of $\chi_\lambda$ and we conclude since $\alpha < 1$. For the second term we have

$$\begin{aligned}\|\chi_\lambda'\nabla\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big)\|^2_{L^2(\Omega)} &= -\int \chi_\lambda'^2(\Delta_D u)\bar{u} - 2\lambda^\alpha Re\int \chi_\lambda'\chi_\lambda''(\nabla u)\bar{u} \\ &\leq \lambda^2\int \chi_\lambda'^2\Big|\Big(e^{it\Delta_D}\tilde{\psi}(\frac{-\Delta_D}{\lambda^2})u_0\Big)\bar{u}\Big| + 2\lambda^{\alpha+1/2}\|\tilde{\chi}_\lambda\|^2_{L^2(\Omega)} \\ &\lesssim \lambda^2\|\tilde{\chi}_\lambda u\|^2_{L^2(\Omega)}, \quad (8.66)\end{aligned}$$

where we have set $\tilde{\psi}(x) = x\psi(x)$. From (8.2) with $s = 0$ that we have already established, we find, since $\alpha < 1$,

$$\begin{aligned}\lambda^\alpha\|\chi_\lambda'\nabla\Big(e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\Big)\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)} &\lesssim \lambda^\alpha\|\tilde{\chi}_\lambda u\|_{L^2_T H^{\frac{1}{2}}(\Omega)} \\ &\lesssim \lambda^{\frac{3\alpha}{4}}\|\tilde{\chi}_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)} \\ &\lesssim \lambda^{1-\frac{\alpha}{4}}\|\tilde{\chi}_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (8.67)\end{aligned}$$

□

**Neumann problem:** *All of the above results remain valid if we consider the Neumann Laplacian* $\Delta_N$*: indeed, let* $w$ *and* $f$ *be such that* (8.9) *holds. Using the same strategy as before we get:*

$$\begin{aligned}\int_{\partial\Omega}\partial_n u(\chi_\lambda\bar{u})d\sigma - \int_\Omega \chi_\lambda|\nabla w|^2dx + \lambda^2\int_\Omega \chi_\lambda|w|^2dx \\ - \lambda^\alpha\int_\Omega < \nabla w, \nabla\chi_\lambda > \tilde{\chi}_\lambda\bar{w}dx = \int_\Omega \chi_\lambda^2 f\bar{w}dx. \quad (8.68)\end{aligned}$$

*Notice that, due to the Neumann boundary conditions, the first term in the left hand side vanishes, so the computations are almost the same as in the previous case. However we must pay a little attention to the spectrum of the Neumann Laplacian: for, we have to introduce a spectral cutoff* $\phi \in C_0^\infty$ *equal to* 1 *near* 0 *and decompose*

$$w = \phi(-\Delta_N)w + (1-\phi(-\Delta_N))w, \quad f = \phi(-\Delta_N)f + (1-\phi(-\Delta_N))f, \quad (8.69)$$
$$u_0 = \phi(-\Delta_N)u_0 + (1-\phi(-\Delta_N))u_0.$$

*We can then rewrite the above proof with* $w$, $f$, $u_0$ *replaced by* $(1-\phi(-\Delta_N))w$, $(1-\phi(-\Delta_N))f$, $(1-\phi(-\Delta_N))u_0$ *in order to obtain similar estimates in Theorem 4.1. To deal with the contributions of the remaining terms we use the fact that in this situation the* $L^2$ *and* $H^k_N$ *norms are equivalent.*

## 8.4 Estimates away from the obstacle

In this section we obtain bounds away from the obstacle. The main idea is to construct a new function $v = \phi u$ which will solve a problem with a nonlinearity supported in a compact set; under these assumptions, it is proved that the free evolution satisfies the usual Strichartz bounds (see [98]). However, we have to take into account the fact that the neighborhood outside of which we will apply this result is of size $\lambda^{-\alpha}$ and thus we will "lose" $\alpha$ derivative; however, this will not pose any problem if $\alpha$ is chosen small enough, since it will be covered by the loss of derivatives near the boundary. After a change of variables we can assume that $\Omega = \{x_3 > 0\}$ and thus $x_3$ defines the distance to the boundary. We define $\phi \in C^\infty(\bar{\Omega})$ by $\phi(x)|_{x_3 \leq 1} = x_3$, $\phi(x)|_{x_3 \geq 2} = 1$.

**Proposition 8.7.** *We have*

$$\|(1-\chi_\lambda)e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T L^{\frac{2d}{d-2}}(\Omega)} \lesssim \lambda^\alpha \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.70}$$

We set $v = \phi u$, where $u = e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0$ solves (9.4). Then $v$ solves the equation

$$\begin{cases} (i\partial_t + \Delta_D)v = (\Delta_D\phi)u + 2\nabla\phi\nabla u + \frac{\partial v}{\partial_n}|_{\partial\Omega} \otimes \delta^{(0)}_{\partial\Omega} - v|_{\partial\Omega} \otimes \delta^{(1)}_{\partial\Omega} = [\Delta_D, \phi]u, \\ v|_{t=0} = \phi u|_{t=0}, \quad v|_{\partial\Omega} = 0, \end{cases} \tag{8.71}$$

where $\delta$ is the Dirac measure on $\partial\Omega$. It can easily be seen that the last two terms vanish. We have $[\Delta_D, \phi]u \in L^2_T H^{\frac{-1}{2}}_{comp}(\Omega)$ and using [26, Prop.2.7]

$$\begin{aligned} \|[\Delta_D, \phi]e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{\frac{-1}{2}}(\Omega)} &\lesssim \|e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H_D^{\frac{1}{2}}(\Omega)} \\ &\lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \end{aligned} \tag{8.72}$$

The inhomogeneous part of the equation (8.71) satisfied by $v$ has compact spatial support and therefore we can employ [98, Thm.3] (for $p = 2$) and [26, Prop.2.10] in order to obtain Strichartz estimates without losses for $v$

$$\|\phi e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}, \tag{8.73}$$

for every $(p, q)$ $d$-admissible pair, which in turn yields

$$\begin{aligned} \lambda^{-\alpha}\|(1-\chi_\lambda)e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} &\lesssim \|(1-\chi_\lambda)\phi e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \\ &\lesssim \|\phi e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \\ &\lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \end{aligned} \tag{8.74}$$

In particular, for $p = 2$ we get (8.70) from (8.74).

**Neumann problem:** *When dealing with the Neumann problem, this approach requires some adjustments, but the above result remains valid (with $\alpha$ slightly modified). Let $\epsilon > 0$ and consider $\phi|_{x_3\leq 1}(x) = x_3^{1+\epsilon}$, $\phi|_{x_3\geq 2}(x) = 1$. Then $v = \phi u$ solves the equation*

$$\begin{cases} (i\partial_t + \Delta_D)v = [\Delta_D, \phi^{1+\epsilon}]u, \quad v|_{t=0} = \phi u|_{t=0}, \\ \frac{\partial v}{\partial \nu}|_{\partial\Omega} = (x_3^\epsilon u)|_{\partial\Omega} + \phi\frac{\partial u}{\partial\nu}|_{\partial\Omega} = 0. \end{cases} \tag{8.75}$$

*It's easy to see that one can obtain similar estimates, with $\alpha$ replaced by $\alpha - \epsilon$; still, this makes no difference for our purpose, since $\epsilon$ can be chosen as small as we like.*

## 8.5 Proof of Theorem 8.2

In this section we achieve the proof of Theorem 8.2. Taking $s = 1/2$ in (8.2) gives

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H^1_D(\Omega)} \lesssim \lambda^{\frac{1}{2}-\frac{\alpha}{4}}\|\chi_\lambda\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}, \tag{8.76}$$

from which we deduce

$$\begin{aligned} \|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T L^{\frac{2d}{d-2}}(\Omega)} &\lesssim \|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H^1_D(\Omega)} \\ &\lesssim \lambda^{\frac{1}{2}-\frac{\alpha}{4}}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \end{aligned} \tag{8.77}$$

An energy argument yields

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^\infty_T L^2(\Omega)} \lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.78}$$

Interpolation between (8.77) and (8.78) with weights $\frac{2}{p}$ and $1-\frac{2}{p}$ respectively yields

$$\|\chi_\lambda e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \lesssim \lambda^{\frac{1}{p}(1-\frac{\alpha}{2})}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.79}$$

We have also obtained estimates away from the boundary

$$\|(1-\chi_\lambda)e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \lesssim \lambda^{\alpha}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.80}$$

If we take $\alpha = \frac{1}{p}(1-\frac{\alpha}{2})$ we find $\alpha = \frac{2}{2p+1}$. Now, for $p = 2$ this gives $\alpha = \frac{2}{5}(\leq \frac{2}{3})$ and consequently we have

$$\|e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T L^{\frac{2d}{d-2}}(\Omega)} \lesssim \lambda^{\frac{2}{5}}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.81}$$

Interpolation between (8.81) and

$$\|e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^\infty_T L^2(\Omega)} \lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}, \tag{8.82}$$

with weights $\frac{2}{p}$ and $1-\frac{2}{p}$ yields, for every $(p,q)$ admissible pair, $p \geq 2$

$$\|e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \lesssim \lambda^{\frac{4}{5p}}\|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.83}$$

**Neumann problem:** The case of the Neumann conditions could be handled in the same way. However in (8.80) we have $\alpha - \epsilon$ instead of $\alpha$ so we find $\alpha = \frac{2}{5} + \frac{4\epsilon}{5}$ and for every $(p, q)$ $d$-admissible pair and every $\epsilon > 0$ we have

$$\|e^{it\Delta_N}\psi(\frac{-\Delta_N}{\lambda^2})u_0\|_{L^p_T L^q(\Omega)} \lesssim \lambda^{\frac{4}{5p}+\frac{8\epsilon}{5p}}\|\psi(\frac{-\Delta_N}{\lambda^2})u_0\|_{L^2(\Omega)}. \tag{8.84}$$

## 8.6 Applications

### 8.6.1 Proof of Theorem 8.3 for non-trapping obstacle

The proof of Theorem 8.3 relies on the contraction principle applied to the equivalent integral equation associated to (8.6) with Dirichlet boundary conditions

$$u(t) = e^{it\Delta_D}u_0 - i\int_0^t e^{i(t-\tau)\Delta_D}F(u(\tau))d\tau. \tag{8.85}$$

The assumptions on $F$ and on the potential $V$ imply the following pointwise estimates:

$$|F(u)| \lesssim |u|(1 + |u|^2), \quad |\nabla F(u)| \lesssim |\nabla u|(1 + |u|^2), \tag{8.86}$$

$$|F(u) - F(v)| \lesssim |u - v|(1 + |u|^2 + |v|^2), \tag{8.87}$$

$$\begin{aligned}|\nabla(F(u) - F(v))| \lesssim\ & |\nabla(u - v)|(1 + |u|^2 + |v|^2)\\ &+ |u - v|(|\nabla u| + |\nabla v|)(1 + |u| + |v|).\end{aligned} \tag{8.88}$$

The aim is to show that for sufficiently small $T > 0$ we can solve the integral equation by a Picard iteration scheme in the Banach space $X_T := L^\infty_T H^1_0(\Omega) \cap L^p_T W^{\sigma,q}(\Omega)$, where $2 < p < \frac{12}{5}$ and $\sigma = 1 - \frac{4}{5p}$. We equip $X_T$ with the norm

$$\|u\|_{X_T} := \|u\|_{L^\infty_T H^1_0(\Omega)} + \|(1 - \Delta_D)^{\frac{\sigma}{2}}u\|_{L^p_T L^q(\Omega)}.$$

*Remark* 8.5. For $2 < p < \frac{12}{5}$ one has the continuous embedding $W^{1-\frac{4}{5p},q}(\Omega) \subset L^\infty(\Omega)$ where $(p, q)$ is an admissible pair, since in this case $1 - \frac{4}{5p} > \frac{3}{q}$ and Proposition 8.1 implies $u \in L^p_T L^\infty(\Omega)$ for any $u \in X_T$.

Define a nonlinear map $\Phi$ as follows

$$\Phi(u)(t) := e^{it\Delta_D}u_0 - i\int_0^t e^{i(t-\tau)\Delta_D}F(u(\tau))d\tau. \tag{8.89}$$

We have to show that $\Phi$ is a contraction in a suitable ball $B(0,R)$ of $X_T$. Using the Minkowski integral inequality together with an energy argument we have

$$\begin{aligned}\|\Phi(u)\|_{L^\infty_T H^1_0(\Omega)} &\lesssim \|u_0\|_{H^1_0(\Omega)} + \|F(u)\|_{L^1_T H^1_0(\Omega)}\\ &\lesssim (|u_0\|_{H^1_0(\Omega)} + \|u\|_{L^1_T H^1_0(\Omega)}\|u\|^2_{L^\infty_T H^1_0(\Omega)}\\ &+ T\|u\|_{L^\infty_T H^1_0(\Omega)} + T^{1-\frac{2}{p}}\|u\|_{L^\infty_T H^1_0(\Omega)}\|u\|^2_{L^p_T L^\infty(\Omega)})\\ &\leq C\Big(\|u_0\|_{H^1_0(\Omega)} + T\|u\|_{X_T} + (T^{1-\frac{2}{p}} + T)\|u\|^3_{X_T}\Big), \end{aligned} \tag{8.90}$$

and on the other hand

$$\begin{aligned}\|(1-\Delta_D)^{\frac{\sigma}{2}}\Phi(u)\|_{L^p_T L^q(\Omega)} &\lesssim \Big(\|(1-\Delta_D)^{\frac{\sigma}{2}}u_0\|_{H_D^{\frac{4}{5p}}(\Omega)}\\ &+ \int_0^T \|(1-\Delta_D)^{\frac{\sigma}{2}}(1+|u(s)|^2)u(s)\|_{H_D^{\frac{4}{5p}}(\Omega)}\Big)\\ &\lesssim (\|u_0\|_{H^1_0(\Omega)} + \int_0^T (1+\|u(s)\|^2_{L^\infty})\|u(s)\|_{H^1_0(\Omega)})\\ &\leq C\Big(\|u_0\|_{H^1_0(\Omega)} + T\|u\|_{X_T} + T^{1-\frac{2}{p}}\|u\|^3_{X_T}\Big). \end{aligned} \tag{8.91}$$

In a similar way we obtain

$$\|\Phi(u)-\Phi(v)\|_{X_T} \leq C\|u-v\|_{X_T}\Big(T + (T+T^{1-\frac{2}{p}})(1+\|u\|^2_{X_T} + \|v\|^2_{X_T})\Big). \tag{8.92}$$

We chose $R > 4C\|u_0\|_{H^1_0(\Omega)}$ and $T$ such that $4C\Big(T + (T+T^{1-\frac{2}{p}})R^2\Big) < 1$. Taking $T$ eventually smaller such that $C\Big(T + (T+T^{1-\frac{2}{p}})(1+2R^2)\Big) < 1$, we deduce from (8.90), (8.91) and (8.92) that $\Phi$ in a contraction from $B(0,R) \subset X_T$ to $B(0,R)$. Therefore, if we consider the sequence $\{v_n\}_{n\in\mathbb{N}} \subset X_T$ such that $v_0 = u_0 \in B(0,R)$, $v_{n+1} = \Phi(v_n)$, then $v_n$ converges in $X_T$ to the unique solution in $X_T$ of the integral equation

$$u(t) = e^{it\Delta_D}u_0 - i\int_0^t e^{i(t-\tau)\Delta_D}F(u(\tau))d\tau \tag{8.93}$$

which yields the local well-posedness result. From (8.92) we obtain the Lipschitz property of the flow map. Using a standard approximation argument we can derive the conservation laws. Next due to (8.7), the assumption on $V$ ($V(|u|^2) \geq 0$) and the Gagliardo-Nirenberg inequality we can extend the local solution to an arbitrary time interval by reiterating the local-posedness argument.

**Neumann problem:** When we consider the Neumann Laplacian we take $X_T$ like before and we choose $\epsilon$ so small such that the embedding $W^{1-\frac{4}{5p}-\frac{8\epsilon}{5p},q}(\Omega) \subset L^\infty(\Omega)$ still holds: for $\epsilon < \frac{12-5p}{16}$ we are led to the same conclusion.

### 8.6.2 Proof Theorem 8.2 for a class of trapping obstacles

In this part we prove Theorem 8.2 for a class of trapping obstacles. In this case, since there are trapped trajectories (e.g. any line minimizing the distance between two obstacles has an unbounded sojourn time), the plain smoothing effect $H^{\frac{1}{2}}$ does not hold. However, one can obtain a smoothing effect with a logarithmic loss (see [25, Thm1.7,Thm.4.2]).

*Assumption* 8.1. We suppose here $\Theta = \cup_{i=1}^{N}\Theta_i$ is the disjoint union of a finite number of balls $\Theta_i = B_i(o_i, r_i)$ of radius $r_i > 0$ in $\mathbb{R}^3$. We denote by $k$ the minimum of the curvatures of the spheres $\mathbb{S}(r_i) = \partial\Theta_i$, that is $k = \min\{\frac{1}{\sqrt{r_i}}, i = \overline{1, N}\}$. Denote by $L$ the minimum of the distances between two balls. Then, if $N > 2$, we assume that $kL > N$ ($L \geq l > 0$ if $N = 2$ for some strictly positive $l$). We keep the notations of the previous sections.

Let $\chi \in C_0^\infty((-1, 1))$, $\chi = 1$ close to 0 and for every $i$, set $\chi_i(x) := \chi(\frac{|x-o_i|}{r_i} - 1)$ and $\chi_{i,\lambda}(x) = \chi(\lambda^\alpha(\frac{|x-o_i|}{r_i} - 1))$ ($\chi_{i,\lambda}$ vanishes outside a neighborhood of size $\lambda^{-\alpha}$ of the ball $\Theta_i$). Set $u = e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0$ and $v_i = \chi_i u$. Then $v_i$ satisfies

$$(i\partial_t + \Delta_D)v_i = [\Delta_D, \chi_i]u, \quad v_i|_{t=0} = \chi_i\psi(\frac{-\Delta_D}{\lambda^2})u_0. \tag{8.94}$$

We introduce the Dirichlet Laplacian $\Delta_D^i$ acting on $\Omega_i := \mathbb{R}^3 \setminus \Theta_i$ and continue to denote $\Delta_D$ the Dirichlet Laplacian outside the obstacle $\Theta$. Writing Duhamel's formula we get

$$v_i(t) = e^{it\Delta_D^i}\chi_i\psi(\frac{-\Delta_D}{\lambda^2})u_0 - i\int_0^t e^{it\Delta_D^i}\psi(\frac{-\Delta_D}{\lambda^2})e^{-it\tau\Delta_D}[\Delta_D, \chi_i](u(\tau))d\tau. \tag{8.95}$$

The solution $v_i$ of (8.94) satisfies

$$\begin{aligned}\|\chi_{i,\lambda}v_i\|_{L^2_T H_D^{\frac{1}{2}}(\Omega)} \leq{}& \|\chi_{i,\lambda}e^{it\Delta_D^i}\chi_i\psi(\frac{-\Delta_D^i}{\lambda^2})u_0\|_{L^2_T H_D^{\frac{1}{2}}(\Omega)} \\ &+ \|\chi_{i,\lambda}\int_0^t e^{it\Delta_D^i}\psi(\frac{-\Delta_D}{\lambda^2})e^{-it\tau\Delta_D}[\Delta_D, \chi_i](u(\tau))d\tau\|_{L^2_T H_D^{\frac{1}{2}}(\Omega)}.\end{aligned} \tag{8.96}$$

From [25, Thm.4.2] we know that the operator

$$A_i^* f_i := \int_0^T \psi(\frac{-\Delta_D}{\lambda^2})e^{-i\tau\Delta_D}\chi_i f_i(\tau)d\tau \tag{8.97}$$

is bounded from $L^2_T H_D^{-\frac{1}{2}}(\Omega)$ to $H_D^{-\epsilon}(\Omega)$. Notice that in (8.97) we can introduce a cutoff function $\tilde{\chi}_i \in C_0^\infty$ equal to 1 on the support of $\chi_i$ without changing the integral modulo smoothing terms. We need the next lemma:

**Lemma 8.2.** *Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+^*)$ be a smooth function such that $\tilde{\psi}(\tau) = 1$ if $\psi(\tau) = 1$ and such that $\sum_{k\geq 0}\tilde{\psi}(2^{-2k}\tau) = 1$. Let $\tilde{\chi}_i \in C_0^\infty$ be equal to 1 on the support of $\chi_i$. Then for $f \in L^2(\Omega)$*

$$\tilde{\psi}(\frac{-\Delta_D^i}{\lambda^2})\tilde{\chi}_i\psi(\frac{-\Delta_D}{\lambda^2})(\chi_i f) = \tilde{\chi}_i\psi(\frac{-\Delta_D}{\lambda^2})(\chi_i f) + O_{L^2(\Omega)}(\lambda^{-\infty})\chi_i f. \tag{8.98}$$

We postpone the proof of Lemma 8.2 for the end of this section.

*End of the proof of Theorem 4.2:* We introduce the operator

$$A_{i,\lambda}u_0(t,x) := \chi_{i,\lambda}e^{it\Delta^i_D}\tilde{\psi}(\frac{-\Delta^i_D}{\lambda^2})u_0,$$

which is continuous from $L^2(\Omega_i)$ to $L^2_T H^{\frac{1}{2}}_D(\Omega_i)$ with the norm bounded by $\lambda^{-\alpha/4}$. Indeed, since $\chi_{i,\lambda}$ vanishes outside a small neighborhood of $\Theta_i$ we can apply Theorem 4.1 in $\mathbb{R}^3\setminus\Theta_i$. If we take $f_i = [\Delta_D, \chi_i]u$, then in view of Lemma 8.2 the last term in the right hand side of (8.96) writes $A_{i,\lambda}A^*_i f_i + O_{L^2(\Omega)}(\lambda^{-\infty})$.

If $\check{\chi}_i \in C^\infty_0$ is equal to 1 on the support of $\chi_i$ we can estimate

$$\|[\Delta_D,\chi_i]u\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)} \lesssim \|[\Delta_D,\chi_i]\check{\chi}_i u\|_{L^2_T H_D^{-\frac{1}{2}}(\Omega)} \lesssim \|\check{\chi}_i e^{it\Delta_D}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T H^{\frac{1}{2}}_D(\Omega)} \lesssim \|\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{H^\epsilon_D(\Omega)}, \quad (8.99)$$

where in the last inequality we applied [25, Thm.1.7]. Hence (8.96) becomes

$$\|\chi_{i,\lambda}u\|_{L^2_T H^{\frac{1}{2}}_D(\Omega)} \lesssim \lambda^{-\frac{\alpha}{4}}\|\chi_{i,\lambda}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2_T(\Omega)} + \lambda^{-\frac{\alpha}{4}+2\epsilon}\|\chi_{i,\lambda}\psi(\frac{-\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (8.100)$$

Set $\chi_\lambda := \sum_{i=1}^N \chi_{i,\lambda}$. Since $\{\chi_{i,\lambda}\}$ have disjoint supports, (8.100) remains valid for $\chi_{i,\lambda}$ replaced by $\chi_\lambda$. We have thus obtained a smoothing effect with a gain $\alpha/4 - 2\epsilon$ and by interpolation with the energy estimate we have

$$\|\chi_\lambda u\|_{L^p_T L^q(\Omega)} \lesssim \lambda^{\frac{1}{p}(1-\frac{\alpha}{2}+\frac{\epsilon}{4})}\|\Psi(-\frac{\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (8.101)$$

Away from $\partial\Theta$ we can use the same arguments as in Section 8.4 and find

$$\|(1-\chi_\lambda)u\|_{L^p_T L^q(\Omega)} \lesssim \lambda^\alpha\|\Psi(-\frac{\Delta_D}{\lambda^2})u_0\|_{L^2(\Omega)}. \quad (8.102)$$

Let $p = 2+\epsilon$ (for $\epsilon > 0$ sufficiently small) and take $\alpha$ such that $\frac{1}{2+\epsilon}(1-\frac{\alpha}{2}+\frac{\epsilon}{4}) = \alpha$.

*Proof. (of Lemma 8.2):* We fix $\lambda = 2^{k_0}$, $k_0 \geq 1$. We write

$$\tilde{\chi}_i\psi(-2^{-2k_0}\Delta_D) = \sum_{k\geq 0}\tilde{\psi}(-2^{-2k}\Delta^i_D)\tilde{\chi}_i\psi(-2^{-2k_0}\Delta_D)$$
$$= \tilde{\psi}(-2^{-2k_0}\Delta^i_D)\tilde{\chi}_i\psi(-2^{-2k_0}\Delta_D) + \sum_{k\neq k_0}\tilde{\psi}(-2^{-2k_0}(2^{-2(k-k_0)}\Delta^i_D))\tilde{\chi}_i\psi(-2^{-2k_0}\Delta_D). \quad (8.103)$$

Let $\tilde{B} \subset \mathbb{R}^3$ be a ball of sufficiently large radius such that $\cup_{i=1}^N \text{supp}\tilde{\chi}_i \subset\subset B$ and $\tilde{B}_i \subset \mathbb{R}^3$ be balls such that $\text{supp}\tilde{\chi}_i \subset \tilde{B}_i$. Suppose that $\tilde{\chi}_i$ are suitably chosen such that the distances between any two such balls $\tilde{B}_i$ be strictly positive (this is always possible, eventually shrinking the supports of $\chi_i$). If we denote $\tilde{\Delta}_D$, resp. $\tilde{\Delta}^i_D$ the Laplace operators in the bounded domains $\tilde{B}\cap\Omega$ and resp. $\tilde{B}_i\cap\Omega$, we notice that

$$\Delta_D(\chi_i f) = \tilde{\Delta}_D(\chi_i f) = \Delta^i_D(\chi_i f) = \tilde{\Delta}^i_D(\chi_i f), \quad \forall f \in L^2(\Omega). \tag{8.104}$$

Let $\phi_n$, $n\geq 1$, resp. $\phi^i_m$, $m\geq 1$ denote an orthonormal system of eigenfunctions of $\tilde{\Delta}_D$, resp. $\tilde{\Delta}^i_D$ associated to the eigenvalues $\lambda^2_n$, resp. $\lambda^{i2}_m$, i.e. such that $-\tilde{\Delta}_D\phi_n = \lambda^2_n\phi_n$, $-\tilde{\Delta}^i_D\phi^i_m = \lambda^{i2}_m\phi^i_m$ for $n,m\geq 1$. Consequently if $f_i\in L^2(\Omega)$ is supported in $\tilde{B}_i\cap\Omega$ we have

$$\psi(-2^{-2k_0}\Delta_D)f_i = \sum_{n\geq 1} < f_i,\phi_n > \psi(2^{-2k_0}\lambda^2_n)\phi_n,$$

and if $g_i\in L^2(\Omega)$ is supported in $\tilde{B}_i\cap\Omega$ and we set $A_k = 2^{-2(k-k_0)}$, $k\geq 0$ than

$$\tilde{\psi}(-2^{-2k_0}A_k\Delta^i_D)g_i = \sum_{m\geq 1} < g_i,\phi^i_m > \tilde{\psi}(2^{-2k_0}A_k\lambda^{i2}_m)\phi^i_m,$$

If the support of $\tilde{\psi}$ is sufficiently large we show that the contribution of the sum in the second line of (8.103) for $k\neq k_0$ is $O_{L^2(\Omega)}(\lambda^{-\infty})$. We consider separately the cases $k>k_0$, resp. $k<k_0$. Let first $k>k_0$: we distinguish two case, according to $2^{\epsilon k_0}\lesssim 2^k < 2^{k_0}$ for some $\epsilon\geq 1/4$ or $2^k\lesssim 2^{k_0/4}$.

— Let $2^k < 2^{k_0/4}$ and set $A_k = 2^{2(k_0-k)}$, then $A_k > 2^{k_0} = \lambda\simeq\lambda_n$.

$$\begin{aligned}\tilde{\psi}(-2^{-2k_0}A_k\Delta^i_D)\tilde{\chi}_i\psi(-2^{-2k_0}\Delta_D)\chi_i f = \sum_{n,m\geq 1}&\tilde{\psi}(2^{-2k_0}A_k\lambda^{i2}_m)\psi(2^{-2k_0}\lambda^2_n)\\ &\times < \chi_i f,\phi_n >< \tilde{\chi}_i\phi_n,\phi^i_m > \phi^i_m,\end{aligned} \tag{8.105}$$

$$\begin{aligned}< \tilde{\chi}_i\phi_n,\phi^i_m > &= \frac{1}{\lambda^2_n} < -\tilde{\chi}_i\tilde{\Delta}_D\phi_n,\phi^i_m >\\ &= \frac{1}{\lambda^2_n} < -\tilde{\Delta}_D(\tilde{\chi}_i\phi_n) + [\tilde{\Delta}_D,\tilde{\chi}_i]\phi_n,\phi^i_m > .\end{aligned} \tag{8.106}$$

Since $-\tilde{\Delta}_D(\tilde{\chi}_i\phi_n) = -\tilde{\Delta}^i_D(\tilde{\chi}_i\phi_n)$ is self-adjoint, the first term in the last line writes

$$\frac{1}{\lambda^2_n} < \tilde{\chi}_i\phi_n, -\tilde{\Delta}^i_D\phi^i_m > = \frac{\lambda^{i2}_m}{\lambda^2_n} < \tilde{\chi}_i\phi_n,\phi^i_m >,$$

and hence its contribution in the sum in (8.105) is

$$\frac{\lambda^{i2}_m}{\lambda^2_n}\tilde{\psi}(2^{-2k_0}A_k\lambda^{i2}_m)\psi(2^{-2k_0}\lambda^2_n) < \tilde{\chi}_i\phi_n,\phi^i_m > . \tag{8.107}$$

Since $\psi$, $\tilde{\psi}$ are compactly supported away from 0, the only nontrivial contribution in the sum (8.105) comes from indices $n$, $m$ such that $\lambda^2_n\simeq A_k\lambda^{i2}_m\simeq 2^{2k_0}$ and from

(8.107) this will be $O_{L^2(\Omega)}(A_k^{-1}) = O_{L^2(\Omega)}(1/\lambda_n)$ which follows from the assumption $2^{\epsilon k_0} \lesssim 2^k$ for some $\epsilon \geq 1/4$. We estimate the last term in the right side of (8.106)

$$\frac{1}{\lambda_n^2} < [\tilde{\Delta}_D, \tilde{\chi}_i]\phi_n, \phi_m^i >= O_{L^2(\Omega)}(\lambda_n^{-1}) = O_{L^2(\Omega)}(2^{-k_0}),$$

since on the support of $\psi$, $\lambda_n \simeq 2^{k_0}$. By iterating these arguments $M \geq 1$ times we deduce that the contribution of the sum (8.103) is $O_{L^2(\Omega)}(2^{-Mk_0})$ for every $M \geq 1$.

— Let now $2^{\epsilon k_0} \lesssim 2^k < 2^{k_0}$, $\epsilon \geq 1/4$: in this case a simple integration by part is useless since the "error" is a multiple of the number of integrations. If the support of $\tilde{\chi}_i$ is sufficiently small then (8.104) holds and we have

$$< \tilde{\chi}_i\phi_n, \phi_m^i >= \int (\tilde{\chi}_i - 1)\phi_n \overline{\phi}_m^i dx + \delta_{n=m}, \qquad (8.108)$$

where $\delta_{n=m}$ is the Dirac distribution. In the sum (8.105) we see that the contribution from $n = m$ is zero, since the support of $\tilde{\psi}(A_k.)$ and $\psi(.)$ are disjoint. For first term in the right hand side of (8.108) we use an argument of N.Burq, P.Gérard and N.Tzvetkov [20, Lemma 2.6]. Let $\kappa \in \mathcal{S}(\mathbb{R})$ be a rapidly decreasing function such that $\kappa(0) = 1$. From a result of Sogge [97, Chp.5.1] we can write, on the support of $\tilde{\chi}$ where (8.104) holds

$$\kappa(\sqrt{-\tilde{\Delta}_D} - \lambda)f(x) = \lambda^{1/2} \int e^{i\lambda\varphi(x,y)} a(x, y, \lambda) f(y) dy + R_\lambda f(x),$$

where $a(x, y, \lambda) \in C_0^\infty$ is an asymptotic assumption in $1/\lambda$ and $-\varphi(x, y)$ is the geodesic distance between $x$ and $y$, and where

$$\forall p, s \in \mathbb{N} : \quad \|R_\lambda f\|_{H^s(\text{supp}\tilde{\chi}_i)} \leq C_{p,s}\lambda^{-p}\|f\|_{L^2}.$$

We use (8.104) and $\kappa(\sqrt{-\Delta_D} - \lambda_n)\phi_n = \phi_n$, $\kappa(\sqrt{-\Delta_D^i} - \lambda_m^i)\phi_m^i = \phi_m^i$ in order to write $< (\tilde{\chi}_i - 1)\phi_n, \phi_m^i >$ as an integral (modulo a remaining term, small)

$$\int_{x,y,z} e^{i\lambda_n^{1/2}\Phi(x,y,z)} a(x, y, \lambda_n)\bar{a}(x, z, \lambda_m^i)\phi_n(y)\bar{\phi}_m^i(z) dydzdx,$$

with $\Phi(x, y, z) = \varphi(x, y) + \sqrt{\lambda_m^i/\lambda_n}\varphi(x, z)$. Since $|\nabla_x\varphi|$ is uniformly bounded from below and bounded from above together with all its derivatives, we obtain that there exist $c > 0$, $C_\beta > 0$ such that $|\nabla_x\Phi| \geq c$ and $\partial^\beta\Phi \leq C_\beta$. It remains to perform integrations by parts in the $x$ variable as many times as we want, each such integration providing a gain of a power $\lambda_n^{-1/2}$.

For $k > k_0$ and $2^{k_0} \lesssim 2^{k/4}$, we write

$$< \tilde{\chi}_i\phi_n, \phi_m^i >= \frac{1}{\lambda_m^{i2}} < -\tilde{\chi}_i\phi_n, \tilde{\Delta}_D^i\phi_m^i >$$

$$= \frac{1}{\lambda_m^{i2}} < -\tilde{\Delta}_D(\tilde{\chi}_i\phi_n) + [\tilde{\Delta}_D, \tilde{\chi}_i]\phi_n, \phi_m^i >, \qquad (8.109)$$

in which case, using the spectral localizations $\psi$, $\tilde{\psi}$, we gain a factor $A_k$ from the first term in (8.109) and a factor $A_k(\lambda_m^i)^{-1}$ from the second one; iterating the argument as many times as we want we obtain contribution $O_{L^2(\Omega)}(A_k^M)$. In the last case $2^{\epsilon k} \lesssim 2^{k_0} < 2^k$, $\epsilon \geq 1/4$ we use again the arguments of [20, Lemma 2.6]. The proof is complete. □

## 8.7 Appendix

We will start by recalling some properties of Hankel functions that will be useful in determining the behavior of the solution to (8.9) (see [1]). The Hankel function of order $\nu$ is defined by

$$H_\nu(z) = \int_{-\infty}^{+\infty-i\pi} e^{z\sinh t - \nu t} dt. \tag{8.110}$$

For all values of $\nu$, $\{H_\nu(z), \bar{H}_\nu(z)\}$ form a pair of linearly independent solutions of the Bessel's equation $z^2 y'' + zy' + (z^2 - \mu^2)y = 0$.

**Proposition 8.8.** *We have the following asymptotic expansions (see [1, Chp.9]):*

1. *If $\nu$ is fixed, bounded and $z = r\lambda \gg \nu > \frac{1}{2}$, $\frac{\nu}{z} \ll 1$ then*

$$H_\nu(z) \simeq \sqrt{\frac{2}{\pi z}} e^{i(\lambda r - \frac{\pi\nu}{2} - \frac{\pi}{4})}(1 + O(\lambda^{-1})), \quad \bar{H}_\nu(z) \simeq \sqrt{\frac{2}{\pi z}} e^{-i(\lambda r - \frac{\pi\nu}{2} - \frac{\pi}{4})}(1 + O(\lambda^{-1})). \tag{8.111}$$

2. *If $z = r\lambda > \nu \gg 1$ and $\frac{\nu}{z} \in [\epsilon_0, 1 - \epsilon_1]$ for some small, fixed $\epsilon_0 > 0$, $\epsilon_1 > 0$, then writing $\frac{z}{\nu} = \frac{1}{\cos\beta}$ we have*

$$H_\nu(z) = H_\nu(\frac{\nu}{\cos\beta}) \simeq \sqrt{\frac{2}{\pi\nu\tan\beta}} e^{i\lambda(\tan\beta - \beta) + \frac{i\pi}{4}}(1 + O(\nu^{-1})), \tag{8.112}$$

$$\bar{H}_\nu(z) = \bar{H}_\nu(\frac{\nu}{\cos\beta}) \simeq \sqrt{\frac{2}{\pi\nu\tan\beta}} e^{i\lambda(\tan\beta - \beta) + \frac{i\pi}{4}}(1 + O(\nu^{-1})). \tag{8.113}$$

3. *If $z, \nu \gg 1$ are nearly equal we have the following formulas*
   (a) *If $z - \nu = \tau\nu^{1/3}$ with fixed $\tau$, bounded, then*

$$J_\nu(\nu + \tau\nu^{\frac{1}{3}}) \simeq \frac{\sqrt[3]{2}}{\sqrt[3]{\nu}} Ai(-\sqrt[3]{2}\tau)(1 + O(\nu^{-1})), \tag{8.114}$$

$$Y_\nu(\nu + \tau\nu^{\frac{1}{3}}) \simeq -\frac{\sqrt[3]{2}}{\sqrt[3]{\nu}} (Bi(-\sqrt[3]{2}\tau)(1 + O(\nu^{-1})), \tag{8.115}$$

*where for $|\tau|$ large and $\xi = \frac{2}{3}\tau^{\frac{3}{2}}$ we have*

$$Ai(-\tau) \simeq \frac{1}{\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \sin(\xi + \frac{\pi}{4})(1 + O(\xi^{-1})), \tag{8.116}$$

$$Bi(-\tau) \simeq \frac{1}{\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}} \cos(\xi + \frac{\pi}{4})(1 + O(\xi^{-1})). \tag{8.117}$$

(b) *If $z = \nu$, then*

$$J_\nu(\nu) \simeq \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)}\nu^{-1/3}(1 + O(\nu^{-1})), \tag{8.118}$$

$$Y_\nu(\nu) \simeq -\frac{2^{1/3}}{3^{1/6}\Gamma(2/3)}\nu^{-1/3}(1 + O(\nu^{-1})). \tag{8.119}$$

(c) *If $|\nu - r\lambda| \leq C_0|r\lambda|$ then if $\nu z = r\lambda$ we have for $z < 1$ (resp. $z > 1$)*

$$J_\nu(\nu z) \simeq (\frac{4\zeta}{1 - z^2})^{1/4}\Big(\frac{Ai(\nu^{2/3}\zeta)}{\nu^{1/3}} + \frac{\exp(\frac{2}{3}\nu\zeta^{3/2})}{1 + \nu^{1/6}|\zeta|^{1/4}}O(1/\nu^{4/3})\Big), \tag{8.120}$$

$$Y_\nu(\nu z) \simeq -(\frac{4\zeta}{1 - z^2})^{1/4}\Big(\frac{Bi(\nu^{2/3}\zeta)}{\nu^{1/3}} + \frac{\exp(|Re(\frac{2}{3}\nu\zeta^{3/2})|)}{1 + \nu^{1/6}|\zeta|^{1/4}}O(1/\nu^{4/3})\Big), \tag{8.121}$$

*where the function $\zeta$ is defined by*

$$\frac{2}{3}\zeta^{3/2} = \int_z^1 \frac{\sqrt{1 - t^2}}{t}dt = \ln[(1 + \sqrt{1 - z^2})/z] - \sqrt{1 - z^2}, \quad z \leq 1, \tag{8.122}$$

$$\frac{2}{3}(-\zeta)^{3/2} = \int_1^z \frac{\sqrt{t^2 - 1}}{t}dt = \sqrt{z^2 - 1} - \arccos(1/z), \quad z \geq 1, \tag{8.123}$$

*and where for $\tau$ large and $\xi = \frac{2}{3}\tau^{3/2}$ we have*

$$Ai(\tau) = \frac{1}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}}e^{-\xi}(1 + O(\xi^{-1})), \quad Bi(\tau) = \frac{1}{2\pi^{\frac{1}{2}}\tau^{\frac{1}{4}}}e^{\xi}(1 + O(\xi^{-1})). \tag{8.124}$$

*Taking $\tau = \nu^{2/3}\zeta$, $\xi = \frac{2}{3}\zeta^{3/2}\nu$ we compute $Ai(\nu^{2/3}\zeta)$ using* (8.116), (8.117), (8.124).

*Here $J_\nu$ and $Y_\nu$ are the Bessel functions of the first kind and $H_\nu(z) = J_\nu(z) + iY_\nu(z)$.*

*Remark* 8.6. We remark that $\zeta$ defined in (8.122), (8.123) is analytic in $z$, even at $z = 1$ and $d\zeta/dz < 0$ there; also, at $z = 1$, $\zeta = 0$ and $(1 - z^2)^{-1}\zeta = 2^{-2/3}$. The formulas (8.120), (8.121) are among the deepest and most important results in the theory of Bessel functions. In the paper we use a simpler form of these asymptotic expansions for which we give an idea of the proof inspired from [34].

**Proposition 8.9.** *For $z > 1$ close to 1 and $\nu \gg 1$ large enough we have*

$$J_\nu(\nu z) \simeq \frac{\sqrt{2}}{\nu^{1/3}}(\frac{\zeta}{z^2 - 1})^{1/4}Ai(-\nu^{2/3}\zeta), \tag{8.125}$$

$$Y_\nu(\nu z) \simeq -\frac{\sqrt{2}}{\nu^{1/3}}(\frac{\zeta}{z^2 - 1})^{1/4}Bi(-\nu^{2/3}\zeta), \tag{8.126}$$

*uniformly in $z$, where $\frac{2}{3}\zeta^{3/2} = \sqrt{z^2 - 1} - \arccos(1/z)$ (see [84]).*

*Proof.* With a suitable choice of contour we have

$$J_\nu(\nu z) = \frac{1}{2\pi} \int e^{i\nu\phi(t,z)} dt, \quad \phi(t,z) = z \sin t - t. \tag{8.127}$$

For $\nu \gg 1$ and $z > 1$ the saddle points are real, $\tilde{t} = \arccos(1/z)$ and the critical value is $\phi(t(z), z) = \sqrt{z^2 - 1} - \arccos(1/z)$. Now if $\zeta(z)$ is defined in terms of the exponent in the Debye expansion, it is analytic in $z$ for $z$ near 1 and we have

$$Ai(-\nu^{2/3}\zeta) = \frac{1}{2\pi} \int e^{-is\nu^{2/3}\zeta + is^3/3} ds = \frac{\nu^{1/3}}{2\pi} \int e^{i\nu(-t\zeta + t^3/3)} dt \tag{8.128}$$

with critical points $t^2 = \zeta$, thus we get (8.125) applying the stationary phase. We obtain the Debye approximations

$$J_\nu(\nu z) \simeq \Big(\frac{2}{\pi\nu\sqrt{z^2-1}}\Big)^{1/2} \cos[\nu\sqrt{z^2-1} - \nu \arccos(1/z) - \pi/4] \tag{8.129}$$

by replacing the Airy function by its asymptotic expansion, thus the proper condition for its validity is $\nu^{2/3}\zeta \gg 1$. For small $\nu^{2/3}\zeta$ we are in the regime $\nu(z-1) = \tau\nu^{1/3}$ for which we have the estimations (8.114), (8.115). An extension of this result giving (8.120), (8.121) uses a result of Chester, Friedman and Ursell [34] who showed that a similar reduction is possible whenever two saddle points coalesce: if $\partial_t \phi(\tilde{t}, 1) = 0$ and $\partial^2_{tt}\phi(\tilde{t}, 1) = 0$ but $\partial^3_{ttt}\phi(\tilde{t}, 1) \neq 0$, then an integral of the form (8.127) has a uniform asymptotic expansion in terms of the Airy function and its derivative. Their method was to make a change of variables so that

$$\phi(t, z) = \zeta\tau - \tau^3/3, \tag{8.130}$$

which holds exactly and uniformly; it is not merely an approximation for $z$ near 1. □

# 9 On the energy critical Schrödinger equation in $3D$ non-trapping domains

We prove that the quintic Schrödinger equation with Dirichlet boundary conditions is locally well posed for $H_0^1(\Omega)$ data on any smooth, non-trapping domain $\Omega \subset \mathbb{R}^3$. The key ingredient is a smoothing effect in $L_x^5(L_t^2)$ for the linear equation. We also derive scattering results for the whole range of defocusing subquintic Schrödinger equations outside a star-shaped domain.

## 9.1 Introduction

The Cauchy problem for the semilinear Schrödinger equation in $\mathbb{R}^3$ is by now relatively well-understood: after seminal results by Ginibre-Velo [49] in the energy class for energy subcritical equations, the issue of local well-posedness in the critical Sobolev spaces ($\dot{H}^{\frac{3}{2}-\frac{2}{p-1}}$) was settled in [33]. Scattering for large time was proved in [49] for energy subcritical defocusing equations, while the energy critical (quintic) defocusing equation was only recently successfully tackled in [36]. The local well-posedness relies on Strichartz estimates, while scattering results combine these local results with suitable non concentration arguments based on Morawetz estimates. On domains, the same set of problems remains an elusive target, due to the difficulty in obtaining Strichartz estimates in such a setting. In [26], the authors proved Strichartz estimates with an half-derivative loss on non trapping domains: the non trapping assumption is crucial in order to rely on the local smoothing estimates. However, the loss resulted in well-posedness results for strictly less than cubic nonlinearities; this was later improved to cubic nonlinearities in [6] (combining local smoothing and semiclassical Strichartz near the boundary) and in [59] (on the exterior of a ball, through precised smoothing effects near the boundary). Recently there were two significant improvements, following different strategies:

— in [87], Luis Vega and the second author obtain an $L^4_{t,x}$ Strichartz estimate which is scale invariant. However, one barely misses $L_t^4(L^\infty(\Omega))$ control for $H_0^1$ data, and therefore local wellposedness in the energy space was improved to all subcritical (less than quintic) nonlinearities, but combining this Strichartz estimate with local smoothing close to the boundary and the full set of Strichartz estimates in $\mathbb{R}^3$ away from it. Scattering was also obtained for the cubic defocusing equation, but the lack of a good local wellposedness theory at the scale invariant level ($\dot{H}^{\frac{1}{2}}$) led to a rather intricate incremental argument, from scattering in $\dot{H}^{\frac{1}{4}}$ to scattering in $H_0^1$;

— in [61], the first author proved the full set of Strichartz estimates (except for the endpoint) outside stricly convex obstacles, by following the strategy pioneered in [95] for the wave equation, and relying on the Melrose-Taylor parametrix. In the case of the Schrödinger equation, one obtains Strichartz estimates on a semiclassical time scale (taking advantage of a "finite speed of propagation" principle at this scale), and then upgrading to large time results from combining them with the smoothing effect (see [23] for a nice presentation of such an argument, already implicit in [98]). Therefore, one obtains the exact same local wellposedness theory as in the $\mathbb{R}^3$ case, including the quintic nonlinearity, and scattering holds for all subquintic defocusing nonlinearities, taking advantage of the a priori estimates from [87].

In the present work, we aim at providing a local wellposedness theory for the quintic nonlinearity outside non trapping obstacles, a case which is not covered by [61]. From explicit computations with gallery modes ([60]), one knows that the full set of optimal Strichartz estimates does not hold for the Schrödinger equation on a domain whose boundary has at least one geodesically convex point; while this does not preclude a scale invariant Strichartz estimate with a loss (like the $L^4_t(L^\infty_x)$ estimate in $\mathbb{R}^3$ which is enough to solve the quintic NLS), it suggests to bypass the issue and use a different set of estimates, which we call smoothing estimates: in $\mathbb{R}^3$, these estimates may be stated as follows,

$$\|\exp(it\Delta)f\|_{L^4_x(L^2_t)} \lesssim \|f\|_{\dot{H}^{-\frac{1}{4}}}, \tag{9.1}$$

from which one can infer various estimates by using Sobolev in time and/or in space. Formally, (9.1) is an immediate consequence of the Stein-Tomas restriction theorem in $\mathbb{R}^3$ (or, more accurately, its dual version, on the extension): let $\tau > 0$ be a fixed radius, one sees $\hat{f}(\xi)$ as a function on $|\xi| = \sqrt{\tau}$, and applies the extension estimate, with $\delta$ the Dirac function and $\mathcal{F}$ the space Fourier transform

$$\|\mathcal{F}^{-1}(\delta(\tau - |\xi|^2)\hat{f}(\xi))\|_{L^4_x} \lesssim \|\hat{f}(\xi)\|_{L^2(|\xi|=\sqrt{\tau})}.$$

Summing over $\tau$ yields the $L^2$ norm of $f$ on the RHS, while on the left we use Plancherel in time and Minkowski to get (9.1). A similar estimate holds for the wave equation, replacing $\sqrt{\tau} = |\xi|$ by $\tau = \pm|\xi|$, and usually goes under the denomination of square function (in time) estimates. In a compact setting (e.g. compact manifolds) a substitute for the Stein-Tomas theorem is provided by $L^p$ eigenfunction estimates, or better yet, spectral cluster estimates. In the context of a compact manifold with boundaries, such spectral cluster estimates were recently obtained by Smith and Sogge in [96], and provided a key tool for solving the critical wave equation on domains, see [28, 30]. In this paper, we apply the same strategy to the Schrödinger equation:

— we derive an $L^5(\Omega; L^2_I)$ "smoothing" estimate for spectrally localized data on compact manifolds with boundaries, from the spectral cluster $L^5(\Omega)$ estimate; here $I$ is a time interval whose size is such that $|I||\sqrt{-\Delta_D}| \sim 1$;

— we decompose the solution to the linear Schrödinger equation on a non trapping domain into two main regions: close to the boundary, where we can view the region as embedded into a 3$D$ punctured torus, to which the previous semi-classical estimate may be applied, and then sumed up using the local smoothing effect; and far away from the boundary where the $\mathbb{R}^3$ estimates hold.

— Finally, we patch together all estimates to obtain an estimate which is valid on the whole exterior domain. Local wellposedness in the critical Sobolev space $\dot{H}^{\frac{3}{2}-\frac{2}{p-1}}$ immediatly follows for $3 + 2/5 < p \leq 5$, and together with the a priori estimates from [87], this implies scattering for the defocusing equation for $3 + 2/5 < p < 5$.

The remaining range $3 \leq p \leq 3 + 2/5$ is sufficiently close to 3 that, as alluded to in [87], a suitable modification of the arguments from [87] yields scattering as well.

*Remark* 9.1. Clearly, such smoothing estimates are better suited to "large" values of $p$: the restriction $3 + 2/5 < p$ for the critical wellposedness is directly linked to the exponent 5 in the spectral cluster estimates; in $\mathbb{R}^3$, where the correct (and optimal !) exponent is 4, one may solve down to $p = 3$ by this method, while the Strichartz estimates allow to solve at scaling level all the way to the $L^2$ critical value $p = 1 + 4/3$.

## 9.2 Statement of results

Let $\Theta$ be a compact, non-trapping obstacle in $\mathbb{R}^3$ and set $\Omega = \mathbb{R}^3 \setminus \Theta$. By $\Delta_D$ we denote the Laplace operator with constants coefficients on $\Omega$. For $s \in \mathbb{R}$, $p, q \in [1, \infty]$ we denote by $\dot{B}^{s,q}_p(\Omega) = \dot{B}^{s,q}_p$ the Besov spaces on $\Omega$, where the spectral localization in their definition is meant to be with respect to $\Delta_D$. We write $L^p_x = L^p(\Omega)$ and $\dot{H}^\sigma = \dot{B}^{s,2}_2$ for the Lebesgue and Sobolev spaces on $\Omega$. It will be useful to introduce the Banach-valued Besov spaces $\dot{B}^{s,q}_p(L^r_t)$, and we refer to the Appendix for their definition. Whenever $L^p_t$ is replaced by $L^p_T$, it is meant that the time integration is restricted to the interval $(-T, T)$.

We aim at studying wellposedness for the energy critical equation on $\Omega \times \mathbb{R}$, with Dirichlet boundary condition,

$$i\partial_t u + \Delta_D u = \pm |u|^4 u, \quad u_{|\partial\Omega} = 0, \quad u_{|t=0} = u_0 \tag{9.2}$$

and more generally

$$i\partial_t u + \Delta_D u = \pm |u|^{p-1} u, \quad u_{|\partial\Omega} = 0, \quad u_{|t=0} = u_0 \tag{9.3}$$

with $p < 5$.

**Theorem 9.1.** *(Well-posedness for the quintic Schrödinger equation) Let $u_0 \in H^1_0(\Omega)$. There exists $T(u_0)$ such that the quintic nonlinear equation (9.2) admits a unique solution $u \in C([-T,T], H^1_0(\Omega)) \cap \dot{B}^{1,2}_5(L^{\frac{20}{11}}_T)$. Moreover, the solution is global in time and scatters in $H^1_0$ if the data is small.*

The previous theorem extends to the following subcritical range:

**Theorem 9.2.** *Let $3 + \frac{2}{5} < p < 5$, $s_p = \frac{3}{2} - \frac{2}{p-1}$ and $u_0 \in \dot{H}^{s_p}(\Omega)$. There exists $T(u_0)$ such that the nonlinear equation (9.3) admits a unique solution $u \in C([-T,T], \dot{H}^{s_p}(\Omega)) \cap \dot{B}^{s_p,2}_5(L^{\frac{20}{11}}_T)$. Moreover the solution is global in time and scatters in $\dot{H}^{s_p}$ if the data is small.*

*Remark* 9.2. We elected to state both theorems for Dirichlet boundary conditions mostly for sake of simplicity. Indeed, both results hold with Neuman boundary conditions: the key ingredients for our linear estimates are known to hold for Neuman, see [96, 26], while the nonlinear mappings from our appendix rely on [63] (where all relevant estimates can be proved to hold in the Neuman case).

Finally, we consider the long time asymptotics for (9.3) in the defocusing case, namely the + sign on the left; in this situation, we are indeed restricted to the Dirichlet boundary conditions, as we rely on a priori estimates from [87].

**Theorem 9.3.** *Assume the domain $\Omega$ to be the exterior of a star-shaped compact obstacle (which implies $\Omega$ is non trapping). Let $3 \leq p < 5$, and $u_0 \in H^1_0(\Omega)$. There exists a unique global in time solution $u$, which is in the energy class, $C(\mathbb{R}, H^1_0(\Omega))$, to the nonlinear equation* (9.3) *in the defocusing case (+ sign in* (9.3)*). Moreover, this solution scatters for large times: there exists two scattering states $u^\pm \in H^1_0(\Omega)$ such that*

$$\lim_{t\to\pm\infty} \|u(x,t) - e^{it\Delta_D} u^\pm\|_{H^1_0(\Omega)} = 0.$$

As mentioned in the introduction, the (global) existence part was dealt with in [87]; for the scattering part, the $p = 3$ case was also dealt with in [87]. In the setting of Theorem 9.2, one may adapt the usual argument from the $\mathbb{R}^n$ case, combining a priori estimates and a good Cauchy theory at the critical regularity; this provides a very short argument in the range $3+2/5 < p < 5$. In the remaining range, namely $3 < p \leq 3+2/5$, one unfortunately needs to adapt the intricate proof from [87], and this leads to a much lenghtier proof; we provide it mostly for the sake of completeness. This type of argument may however be of relevance in other contexts.

## 9.3 Smoothing type estimates

We start with definitions and notations. Let $\psi(\xi^2) \in C^\infty_0(\mathbb{R} \setminus \{0\})$ and $\psi_j(\xi^2) = \psi(2^{-2j}\xi^2)$. On the domain $\Omega$, one has the spectral resolution of the Dirichlet Laplacian, and we may define smooth spectral projections $\Delta_j = \psi_j(-\Delta_D)$ as continuous operators on $L^2$. Moreover, these operators are continuous on $L^p$ for all $p$, and if $f$ is Hilbert-valued and such that $\|\|f\|_H\|_{L^p(\Omega)} < +\infty$, then the operators $\Delta_j$ are continuous as well on $L^p(H)$. We refer to [63] for an extensive discussion and references. We simply point out that if $H = L^2_t$, then $\Delta_j$ is continuous on all $L^p_x L^q_t$ by interpolation with the obvious $L^p_t(L^p_x)$ bound and duality.

In this section we concentrate on estimates for the linear Schrödinger equation on $\Omega \times \mathbb{R}$ with Dirichlet boundary conditions,

$$i\partial_t u_L + \Delta_D u_L = 0, \quad u_{L|\partial\Omega} = 0, \quad u_{L|t=0} = u_0 \tag{9.4}$$

**Theorem 9.4.** *The following local smoothing estimate holds for the homogeneous linear equation* (9.4),

$$\|\Delta_j u_L\|_{L^5_x L^2_t} \lesssim 2^{-\frac{j}{10}} \|\Delta_j u_0\|_{L^2_x}. \tag{9.5}$$

*Moreover, let $2 \leq q \leq \infty$, then*

$$\|\Delta_j u_L\|_{L^5_x L^q_t} \lesssim 2^{-j(\frac{2}{q}-\frac{9}{10})} \|\Delta_j u_0\|_{L^2_x}. \tag{9.6}$$

Consider now the inhomogeneous equation,

$$i\partial_t v + \Delta_D v = F, \quad v_{|\partial\Omega} = 0, \quad v_{|t=0} = 0. \tag{9.7}$$

From Theorem 9.4, we will obtain the following set of estimates:

**Theorem 9.5.** *Let* $2 \leq q < r \leq +\infty$, *then*

$$\|\Delta_j v\|_{C_t(L^2_x)} + 2^{j(\frac{2}{q}-\frac{9}{10})}\|\Delta_j v\|_{L^5_x L^q_t} \lesssim 2^{-j(\frac{4}{r}-\frac{9}{5})}\|\Delta_j F\|_{L^{\frac{5}{4}}_x L^{r'}_t}, \tag{9.8}$$

*with* $1/r + 1/r' = 1$.

Combining the previous theorems with the results from [87], we finally state the set of estimates which will be used later for

$$i\partial_t u + \Delta_D u = F_1 + F_2, \quad u_{|\partial\Omega} = 0, \quad v_{|t=0} = u_0. \tag{9.9}$$

**Theorem 9.6.** *Let* $2 < r \leq +\infty$, *then*

$$\|\Delta_j u\|_{C_t(L^2_x)} + 2^{\frac{j}{10}}\|\Delta_j u\|_{L^5_x L^2_t} + 2^{-\frac{3}{4}j}\|\Delta_j u\|_{L^4_{t,x}} \lesssim \|\Delta_j u_0\|_{L^2_x} 2^{-j(\frac{4}{r}-\frac{9}{5})}\|\Delta_j F_1\|_{L^{\frac{5}{4}}_x L^{r'}_t} + 2^{-\frac{1}{4}j}\|\Delta_j F_2\|_{L^{\frac{4}{3}}_{t,x}}, \tag{9.10}$$

*with* $1/r + 1/r' = 1$.

### 9.3.1 Proof of Theorem 9.4

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be such that $\tilde{\psi} = 1$ on the support of $\psi$: hence, if $\tilde{\Delta}_j$ denotes the corresponding localization operator, $\tilde{\Delta}_j\Delta_j = \Delta_j$. We now split the solution of the linear equation $\Delta_j u_L(t,x) = \tilde{\Delta}_j\Delta_j u_L$ as a sum of two terms $\tilde{\Delta}_j\chi\Delta_j u_L + \tilde{\Delta}_j(1-\chi)\Delta_j u_L$, where $\chi \in C_0^\infty(\mathbb{R}^3)$ is compactly supported and it is equal to 1 near the boundary $\partial\Omega$.

**"Far" from the boundary:** $\tilde{\Delta}_j(1-\chi)\Delta_j u_L$ Set $w_h(t,x) = (1-\chi)\Delta_j e^{it\Delta_D}u_0(x)$. Then $w_h$ satisfies

$$\begin{cases} i\partial_t w_h + \Delta_D w_h = -[\Delta_D, \chi]\Delta_j u_L, \\ w_h|_{t=0} = (1-\chi)\Delta_j u_0. \end{cases} \tag{9.11}$$

Since $\chi$ is equal to 1 near the boundary $\partial\Omega$, we can view the solution to (9.11) as the solution of a problem in the whole space $\mathbb{R}^3$. Consequently, the Duhamel formula writes

$$w_h(t,x) = e^{it\Delta}(1-\chi)\Delta_j u_0 - \int_0^t e^{i(t-s)\Delta}[\Delta_D, \chi]\Delta_j u_L(s)ds, \tag{9.12}$$

where $\Delta$ is the free Laplacian on $\mathbb{R}^3$ and therefore the contribution of $e^{it\Delta}(1-\chi)\Delta_j u_0$ satisfies the usual Strichartz estimates. We have thus reduced the problem to the study of the second term in the right hand-side of (9.12). Ideally, one would like to remove the time restriction $s < t$ and use a variant of the Christ-Kiselev lemma. However, this would miss the endpoint case $q = 2$. Instead, we recall the following lemma:

**Lemma 9.1** (Staffilani-Tataru [98]). *Let $x \in \mathbb{R}^n$, $n \geq 3$ and let $f(x,t)$ be compactly supported in space, such that $f \in L^2_t(H^{-\frac{1}{2}})$. Then the solution $w$ to $(i\partial_t + \Delta_x)w = f$ with $w_{|t=0} = 0$, is such that*

$$\|w\|_{L^2_t(L_x^{\frac{2n}{n-2}})} \lesssim \|f\|_{L^2_t(H^{-\frac{1}{2}})}. \tag{9.13}$$

In fact, one may shift regularity in (9.13) without difficulty. Now, the proof in [98] relies on a decomposition into traveling waves, to which homogeneous estimates are then applied. We can therefore use the $L^4_x(L^2_t)$ smoothing estimate, Sobolev in space, and extend the conclusion of Lemma 9.1 to

$$\|w\|_{L^5_x(L^2_t)} \lesssim \|f\|_{L^2_t(H^{-\frac{1}{2}-\frac{1}{10}})}, \tag{9.14}$$

where we chose to conveniently shift the regularity to the right handside.

We now take $f = -[\Delta_D, \chi]\Delta_j u_L \in L^2 H^{-1/2-1/10}_{\text{comp}}(\Omega)$ and

$$\|[\Delta_D, \chi]\Delta_j u_L\|_{L^2 H^{-1/2-1/10}_{\text{comp}}} \lesssim \|\Delta_j u_L\|_{L^2 \dot{H}^{1/2-1/10}(\Omega)} \lesssim \|\Delta_j u_0\|_{\dot{H}^{1/10}(\Omega)},$$

from which the smoothing estimates follow

$$\begin{aligned}\|(1-\chi)\Delta_j u_L\|_{L^5(\mathbb{R}^3)L^2_t} &\lesssim \|(1-\chi)\Delta_j u_0\|_{\dot{H}^{-\frac{1}{10}}(\mathbb{R}^3)} + \|[\Delta_D, \chi]\Delta_j u_L\|_{L^2 H^{-1/2-1/10}_{\text{comp}}} \\ &\lesssim \|\Delta_j u_0\|_{\dot{H}^{-\frac{1}{10}}(\Omega)}.\end{aligned} \tag{9.15}$$

We conclude using the continuity properties of $\tilde{\Delta}_j$ which were recalled at the beginning of Section 9.3 (e.g. see [63, Cor.2.5]). In fact, using (9.15), we get

$$\begin{aligned}\|\tilde{\Delta}_j(1-\chi)\Delta_j u_L\|_{L^5_x L^2_t} &\lesssim \|(1-\chi)\Delta_j u_L\|_{L^5_x L^2_t} \\ &\lesssim 2^{-\frac{j}{10}}\|\Delta_j u_0\|_{L^2(\Omega)},\end{aligned}$$

where we have used the spectral localization $\Delta_j$ to estimate

$$\|\Delta_j u_0\|_{\dot{H}^\sigma(\Omega)} \simeq 2^{\sigma j}\|\Delta_j u_0\|_{L^2(\Omega)}.$$

**"Close" to the boundary:** $\tilde{\Delta}_j \chi \Delta_j u_L$ For $l \in \mathbb{Z}$ let $\varphi_l \in C_0^\infty(((l-1/2)\pi, (l+1)\pi))$ equal to 1 on $[l\pi, (l+1/2)\pi]$. We set $v_j = \tilde{\Delta}_j \chi \Delta_j u_L$ and for $l \in \mathbb{Z}$ we set $v_{j,l} = \varphi_l(2^j t)v_j$. We have

$$\begin{aligned}\|v_j\|^2_{L^5(\Omega)L^2(\mathbb{R})} = \|\sum_{l\in\mathbb{Z}} v_{j,l}\|^2_{L^5_x L^2_t} &\simeq \|\|\sum_{l\in\mathbb{Z}} v_{j,l}\|^2_{L^2_t}\|_{L^{5/2}_x} \\ &\lesssim \|\sum_{l\in\mathbb{Z}} \|v_{j,l}\|^2_{L^2_t}\|_{L^{5/2}_x} \leq \sum_{l\in\mathbb{Z}} \|v_{j,l}\|^2_{L^5_x L^2_t},\end{aligned} \tag{9.16}$$

where for the first inequality we used the fact that the supports in time of $\varphi_l$ are almost orthogonal. In order to estimate $\|v_j\|^2_{L^5_x L^2_t}$ it will be thus sufficient to estimate each $\|v_{j,l}\|^2_{L^5_x L^2_t}$. The equation satisfied by $\tilde{v}_{j,l} := \varphi_l(2^j t)\chi\Delta_j u_L$ is

$$i\partial_t \tilde{v}_{j,l} + \Delta_D \tilde{v}_{j,l} = -(\varphi_l(2^j t)[\Delta_D, \chi]\Delta_j u_L - i2^j \varphi_l'(2^j t)\chi\Delta_j u_L), \tag{9.17}$$

where we stress that $\tilde{v}_{j,l}$ vanishes outside the time interval $(2^{-j}(l-1/2)\pi, 2^{-j}(l+1)\pi)$. We denote $V_{j,l}$ the right hand side in (9.17), namely

$$V_{j,l} := -\varphi_l(2^j t)[\Delta_D, \chi]\Delta_j u_L + i2^j \varphi_l'(2^j t)\chi\Delta_j u_L. \tag{9.18}$$

Let $Q \subset \mathbb{R}^3$ be an open cube sufficiently large such that $\partial\Omega$ is contained in the interior of $Q$. We denote by $S$ the punctured torus obtained from removing the obstacle $\Theta$ (recall that $\Omega = \mathbb{R}^3 \setminus \Theta$) in the compact manifold obtained from $Q$ with periodic boundary conditions on $\partial Q$. Notice that defined in this way $S$ coincides with the Sinaï billiard. Let also $\Delta_S := \sum_{j=1}^3 \partial_j^2$ denote the Laplace operator on the compact domain $S$.

On $S$, we may define a spectral localization operator using eigenvalues $\lambda_k$ and eigenvectors $e_k$ of $\Delta_S$: if $f = \sum_k c_k e_k$, then

$$\Delta_j^S f = \psi(2^{-2j}\Delta_S)f = \sum_k \psi(2^{-2j}\lambda_k^2)c_k e_k. \tag{9.19}$$

*Remark* 9.3. Notice that in a neighborhood of the boundary, the domains of $\Delta_S$ and $\Delta_D$ coincide, thus if $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ is supported near $\partial\Omega$ then

$$\Delta_S \tilde{\chi} = \Delta_D \tilde{\chi}.$$

In order to apply estimates on the manifold $S$, we will need to relocalize close to the obstacle. Consider $\chi_1 \in C_0^\infty(\mathbb{R}^3)$ supported near the boundary and equal to 1 on the support of $\tilde{\chi}$, we will write

$$\chi_1 \tilde{\Delta}_j \tilde{\chi} = \chi_1 \tilde{\Delta}_j^S \tilde{\chi} + \chi_1(\tilde{\Delta}_j - \tilde{\Delta}_j^S)\tilde{\chi}, \tag{9.20}$$

with the expectation that the difference term is smoothing.

In what follows let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ be equal to 1 on the support of $\chi$ and be supported in a neighborhood of $\partial\Omega$ such that on its support the operator $-\Delta_D$ coincide with $-\Delta_S$. From their respective definition, $\tilde{v}_{j,l} = \tilde{\chi}\tilde{v}_{j,l}$, $V_{j,l} = \tilde{\chi}V_{j,l}$, consequently $\tilde{v}_{j,l}$ will also solve the following equation on the compact manifold $S$

$$\begin{cases} i\partial_t \tilde{v}_{j,l} + \Delta_S \tilde{v}_{j,l} = V_{j,l}, \\ \tilde{v}_{j,l}|_{t<h(l-1/2)\pi} = 0, \quad \tilde{v}_{j,l}|_{t>h(l+1)\pi} = 0. \end{cases} \tag{9.21}$$

Therefore we can write the Duhamel formula either for the last equation (9.21) on $S$, or for the equation (9.17) on $\Omega$. We now apply $\tilde{\Delta}_j$ and use that $v_{j.l} = \tilde{\Delta}_j \tilde{v}_{j,l}$, $\tilde{\chi}\tilde{v}_{j,l} = \tilde{v}_{j,l}$ and

$\tilde{\Delta}_j\tilde{\chi} = \chi_1\tilde{\Delta}_j^S\tilde{\chi} + (1-\chi_1)\tilde{\Delta}_j\tilde{\chi} + \chi_1(\tilde{\Delta}_j - \tilde{\Delta}_j^S)\chi$, which yields

$$v_{j,l}(t,x) = \chi_1\int_{h(l-1/2)\pi}^t e^{i(t-s)\Delta_S}\tilde{\Delta}_j^S V_{j,l}(s,x)ds \\ + (1-\chi_1)\int_{h(l-1/2)\pi}^t e^{i(t-s)\Delta_D}\tilde{\Delta}_j V_{j,l}(s,x)ds \\ + \chi_1(\tilde{\Delta}_j - \tilde{\Delta}_j^S)\tilde{v}_{j,l}, \quad (9.22)$$

where we conveniently chose to write Duhamel on $S$ for the first term and Duhamel on $\Omega$ for the second one, which allows to commute the flow under the time integral. Denote by $v_{j,l,m}$ the first term in the second line of (9.22) by $v_{j,l,f}$ the second one and $v_{j,l,s}$ the last one. We deal with them separately. To estimate the $L^5_xL^2_t$ norm of the $v_{j,l,f}$ we notice that its support is far from the boundary: as such, estimates on the $L^5_xL^2_t$ norm will follow from Section 9.3.1. Indeed, we get

$$\|(1-\chi_1)\tilde{\Delta}_j e^{i(t-s)\Delta_D}V_{j,l}\|_{L^5_xL^2_t} \lesssim \|\tilde{\Delta}_j V_{j,l}\|_{\dot{H}^{-1/10}(\Omega)} \simeq 2^{-\frac{j}{10}}\|\tilde{\Delta}_j V_{j,l}\|_{L^2(\Omega)}. \quad (9.23)$$

We then apply the Minkowski inequality to deduce

$$\|(1-\chi_1)\int_{h(l-1/2)\pi}^t \tilde{\Delta}_j e^{i(t-s)\Delta_D}V_{j,l}(s,x)ds\|_{L^5_xL^2_t} \\ \leq 2^{-j/2}(\int_{I_{j,l}} \|(1-\chi_1)\tilde{\Delta}_j e^{i(t-s)\Delta_D}V_{j,l}(s,.)\|^2_{L^5(\Omega)L^2(I_{j,l})}ds)^{1/2}, \quad (9.24)$$

where we denoted $I_{j,l} = [2^{-j}(l-1/2)\pi, 2^{-j}(l+1)\pi]$ and we used the Cauchy-Schwartz inequality. Using (9.23) we finally get

$$\|v_{j,l,f}\|_{L^5(\Omega)L^2(I_{j,l})} \leq 2^{-j(1/2+1/10)}\|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)}. \quad (9.25)$$

To estimate the $L^5_xL^2_t$ norm of the main contribution $v_{j,l,m}$ we need the following:

**Proposition 9.1.** *Let $j \geq 0$, $I_j = (-\pi 2^{-j}, \pi 2^{-j})$, $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ be supported near $\partial\Omega$ and $V_0 \in L^2(\Omega)$. Then there exists $C > 0$ independent of $j$ such that for the solution $e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0$ of the linear Schrödinger equation on $S$ with initial data $\tilde{\Delta}_j^S\tilde{\chi}V_0$ we have*

$$\|e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^5(S)L^2_t(I_j)} \leq C2^{-\frac{j}{10}}\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}. \quad (9.26)$$

We postpone the proof of Proposition 9.1 to Subsection 9.3.3.

Using the fact that $v_{j,l}$ is supported in time in $I_{j,l} = [2^{-j}(l-1/2)\pi, 2^{-j}(l+1)\pi]$, the Minkowski inequality, Proposition 9.1 with $\tilde{\chi} = 1$ on the support of $\chi$ and with $V_0 = V_{j,l}$, and since $\tilde{\chi}_1 v_{j,l,m} = v_{j,l,m}$ for any $\tilde{\chi}_1 \in C^\infty(\mathbb{R}^3)$ with $\tilde{\chi}_1 = 1$ on the support of $\chi_1$, we obtain

$$\begin{aligned}
\|v_{j,l,m}\|_{L^5(\Omega)L^2(I_{j,l})} =&\|\tilde{\chi}_1 v_{j,l,m}\|_{L^5(\Omega)L^2(I_{j,l})} = \|v_{j,l,m}\|_{L^5(S)L^2(I_{j,l})} \\
\leq& \int_{2^{-j}(l-1)\pi}^{2^{-j}(l+1)\pi} \|e^{i(t-s)\Delta_S} \tilde{\Delta}_j^S V_{j,l}(s,.)\|_{L^5(S)L^2(I_{j,l})} ds \\
\leq& 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\Delta}_j^S V_{j,l}(s)\|_{L^2(S)} ds \\
\leq& 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\chi} V_{j,l}(s)\|_{L^2(S)} ds \\
\leq& 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\chi} V_{j,l}(s)\|_{L^2(\Omega)} ds \qquad (9.27)
\end{aligned}$$

where we used again $V_{j,l} = \tilde{\chi} V_{j,l}$ to switch $S$ and $\Omega$ and continuity of $\Delta_j^S$ on $L^2(S)$. Using the Cauchy-Schwartz inequality in (9.27) yields

$$\|v_{j,l,m}\|_{L^5(\Omega)L^2(I_{j,l})} \lesssim 2^{-j(1/2+1/10)} \|V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} \qquad (9.28)$$

We deal with the right handside in (9.28). Using the explicit expression of $V_{j,l}$ given in (9.18),

$$\begin{aligned}
\|V_{j,l}(s)\|_{L^2(I_{j,l})L^2(\Omega)} \lesssim (&\|\varphi_l(2^j t)[\Delta_D, \chi]\Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)} \\
&+ 2^j \|\varphi_l'(2^j t)\chi \Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)}). \qquad (9.29)
\end{aligned}$$

As $[\Delta_D, \chi]$ is bounded from $H_0^1$ to $L^2$, we get

$$\|\tilde{\Delta}_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} \lesssim \|\chi_1 \Delta_j u_L\|_{L^2(I_{j,l})H_0^1(\Omega)} + 2^j \|\chi \Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)} \qquad (9.30)$$

Let us recall the following local smoothing result on a non trapping domain:

**Lemma 9.2.** *(Burq, Gérard, Tzvetkov [26, Prop.2.7]) Assume that* $\Omega = \mathbb{R}^3 \setminus \Theta$*, where* $\Theta \neq \emptyset$ *is a non-trapping obstacle. Then, for every* $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$*, and* $\sigma \in [-1/2, 1]$*,*

$$\|\tilde{\chi}\Delta_j u_L\|_{L^2(\mathbb{R}, \dot{H}^{\sigma+1/2}(\Omega))} \leq C\|\Delta_j u_0\|_{H^\sigma(\Omega)}, \qquad (9.31)$$

*where, as usual,* $u_L(t,x) = e^{-it\Delta_D} u_0(x)$*.*

We now turn to the difference term $v_{j,l,s}$ and prove a smoothing lemma.

**Lemma 9.3.** *Let* $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ *be equal to* 1 *on a fixed neighborhood of the support of* $\tilde{\chi}$*. Then we have for all* $N \in \mathbb{N}$*,*

$$\|v_{j,l,s}\|_{L^5(\Omega)L^2(I_{j,l})} \leq C_N 2^{-Nj} \|V_{j,l}(x,s)\|_{L^2(I_{j,l}, L^2(\Omega))}. \qquad (9.32)$$

In order to prove the lemma, one would like to rewrite $\tilde{\Delta}_j = \tilde{\psi}(2^{-2j}\Delta_D)$ as a solution of the wave equation, using $h = 2^{-j}$ as a time. Then the finite speed of propagation would let us switch $\Delta_D$ and $\Delta_S$. However the inverse Fourier transform (in $|\xi|$) of $\Psi(|\xi|) = \tilde{\psi}(|\xi|^2)$ is only Schwartz class, rather than compactly supported. The tails will eventually account for the right handside of (9.32). We now turn to the details: let $\varphi_0, \varphi(y)$ be even, compactly supported ($\varphi(y)$ away from zero) and such that

$$\varphi_0(y) + \sum_{k\geq 1} \varphi(2^{-k}y) = 1.$$

We decompose $\hat{\Psi}(y)$ using this resolution of the identity, and set with obvious notations

$$\Psi(|\xi|) = \sum_{k\in\mathbb{N}} \phi_k(|\xi|),$$

where the $\phi_k$ have good bounds, say $\hat{\phi}_0 \in L^\infty$ and for $k \geq 1$

$$\forall N \in \mathbb{N}, \ \ \|\hat{\phi}_k\|_\infty = \|\hat{\Psi}(y)\varphi(2^{-k}y)\|_\infty \leq C_N 2^{-kN}. \tag{9.33}$$

At fixed $k$, we write (abusing notation and letting $\Delta$ be either $\Delta_D$ or $\Delta_S$)

$$\phi_k(h\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l} = \frac{1}{2\pi}\int e^{iyh\sqrt{-\Delta}}\tilde{\chi}(x)\tilde{v}_{j,l}(x)\hat{\phi}_k(y)\,dy.$$

Notice that $\phi_k(y)$ is compactly supported, in fact its support is roughly $|y| \in [2^{k-1}, 2^{k+1}]$. As such the $y$ integral is a time average of half-wave operators, which have finite speed of propagation. Therefore if the "time" $|yh| \leq 1$, we can add another cut-off function $\chi_1$ which is equal to one on the domain of dependency of $\tilde{\chi}$ on this time scale, and such that $\chi_1$ is indifferently defined on $S$ or $\Omega$: namely, for $k \lesssim j$,

$$\begin{aligned} \phi_k(h\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l} &= \chi_1(x)\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l} \\ &= \chi_1(x)\frac{1}{2\pi}\int e^{iyh\sqrt{-\Delta}}\tilde{\chi}(x)\tilde{v}_{j,l}(x)\hat{\phi}_k(y)\,dy, \\ \phi_k(2^{-j}\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l} &= \chi_1(x)\phi_k(2^{-j}\sqrt{-\Delta_D})\tilde{\chi}\tilde{v}_{j,l}. \end{aligned} \tag{9.34}$$

From this identity, we obtain

$$v_{j,l,s} = \chi_1(x)\sum_{j\lesssim k}(\phi_k(2^{-j}\sqrt{-\Delta_D}) - \phi_k(2^{-j}\sqrt{-\Delta_S}))\tilde{\chi}(x)\tilde{v}_{j,l}. \tag{9.35}$$

At this point the difference in (9.35) is irrelevant and we estimate both terms using Sobolev embedding and energy estimates. Abusing notations, with $\Delta \in \{\Delta_D, \Delta_S\}$, we have

$$\begin{aligned} \|\chi_1\phi_k(2^{-j}\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l}\|_{L^5(\Omega)L^2_t(I_{j,l})} \leq& \|\chi_1\phi_k(2^{-j}\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l}\|_{L^2_t(I_{j,l})L^5(\Omega)} \\ \leq& 2^{-\frac{j}{2}}\|\chi_1\phi_k(2^{-j}\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l}\|_{L^\infty_t(I_{j,l})L^5(\Omega)} \\ \lesssim& 2^{-\frac{j}{2}}\|\phi_k(2^{-j}\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l}\|_{L^\infty_t(I_{j,l})H^{\frac{1}{2}}(\Omega)} \\ \lesssim& C_N 2^{-\frac{j}{2}-kN}\|\tilde{\chi}\tilde{v}_{j,l}\|_{L^\infty_t(I_{j,l})H^{\frac{1}{2}}(\Omega)} \end{aligned}$$

where we used Minkowski, Hölder, (non sharp !) Sobolev and (9.33). Finally, by the dual estimate of (9.31),

$$\|\tilde{v}_{j,l}\|_{L_t^\infty(I_{j,l})H^{\frac{1}{2}}(\Omega)} \lesssim \|V_{j,l}\|_{L_t^2(I_{j,l},L^2(\Omega))}.$$

Summing in $k$ and relabeling $N$, we have

$$\|v_{j,l,s}\|_{L^5(\Omega)L_t^2(I_{j,l})} \leq C_N 2^{-jN}\|V_{j,l}\|_{L_t^2(I_{j,l},L^2(\Omega))}, \tag{9.36}$$

which concludes the proof of the lemma.

Using this lemma and (9.30), we get for $v_{j,l,s}$ an estimate which matches (9.28): picking $N = 1$ is enough. From there, using (9.16), (9.25), (9.28), we write

$$\begin{aligned}
\|\tilde{\Delta}_j\chi\Delta_j u_L\|^2_{L^5(\Omega)L_t^2} &\lesssim 2^{-2j(\frac{1}{2}+\frac{1}{10})}\sum_{l\in\mathbb{Z}}\|\tilde{\Delta}_j V_{j,l}(s)\|^2_{L^2(I_{j,l})L^2(\Omega)} \\
&\lesssim 2^{-2j(\frac{1}{2}+\frac{1}{10})}\sum_{l\in\mathbb{Z}}(\|\tilde{\chi}\Delta_j u_L\|^2_{L^2(I_{j,l})H_0^1(\Omega)} + 2^{2j}\|\tilde{\chi}\Delta_j u_L\|^2_{L^2(I_{j,l})L^2(\Omega)}) \\
&\lesssim 2^{-\frac{2j}{10}}(2^{-j}\|\tilde{\Delta}_j u_0\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} + 2^j\|\tilde{\Delta}_j u_0\|^2_{\dot{H}^{-\frac{1}{2}}(\Omega)}) \\
&\lesssim 2^{-\frac{2j}{10}}(\|\tilde{\Delta}_j u_0\|^2_{L^2(\Omega)},
\end{aligned}$$

which is the desired result.

**End of the proof of Theorem 9.4** Until now we have prove Theorem 9.4 only for $q = 2$. We shall use the Gagliardo-Nirenberg inequality in order to deduce (9.6) for every $q \geq 2$. We have

$$\|\Delta_j u_L\|_{L_t^\infty} \lesssim \|\Delta_j u_L\|^{1/2}_{L_t^2}\|\Delta_j\partial_t u_L\|^{1/2}_{L_t^2}.$$

which gives, taking the $L_x^5$ norms and using the Cauchy-Schwartz inequality

$$\|\Delta_j u_L\|^5_{L_x^5 L_t^\infty} \lesssim \|\Delta_j u_L\|^{5/2}_{L^5 L_t^2}\|\Delta_j\partial_t u_L\|^{5/2}_{L_x^5 L_t^2}. \tag{9.37}$$

It remains to estimate $\|\Delta_j\partial_t u_L\|_{L_x^5 L_t^2}$: notice that since $u_L = e^{-it\Delta_D}u_0$

$$\Delta_j\partial_t u_L = -i\Delta_D\Delta_j u_L = i2^{2j}\tilde{\Delta}_j u_L,$$

where $\tilde{\Delta}_j$ is defined with $\psi_1(x) = x\psi(x) \in C_0^\infty(\mathbb{R}\setminus\{0\})$. Therefore

$$\|\Delta_j\partial_t u_L\|_{L_x^5 L_t^\infty} \leq C2^{j(2-1/10)}\|\tilde{\Delta}_j u_0\|_{L^2(\Omega)}, \tag{9.38}$$

consequently

$$\|\Delta_j\partial_t u_L\|_{L_x^5 L_t^q} \leq C2^{-j(2/q-9/10)}\|\Delta_j u_0\|_{L^2(\Omega)}$$

and Theorem 9.4 is proved.

### 9.3.2 Proof of Theorems 9.5 and 9.6

We recall a lemma due to Christ and Kiselev [35]. We state the corollary we will use, with only the time variable: we refer to [29] for a simple direct proof of all the different cases we use, with Banach-valued $L^p_t(B)$ spaces or $B(L^p_t)$. Its use in the context of reversed norms $L^q_x(L^p_t)$ goes back to [86] and it greatly simplifies obtaining inhomogeneous estimates from homogeneous ones.

**Lemma 9.4.** *(Christ and Kiselev [35]) Consider a bounded operator*

$$T : L^r(\mathbb{R}) \to L^q(\mathbb{R})$$

*given by a locally integrable kernel $K(t,s)$. Suppose that $r < q$. Then the restricted operator*

$$T_R f(t) = \int_{s<t} K(t,s) f(s) ds$$

*is bounded from $L^r(\mathbb{R})$ to $L^q(\mathbb{R})$ and*

$$\|T_R\|_{L^r(\mathbb{R})\to L^q(\mathbb{R})} \leq C(1 - 2^{-(1/q-1/r)})^{-1} \|T\|_{L^r(\mathbb{R})\to L^q(\mathbb{R})}.$$

From the lemma, the proof of the inhomogeneous set of estimates in Theorem 9.5 is routine from the homogeneous estimates in Theorem 9.4 and the Duhamel formula. Combining both homogeneous and inhomogeneous estimates yields Theorem 9.6.

### 9.3.3 Proof of Proposition 9.1

Let $S$ denote the compact domain defined above. Recall $(e_n)_n$ is the eigenbasis of $L^2(S)$ consisting of eigenfunctions of $-\Delta_S$ associated to the eigenvalues $\lambda_n^2$. Following [28], we define an abstract self adjoint operator on $L^2(S)$ as follows

$$A_h(e_n) := -[h\lambda_n^2] e_n,$$

where $[\lambda]$ is the integer part of $\lambda$. Notice that in some sense $A_h =$ "$[h\Delta_S]$". We first need to establish estimates for the linear Schrödinger equation on the compact domain $S$ with spectrally localized initial data.

We now set $h = 2^{-j}$ and state estimates on the evolution equation where $h\Delta_D$ is replaced by $A_h$.

**Lemma 9.5.** *Let $0 < h \leq 1$, $q \geq 2$, $I_h = (-\pi h, \pi h)$, $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ be supported near $\partial\Omega$ and $V_0 \in L^2(\Omega)$. There exists $C > 0$ independent of $h$ such that*

$$\|e^{i\frac{t}{h}A_h} \tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^5(S)L^q(I_h)} \leq C h^{2/q-9/10} \|\tilde{\Delta}_j^S \tilde{\chi} V_0\|_{L^2(S)}. \tag{9.39}$$

We postpone the proof of Lemma 9.5 and proceed with the proof of Proposition 9.1. Denote by $V_h(t,x) := e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0(x)$, then

$$(ih\partial_t + A_h)V_h = (ih\partial_t + h\Delta_S)V_h + (A_h - h\Delta_S)V_h = (A_h - h\Delta_S)e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0.$$

Writing Duhamel formula for $V_h$ yields

$$V_h(t,x) = e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S\tilde{\chi}V_0(x) - \frac{i}{h}\int_0^t e^{i\frac{(t-s)}{h}A_h}(A_h - h\Delta_S)e^{is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0(x)ds. \tag{9.40}$$

Using (9.39) with $q = 2$, (9.40), the Minkowski inequality and boundedness of the operator

$$\|e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S\|_{L^2(S)\to L^5(S)L^2(I_h)} \lesssim 2^{-\frac{j}{10}} \sim h^{1/10}$$

(which follows from the proof of Lemma 9.5), we obtain

$$\|e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^5(S)L^2(I_h)} \lesssim h^{\frac{1}{10}}\Big(\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)} + \frac{1}{h}\|(A_h - h\Delta_S)e^{is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^1(-h\pi,h\pi)L^2(S)}\Big), \tag{9.41}$$

where to estimate the second term in the right hand side of (9.40) we used the fact that $A_h$ commutes with the spectral localization $\tilde{\Delta}_j^S$. Changing variables $s = h\tau$ in the second term in the right hand side of (9.41) yields

$$\frac{1}{h}\|(A_h - h\Delta_S)e^{is\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^1(-h\pi,h\pi)L^2(S)} = \int_{-\pi}^{\pi}\|(A_h - h\Delta_S)e^{i\tau h\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}d\tau \lesssim 2\pi\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)}, \tag{9.42}$$

where we used the fact that the operator $(A_h - h\Delta_S)$ is bounded on $L^2(S)$ and the mass conservation of the linear Schrödinger flow. If follows from (9.41) and (9.42) that

$$\|e^{it\Delta_S}\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^5(S)L^2(I_h)} \lesssim h^{1/10}\|\tilde{\Delta}_j^S\tilde{\chi}V_0\|_{L^2(S)},$$

which ends the proof of Proposition 9.1.

We now return to Lemma 9.5 for the rest of this section. Writing $\tilde{\Delta}_j^S V_0 = \sum_n \tilde{\psi}(h^2\lambda_n^2)V_{\lambda_n}e_n$, we decompose (for $0 < h \leq 1/4$)

$$e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S V_0(t,x) = \sum_{k\in\mathbb{N}} e^{i\frac{t}{h}k}v_k(x)$$

with

$$v_k(x) = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1}\ \sum_{\lambda_n\in[\lambda,\lambda+1)}\tilde{\Psi}(h^2\lambda_n^2)V_{\lambda_n}e_n = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1}\Pi_\lambda(\tilde{\Delta}_j^S V_0),$$

where $\Pi_\lambda$ denotes the spectral projector $\Pi_\lambda = 1_{\sqrt{-\Delta_S}\in[\lambda,\lambda+1)}$. Let us estimate the $L^5(S)L^q(I_h)$ norm of $e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S V_0$:

$$\begin{aligned}
\|e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S V_0\|^2_{L^5(S)L^q(I_h)} &\lesssim h^{2/q}\| \|e^{isA_h}\tilde{\Delta}_j^S V_0\|^2_{L^q_s(-\pi,\pi)}\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q}\| \|e^{isA_h}\tilde{\Delta}_j^S V_0\|^2_{H^{1/2-1/q}(s\in(-\pi,\pi))}\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q}\| \sum_{k\in\mathbb{N}}(1+k)^{2(\frac{1}{2}-\frac{1}{q})}\|e^{isk}v_k(x)\|^2_{L^2_s(-\pi,\pi)}\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q}\sum_{k\in\mathbb{N}}(1+k)^{1-2/q}\|e^{isk}v_k(x)\|^2_{L^5(S)L^2(-\pi,\pi)} \\
&\lesssim h^{2/q}\sum_{k\in\mathbb{N}}(1+k)^{1-2/q}\|e^{isk}v_k(x)\|^2_{L^2(-\pi,\pi)L^5(S)},
\end{aligned}$$

where we used Sobolev injection in the time variable $H^{1/2-1/q}\subset L^q$ and Plancherel in time. We recall a result of [96] of Smith and Sogge on the spectral projector $\Pi_\lambda$:

**Theorem 9.7.** *(Smith and Sogge [96]) Let $S$ be a compact manifold of dimension 3, then*

$$\|\Pi_\lambda\|_{L^2(S)\to L^5(S)} \leq \lambda^{2/5}.$$

Using Theorem 9.7 we have

$$\begin{aligned}
\|e^{i\frac{t}{h}A_h}\tilde{\Delta}_j^S V_0\|^2_{L^5(S)L^q(I_h)} &\lesssim h^{2/q}\sum_{1/4h-1\leq k\leq 4/h}(1+k)^{1-2/q+4/5}\|\tilde{\Delta}_j^S V_0\|^2_{L^2(S)} \\
&\lesssim \sum_{hk\in[1/4,4]} k^{1-4/q+4/5}\|\tilde{\Delta}_j^S V_0\|^2_{L^2(S)} \\
&\lesssim \|\tilde{\Delta}_j^S V_0\|^2_{\dot{H}^{2/q-9/10}(S)},
\end{aligned}$$

since for $hk>4$ or $h(k+1)<1/4$ and $\lambda_n\in[(k2^j)^{1/2},((k+1)2^j)^{1/2})$ we have $\tilde{\Psi}(h^2\lambda_n^2)=0$ and on the other hand for these values of $k$ we have

$$k/\sqrt{2}\leq(k2^j)^{1/2}\leq\lambda_n\leq((k+1)2^j)^{1/2}\leq\sqrt{2}(k+1), \quad h\leq 5(k+1)^{-1}.$$

This completes the proof of Lemma 9.5.

## 9.4 Local existence

In this section we prove Theorem 9.1.

*Definition* 9.1. Let $u\in\mathcal{S}'(\mathbb{R}\times\Omega)$ and let $\Delta_j=\psi(-2^{-2j}\Delta_D)$ be a spectral localization with respect to the Dirichlet Laplacian $\Delta_D$ in the $x$ variable, such that $\sum_j\Delta_j=Id$ and let $S_j = \sum_{k<j} \Delta_j$. We introduce the "Banach valued" Besov space $\dot{B}^{s,q}_p(L^r_t)$ as follows: we say that $u \in \dot{B}^{s,q}_p(L^r_t)$ if

$$\left(2^{js}\|\Delta_j u\|_{L^p_x L^r_t}\right) \in l^q,$$

and $\sum_j \Delta_j f$ converges to $f$ in $\mathcal{S}'$. If $L^r_t$ is replaced by $L^r_T$, the time integration is meant to be over $(-T,T)$. Moreover, when $s<0$, $\Delta_j$ may be replaced by $S_j$ in the norm and both norms are equivalent.

Consider $u_0 \in \dot{H}^1_0$ and $u_L$ the solution to the linear equation (9.4). Applying Theorem 9.4 with $q=2,5$ and taking $s=1$ in the definition above we obtain

$$u_L \in \dot{B}^{1+\frac{1}{10},2}_5(L^2_t) \cap \dot{B}^{\frac{1}{2},2}_5(L^5_t) \quad \text{and} \quad \partial_t u_L \in \dot{B}^{-\frac{3}{2},2}_5(L^5_t).$$

From this, by Gagliardo-Nirenberg in the time variable, one should have

$$u_L \in \dot{B}^{1,2}_5(L^{\frac{20}{9}}_t) \cap \dot{B}^{3/20,2}_5(L^{40}_t) \subset L^{20/3}_x L^{40}_t,$$

and consequently

$$u_L^4 \in L^{5/3}_x L^{10}_t \text{ as well as } |u_L|^4 u_L \in \dot{B}^{1,2}_{\frac{5}{4}}(L^{\frac{20}{11}}_t)$$

which should be enough to iterate. However, our spaces are Banach valued Besov spaces (if one sees time as a parametrer) and justifying Berstein-like inequalities and Sobolev embedding is not entirely trivial (but doable, using the estimates from [63]). We choose an apparently complicated space in order to set up the fixed point, but the little gain in regularity from the smoothing estimate will turn out to be crucial for subcritical scattering.

*Remark* 9.4. By this choice, we only restrict the uniqueness class. It is likely that one may prove a better result, but there is no immediate benefit in the present setting, except proving additional estimates. We retained, however, the uniqueness class that would be provided by the argument above in the Theorems'statements. Another remark is that one may dispense with the use of Lemma 9.1, miss the endpoint $q=2$ and still get the exact same nonlinear results, as there is room (due to the use of Sobolev embedding) in all mapping estimates. Moreover, as soon as we use an estimate with a (however small) gain in regularity, we do not need Lemma 9.16, as we could use a simpler embedding in a Besov space of negative regularity and play regularities against each other. In fact, in the same spirit as [86] one could replace the critical Sobolev norm by a Besov norm $\dot{B}^{s_p,\infty}_2$.

For $T>0$ let

$$X_T := \{u \,|\, u \in \dot{B}^{1+\frac{1}{10},2}_5(L^2_T) \cap \dot{B}^{\frac{1}{2},2}_5(L^5_T) \text{ and } \partial_t u \in \dot{B}^{-\frac{3}{2},2}_5(L^5_T)\}. \tag{9.43}$$

and for $u \in X_T$ set $F(u) := |u|^4 u$.

**Proposition 9.2.** *Define a nonlinear map* $\phi$ *as follows,*

$$\phi(u)(t) := \int_{s<t} e^{i(t-s)\Delta_D} F(u(s))ds.$$

*Then*

$$\|\phi(u)\|_{C_T(\dot H^1_0)} + \|\phi(u)\|_{X_T} \lesssim \|F(u)\|_{\dot B^{1,2}_{5/4}(L_T^{20/11})} \lesssim \|u\|^5_{X_T}, \tag{9.44}$$

*and*

$$\|\phi(u)-\phi(v)\|_{X_T} \lesssim \|F(u)-F(v)\|_{\dot B^{1,2}_{5/4}(L_T^{20/11})} \lesssim \|u-v\|_{X_T}(\|u\|_{X_T}+\|v\|_{X_T})^4. \tag{9.45}$$

The estimate for the inhomogeneous problem writes

$$\| \int e^{-is\Delta_D} F\|_{L^2_x} \leq C\|F\|_{\dot B^{0,2}_{5/4}(L_t^{20/11})},$$

Shifting the regularity to $s=1$ and using the Christ-Kiselev lemma provides the first step of both estimates 9.44 and 9.45. Now, Lemma 9.15 in the Appendix provides the nonlinear part of both estimates (note however that, as $p=5$ is an integer, one could prove directly the nonlinear mappings by product rules).

One may now set up the usual fixed point argument in $X_T$ if $T$ is sufficiently small of if the data is small. This concludes the proof of Theorem 9.1 (scattering for small data follows the usual way from the global in time space-time estimates).

We now consider local wellposedness for $p<5$, e.g. Theorem 9.2. The critical Sobolev exponent w.r.t. scaling is $s_p = 3/2-2/(p-1)$. We aim at setting up a contraction argument in a small ball of

$$X_T := \{u \,|\, u \in \dot B_5^{s_p+\frac{1}{10},2}(L^2_T) \cap \dot B_4^{s_p-\frac14,2}(L^4_T) \text{ and } \partial_t u \in \dot B_4^{s_p-\frac14-2,2}(L^4_T)\}. \tag{9.46}$$

The important fact (if we were to ignore issues with Banach valued Besov spaces) would be that $X_T \subset \dot B_5^{s_p,2}(L_T^{20/9}) \cap L_x^{5(p-1)/3} L_T^{10(p-1)}$.

*Remark* 9.5. Some numerology is in order: if one were only to have the $L^5_x L^2_t$ smoothing estimate and use Sobolev (in time and in space), it would require $5(p-1)/3 \geq 5$, namely $p \geq 4$. However, we have the Strichartz estimate from [87], which allows $5(p-1)/3 \geq 4$, or $p \geq 3+2/5$.

Again from the Appendix, the nonlinear mapping verifies

$$\|F(u)-F(v)\|_{\dot B^{s_p,2}_{5/4}(L_T^{20/11})} \lesssim \|u-v\|_{X_T}(\|u\|^{p-1}_{X_T}+\|v\|^{p-1}_{X_T})$$

and existence and uniqueness follow by fixed point again.

### 9.4.1 Scattering for $3+2/5<p<5$

We now deal with scattering in the same range of $p \in (3+2/5,5)$: from [87], we have an a priori bound

$$\|S_j u\|^4_{L^4_t L^4_x} \lesssim \|u\|^4_{L^4_t L^4_x} \lesssim \|u_0\|^3_{L^2_x} \sup_t \|u\|_{H^1_0} \leq M^{\frac32} E^{\frac12},$$

where $M$ and $E$ are the conserved charge and hamiltonian,

$$M = \int_\Omega |u|^2\,dx \text{ and } E = \int_\Omega |\nabla u|^2 + \frac{2}{p+1}|u|^{p+1}\,dx. \tag{9.47}$$

Notice how this estimate is below the critical scaling $s_p$, as the RHS regularity is $s = 1/4$. From the energy a priori bound and Sobolev embedding, one has on the other hand

$$\|S_j u\|_{L^\infty_{t,x}} \lesssim 2^{\frac{j}{2}} \sup_t \|u\|_{H^1_0} \lesssim 2^{\frac{j}{2}} E^{\frac{1}{2}}.$$

Interpolating between the two bounds to get the right scaling yields,

$$\|S_j u\|_{L^q_{t,x}} \lesssim C(M,E) 2^{j(\frac{1}{2} - \frac{5-p}{3(p-1)})}, \tag{9.48}$$

where $1/q = (5-p)/6(p-1)$. In order to proceed with the usual scattering argument, we need to revisit the fixed point, or more precisely the nonlinear estimate on $F(u)$: indeed, if we wish to use (9.48), even at a power $\varepsilon$, we cannot afford to use the same regularity on both sides of the Duhamel formula. Fortunately, we have off diagonal inhomogeneous estimates, e.g.

$$\| \int e^{i(t-s)\Delta_D} F \|_{\dot B^{s_p,2}_5 (L^{20/9}_t) \cap \dot B^{s_p - 3/4, 2}_4 (L^4_t)} \leq C \|F(u)\|_{\dot B^{s_p - \frac{1}{10},2}_{5/4} (L^2_t)}.$$

In order to evaluate $F(u)$, one needs to place the $S_j u$ factors in such a way that

$$\|(S_j u)^{p-1}\|_{L^{5/3}_x L^{20}_t} \lesssim 2^{\frac{j}{10}}.$$

However, we have from (9.48)

$$\|(\Delta_j u)^{p-1}\|_{L^{\frac{6}{5-p}}_{t,x}} \lesssim C(M,E) 2^{j(\frac{5p-13}{6})}, \tag{9.49}$$

and $6/(5-p) > 5/3$. As such, one may interpolate with

$$\|\Delta_j u\|_{L^4_x L^4_t} \lesssim 2^{-j(s_p - \frac{1}{4})},$$

to get (after Sobolev embedding)

$$\|(\Delta_j u)^{p-1}\|_{L^{\frac{5}{3}}_x L^{20}_t} \lesssim 2^{\frac{j}{10}}.$$

Suming over low frequencies recovers the desired bound. Notice that scaling dictates the exponents (hence there is no need to compute explicitely the interpolation $\theta$).

### 9.4.2 Scattering for $3 \leq p \leq 3 + 2/5$

In this part we consider the remaining case, e.g. nonlinearities which are close to 3 and for which our main results do not provide a scale-invariant local Cauchy theory. As mentioned before, this case will be dealt with using the approach from [87]. As such, this entire Subsection is somewhat disconnected from the rest of the paper; the combination of several technical difficulties makes it lenghty and cumbersome, but we hope the underlying strategy is clear. We have two a priori bounds on the nonlinear equation at our disposal: local smoothing, which is at the scale of $\dot{H}^{\frac{1}{2}}$ regularity for the data, and an $L^4_{t,x}$ space-time bound, which is at the scale of $\dot{H}^{\frac{1}{4}}$ regularity for the data. Both are below the scale of critical $H^s$ regularity, which is $s_p = \frac{3}{2} - \frac{2}{(p-1)}$. Interpolation with the energy bound provides bounds at the critical level, but the lack of flexible scale-invariant estimates on the inhomogeneous problem make them seemingly useless. As such, one has to improve both the local smoothing bound and the $L^4_{t,x}$ space-time bounds obtained in [87], to reach critical scaling and beyond. This is accomplished through several steps, which we informally summarize as follows:

— improve the space-time bounds by using the equations far and close to the boundary. As the resulting commutator source term can only be handle at $H^{\frac{1}{2}}$ regularity, this will improve estimates from $\dot{H}^{\frac{1}{4}}$ regularity to $\dot{H}^{\frac{1}{2}-\varepsilon}$ regularity, which is still below scale invariance;

— combine this improved estimates with the energy bound to obtain yet again better space-time bounds through the equation (but splitting the source terms in close and far away terms). As an added bonus we also improve our local smoothing estimate; moreover we now go beyond scale-invariance;

— turn the crank a few more times, going back and forth between estimates on the split equations and estimates on the equation with split source terms, until we reach the correct set of estimates to prove scattering at the $H^1_0$ regularity. It is worth noticing that the numerology gets worse with $p > 3+2/5$, and that the forthcoming argument would probably break down before even reaching $p = 4$.

We start by stating a few linear estimates which will be needed in the proof and are simple consequences of our Theorem 9.6 by summing over dyadic frequencies.

**Lemma 9.6.** *(see [87, Lemma 5.4]) Let $\Omega$ be a non trapping domain and denote $u_L = e^{it\Delta_D}$ the linear flow for the Schrödinger equation on $\Omega$ with Dirichlet boundary conditions. Then*

$$\|e^{it\Delta_D} u_0\|_{L^4_t \dot{W}^{s,4}(\Omega)} \lesssim \|u_0\|_{\dot{H}_0^{s+\frac{1}{4}}(\Omega)}. \tag{9.50}$$

*Denote by $w$ the solution of the inhomogeneous equation, e.g. $w = \int_0^t e^{i(t-s)\Delta_D} f(s) ds$, then*

$$\|w\|_{C_t \dot{H}_0^{s+\frac{1}{4}}(\Omega)} + \|w\|_{L^4_t \dot{W}^{s,4}} \lesssim \|f\|_{L_t^{\frac{4}{3}} \dot{W}^{s+\frac{1}{2},\frac{4}{3}}}. \tag{9.51}$$

The next lemma is just the Christ-Kiselev lemma again, stated in a form which is convenient for later use.

**Lemma 9.7.** *(see [87, Lemma 5.6]) Let $U(t)$ be a one parameter group of operators, $1 \leq r < q \leq \infty$, $H$ an Hilbert space and $B_r$ and $B_q$ two Banach spaces. Suppose that*

$$\|U(t)\varphi\|_{L^q_t(B_q)} \lesssim \|\varphi\|_H \quad \textit{and} \quad \|\int_s U(-s)g(s)ds\|_H \lesssim \|g\|_{L^r_t(B_r)},$$

*then*

$$\|\int_{s<t} U(t-s)g(s)ds\|_{L^q_t(B_q)} \lesssim \|g\|_{L^r_t(B_r)}.$$

finally, we recall that we have Lemma 9.1 at our disposal, should we need the end-point Strichartz on the left handside in Lemma 9.7, provided that we used a (dual) local smoothing norm on the right handside.

In what follows we shall write $p = 3 + 2\eta$, with $\eta \in [0, 1/5]$. All the nonlinear mappings which we use can be proved using the appendix and we will no longer refer to it. We recall all a priori bounds at our disposal: the first two are uniform in time bounds for the $L^2(\Omega)$ and $H^1_0(\Omega)$ norms of the solution to the defocusing NLS, irrespective of the power $p$, and were already stated in the previous section, see (9.47). The next two were obtained in [87], again in the defocusing case and irrespective of $p$: a space-time norm estimate

$$\|u\|_{L^4_t(L^4(\Omega))} \leq E^{\frac{1}{8}} M^{\frac{3}{8}}, \tag{9.52}$$

which has the same scaling as $\dot{H}^{\frac{1}{4}}$ for the data; and a local smoothing norm estimate

$$\|\nabla u\|_{L^2_t(L^2(K))} \leq C(K) E^{\frac{1}{4}} M^{\frac{1}{4}}, \tag{9.53}$$

which has the same scaling as $\dot{H}^{\frac{1}{2}}$ for the data; here $K$ is meant to be a compact set which includes the obstacle, and (9.53) holds only under the star-shaped condition on the obstacle, while proving (9.52) makes an essential use of (9.53).

We start with proving

**Proposition 9.3.** *Let $u$ be a solution to the nonlinear problem* (9.3). *Let $\chi \in C^2_0(\mathbb{R}^3)$ be a smooth function equal to* 1 *near $\partial\Omega$. Then*

$$\chi u \in L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega) \quad \textit{and} \quad (1-\chi)u \in L^2_t \dot{B}^{1/2-\eta,2}_6(\Omega). \tag{9.54}$$

*Remark* 9.6. Notice that our cut $\chi$ is only $C^2$ rather thant $C^\infty$, and this will remain so for the rest of the section. This is in no way a difficulty, and it allows to conveniently take $\chi = \chi_1^p$ or $\chi = \chi_1^{p-1}$, where $\chi_1 \in C^2_0$ as an admissible cut if we need, as $p - 1 > 2$. This is particulary convenient for nonlinear mappings where all factors can be considered "equal". Alternatively, one may retain $C^\infty_0$ cuts and play with at least 3 overlapping ones, as was done in [87], at the expense of desymetrizing various nonlinear estimates. These are (mildly ennoying) considerations that the reader should ignore at first read.

*Proof.* In order to prove the Proposition, we split the equation (9.3), treating differently the neighborhood of the boundary (using local smoothing type arguments) and spatial infinity (where the full range of sharp Stricharz estimates holds).

Consider the equation satisfied by $\chi u$,

$$(i\partial_t + \Delta_D)(\chi u) = \chi|u|^{2+2\eta}u - [\chi, \Delta_D]u. \tag{9.55}$$

We need to show that the nonlinear term belongs to $L^2_t H^{-\eta}_{comp}(\Omega)$. The commutator term is controlled by $\|\tilde{\chi}u\|_{L^2_t H^1_{comp}}$ for some $\tilde{\chi} \in C^2_0(\mathbb{R}^3)$ equal to 1 on the support of $\chi$ and it belongs to $L^2_t L^2_{comp}(\Omega) \subset L^2_t H^{-\eta}_{comp}(\Omega)$. We now deal with the nonlinear term: let $q$ be such that $\dot{B}^{1,2}_q(\Omega) \subset H^{-\eta}(\Omega)$, hence $1 - \frac{3}{q} = -\eta - \frac{3}{2}$. Then $\frac{1}{q} = \frac{1}{2} + \frac{2(1+\eta)}{6}$ and

$$\|\chi|u|^{2(1+\eta)}u\|_{L^2_t H^{-\eta}_{comp0}(\Omega)} \lesssim \|\chi|u|^{2(1+\eta)}u\|_{L^2_t \dot{B}^{1,2}_q(\Omega)} \lesssim \|\chi_1 u\|_{L^2_t H^1_0(\Omega)} \|(\chi_1 u)^{1+\eta}\|_{L^\infty_t L^{\frac{6}{1+\eta}}(\Omega)},$$

where $\chi_1^p = \chi$ and we used $u \in L^\infty_t H^1_0(\Omega) \subset L^\infty_t L^6(\Omega)$ on two factors and $u \in L^2_t H^1_{comp}(\Omega)$ on one factor. Hence the right hand side in (9.55) is in $L^2_t H^{-\eta}_{comp}(\Omega)$ and we can apply Lemma 9.7 with $L^q(B_q) := L^4_t \dot{W}^{1/4-\eta,4}(\Omega)$, $H := H^{1/2-\eta}(\Omega)$ and $L^r(B_r) := L^2_t H^{-\eta}_{comp}(\Omega)$. This gives the first assertion in (9.54). Let us deal now with $(1-\chi)u$ which is solution to

$$(i\partial_t + \Delta_D)((1-\chi)u) = (1-\chi)|u|^{2+2\eta}u + [\chi, \Delta]u, \tag{9.56}$$

where $\Delta$ denotes the free Laplacian (notice that we can consider (9.56) in the whole space $\mathbb{R}^3$ since both source terms vanish near the boundary $\partial\Omega$). The commutator term is dealt with exactly as in the previous part and is therefore in $L^2_t L^2_{comp}(\Omega)$.

Let $v := (1-\chi_1)u$ for some $\chi_1 \in C^2_0(\mathbb{R}^3)$ such that $(1-\chi_1)^p = 1 - \chi$. In order to prove (9.54) we only need to prove $|v|^{2+2\eta}v \in L^2_t \dot{B}^{1/2-\eta,2}_{6/5}(\Omega)$, since then we may apply the dual end-point Strichartz estimates (from the $\mathbb{R}^3$ case) on the nonlinear term. Using the embedding $\dot{B}^{1-\eta,2}_1(\Omega) \subset \dot{B}^{1/2-\eta,2}_{6/5}(\Omega)$, it suffices to get $|v|^{2+2\eta}v \in L^2_t \dot{B}^{1-\eta,2}_1(\Omega)$. When evaluating the "product" $|v|^{2+2\eta}v$ we will use for one factor $v$ the energy bound and Sobolev embedding, $L^\infty_t H^1_0(\Omega) \subset L^\infty_t \dot{B}^{1-\eta,2}_q(\Omega)$ with $\frac{1}{q} = \frac{1}{2} - \frac{\eta}{3}$. On the other hand, from our a priori bound from [87], we have $v \in L^4_t L^4(\Omega)$, while $v \in L^\infty_t H^1_0(\Omega) \subset L^\infty_t L^6(\Omega)$ and hence $v^{1+\eta} \in L^{4/(1+\eta)}_t L^{4/(1+\eta)}(\Omega) \cap L^\infty_t L^{6/(1+\eta)}(\Omega)$. Interpolation with weights $1/(1+\eta)$ and $\eta/(1+\eta)$ gives $v^{1+\eta} \in L^4_t L^{12/(3+2\eta)}(\Omega)$. Consequently,

$$\||v|^{2+2\eta}v\|_{L^2_t \dot{B}^{1/2-\eta,2}_{6/5}(\Omega)} \lesssim \||v|^{2+2\eta}v\|_{L^2_t \dot{B}^{1-\eta,2}_1(\Omega)} \lesssim \|v\|_{L^\infty_t \dot{B}^{1-\eta,2}_q(\Omega)} \||v|^{1+\eta}\|^2_{L^4_t L^{12/(3+2\eta))}(\Omega)}.$$

This achieves the proof of Proposition 9.3. □

*Remark* 9.7. One should point out that the proof of this last estimate is slightly incorrect, as it conveniently ignores the situation where low frequencies are on the $v$ factor and high frequencies are on $|v|^{2+2\eta}$. This can be easily fixed by revisiting the proof of Lemma 9.14 and 9.15 in the Appendix, noticing that we may suppose that factors $f$ there are in several different $L^r$ spaces and distribute them when using Hölder on the low frequencies in the proofs. The same situation occurs several times in the present proof and we leave details to the reader.

The next iterative step will be the following lemma:

**Proposition 9.4.** *Let $u$ be a solution to the nonlinear problem* (9.3)*. Then*

$$u \in L^4_t \dot{W}^{1/4+\eta,4}(\Omega) \cap L^2_t H^{1+\eta}_{comp}(\Omega). \tag{9.57}$$

*Proof.* The split of the equation into equations for $\chi u$ and $(1-\chi)u$ is no longer of any use: the resulting commutator source term is no better than $[\chi, \Delta]u \in L^2_t L^2_{comp}(\Omega)$. However we now have estimates from Proposition 9.3 which turn out to be good enough that splitting the nonlinear term in (9.3) in two parts, using the partition $\chi + (1-\chi) = 1$ will allow us to use the somewhat restricted set of inhomogeneous estimates we have for the equation on a domain. Setting $g_1 := \chi|u|^{2+2\eta}u$, $g_2 := (1-\chi)|u|^{2+2\eta}u$ and using Duhamel formula, we have

$$u(t,x) = e^{it\Delta_D}u_0 + \int_0^t e^{i(t-s)\Delta_D} g_1(s)ds + \int_0^t e^{i(t-s)\Delta_D} g_2(s)ds\,; \tag{9.58}$$

the idea is then that one may use (9.51) on the $g_1$ Duhamel term, while the $g_2$ term may be handled in $L^1_t(\dot{H}^s)$ for a suitable $s$.

**Lemma 9.8.** *Let $v := (1-\chi_1)u$, where $\chi_1 \in C^2_0(\mathbb{R}^3)$ is such that $(1-\chi_1)^p = 1-\chi$. We have*

$$g_2 \in L^2_t \dot{B}^{1/2,2}_{6/5}(\Omega) \quad \textit{and} \quad v \in L^2_t \dot{B}^{1/2,2}_{6}. \tag{9.59}$$

*Moreover, $g_2 \in L^1_t(\dot{H}^{\frac12+\eta}(\Omega))$ and*

$$\|\int_0^t e^{i(t-s)\Delta_D} g_2(s)ds\|_{L^4_t \dot{B}^{1/4+\eta,2}_4(\Omega)\cap L^2_t H^{1+\eta}_{comp}(\Omega)} \lesssim \|g_2\|_{L^1_t(\dot{H}^{\frac12+\eta}(\Omega))}. \tag{9.60}$$

*Proof.* From Proposition 9.3, the energy and mass bound, and interpolation, we have

$$v \in L^2_t \dot{W}^{1/2-\eta,6}(\Omega) \cap L^\infty_t(\dot{H}^{\frac12-\eta}(\Omega) \subset L^4_t L^q(\Omega) \quad \text{for} \quad \frac{1}{q} = \frac{1}{6} + \frac{\eta}{3},$$

hence $|v|^{1+\eta} \in L^{4/(1+\eta)}_t L^{q/(1+\eta)}(\Omega) \cap L^\infty_t L^{6/(1+\eta)}(\Omega)$. We now interpolate again and obtain $|v|^{1+\eta} \in L^4_t L^r(\Omega)$, where $\frac{2}{r} = \frac13 + \eta$. Therefore, the nonlinear term $g_2 = |v|^{2+2\eta}v$ belongs to $L^2_t \dot{B}^{1-3\eta,2}_{6/5}(\Omega)$. Indeed, let $\frac{1}{m} = \frac12 + \frac{2}{r} = \frac56 + \eta$, then

$$\|g_2\|_{L^2_t \dot{B}^{1-3\eta,2}_{6/5}(\Omega)} \lesssim \|g_2\|_{L^2_t \dot{B}^{1,2}_m(\Omega)} \lesssim \|v\|_{L^\infty_t \dot{H}^1_0(\Omega)} \||v|^{1+\eta}\|^2_{L^4_t L^r(\Omega)}. \tag{9.61}$$

If $1-3\eta \geq 1/2$, (9.59) follows, but unfortunately this covers only $\eta \leq 1/6$. It remains to deal with the situation $\eta \in (1/6, 1/5]$. In this case we use the equation satisfied by $v$ (obtained by replacing $\chi$ by $\chi_1$ in (9.56)) to get

$$v \in L^2_t \dot{B}^{1-3\eta,2}_6(\Omega). \tag{9.62}$$

In fact, the commutator term $[\chi_1, \Delta]u$ is in $L^2_t L^2(\Omega)$ and, consequently, it also belongs to $L^2_t H^{1/2-3\eta}(\Omega)$ since in this case $1/2 - 3\eta < 0$, while $(1-\chi_1)|v|^{2+2\eta}v \in L^2_t \dot{B}^{1-3\eta,2}_{6/5}(\Omega)$ as shown before. Therefore, with $1 - 3\eta - 3/r = 2(1-3\eta) - 1$,

$$v|v| \in L^1_t \dot{B}^{1-3\eta,2}_r(\Omega) \subset L^1_t \dot{B}^{1-6\eta,2}_\infty(\Omega). \tag{9.63}$$

In order to estimate $g_2$ we use (9.63) for a factor $v|v|$, while for the remaining factor $|v|^{1+2\eta}$ we use $v \in L^\infty_t H^1_0(\Omega)$, which yields

$$|v|^{1+2\eta} \subset L^\infty_t \dot{B}^{1,2}_\lambda(\Omega) \subset L^\infty_t H^{1-\eta}(\Omega) \quad \text{for} \quad \frac{1}{\lambda} = \frac{1}{2} + \frac{\eta}{3}. \tag{9.64}$$

From (9.63), (9.64) and product rules, we get $g_2 \in L^1_t H^{2-7\eta}(\Omega) \subset L^1_t H^{1/2}(\Omega)$ (notice that the regularity is $1 - \eta - (6\eta - 1)$ where $6\eta - 1 > 0$).

Using the equation satisfied by $v$ and Duhamel formula we can write

$$v(t,x) = e^{it\Delta_{\mathbb{R}^3}}(1-\chi_1)u_0 + \int_0^t e^{i(t-s)\Delta_{\mathbb{R}^3}}(g_2 + [\chi_1, \Delta]u)(s)ds. \tag{9.65}$$

Using Lemma 9.6 with $L^q(B_q) := L^2_t \dot{B}^{1/2,2}_6(\Omega)$, $L^r(B_r) := L^1_t H^{1/2}(\Omega)$, the first term in the integral in the right hand side of (9.65) belongs to $L^2_t \dot{B}^{1/2,2}_6(\Omega)$. Using Lemma 9.1, we also obtain

$$\| \int_0^t e^{i(t-s)\Delta}[\chi_1, \Delta]u(s)ds\|_{L^2_t \dot{B}^{1/2,2}_6(\Omega)} \lesssim \|[\chi_1, \Delta]u\|_{L^2_t L^2_{comp}(\Omega)}.$$

Finally, the linear evolution $e^{it\Delta_{\mathbb{R}^3}}(1-\chi_1)u_0$ is evidently in $L^2_t \dot{B}^{1/2,2}_6(\Omega)$ and we obtain (9.59).

*Remark* 9.8. For the last part of the proof of Lemma 9.8 we shall use less information than that, precisely we only need the fact that for $\epsilon > 0$ small enough we have

$$v \in L^2_t \dot{B}^{1/2-\epsilon,2}_6(\Omega) \subset L^2_t(L^{\frac{3}{\epsilon}}(\Omega)) \subset L^2_t \dot{B}^{-\epsilon,\infty}_\infty(\Omega), \tag{9.66}$$

and $|v| \in L^{\frac{3}{\epsilon}}(\Omega) \subset L^2_t \dot{B}^{-\epsilon,\infty}_\infty(\Omega)$ as well.

We refine our knowledge on $g_2 = v|v|v^{1+2\eta}$: using the previous remark, we now have $v|v| \in L^1_t \dot{B}^{-2\epsilon,\infty}_\infty(\Omega)$. From (9.64) we also have $|v|^{1+2\eta} \in L^\infty_t \dot{B}^{1,2}_\lambda(\Omega)$ if $\lambda = \frac{6}{3+2\eta}$. Thus, the source term $g_2$ can be estimated as follows

$$\|g_2\|_{L^1_t H^{1-\eta-2\epsilon}(\Omega)} \lesssim \|g_2\|_{L^1_t \dot{B}^{1-2\epsilon,2}_\lambda(\Omega)} \lesssim \|v|v|\|_{L^1_t \dot{B}^{-2\epsilon,\infty}_\infty(\Omega)} \||v|^{1+2\eta}\|_{L^\infty_t \dot{B}^{1,2}_\lambda(\Omega)}. \tag{9.67}$$

Using again Lemma 9.6, this time with $L^q(B_q) := L^4_t \dot{B}^{3/4-\eta-2\epsilon,2}_4(\Omega)$, $H := H^{1-\eta-2\epsilon}(\Omega)$ and $L^r(B_r) := L^1_t H^{1-\eta-2\epsilon}(\Omega)$, we get by interpolation

$$\| \int_0^t e^{i(t-s)\Delta} g_2(s)ds\|_{L^4_t \dot{B}^{1/4+\eta,2}_4(\Omega)} \lesssim \| \int_0^t e^{i(t-s)\Delta} g_2(s)ds\|^\theta_{L^4_t B^{3/4-\eta-2\epsilon,2}_4(\Omega)} \|u\|^{1-\theta}_{L^4_{t,x}}$$
$$\lesssim \|g_2\|_{L^1_t H^{1-\eta-2\epsilon}(\Omega)} + \|u\|_{L^4_{t,x}}; \tag{9.68}$$

where for the first (interpolation) inequality in (9.68) we used that $3/4-\eta-2\epsilon > 1/4+\eta$ if $\epsilon$ is sufficiently small (take $0<\epsilon\leq 1/20$ for example).

On the other hand, by Lemma 9.7 again,

$$\| \int_0^t e^{i(t-s)\Delta} g_2(s)ds\|_{L_t^2 H^{1+\eta}_{comp}(\Omega)} \lesssim \|g_2\|_{L_t^1 H^{1/2+\eta}(\Omega)} \lesssim \|g_2\|_{L_t^1 H^{1-\eta-2\epsilon}(\Omega)}, \tag{9.69}$$

which finally achieves the proof of Lemma 9.8. □

It remains now to deal with the Duhamel term coming from $g_1$ in (9.58).

**Lemma 9.9.** *Suppose that we know moreover that*

$$u \in L_t^4 \dot{B}_4^{\sigma,2}(\Omega), \quad \textit{where} \quad \sigma = \frac{1}{4}+\frac{\eta}{1+\eta}, \tag{9.70}$$

*then*

$$g_1 \in L_t^{4/3}\dot{B}^{3/4+\eta}_{4/3}(\Omega) \quad \textit{and} \quad \int_0^t e^{i(t-s)\Delta_D} g_1(s)ds \in L_t^4 \dot{B}_4^{1/4+\eta,2} \cap L_t^2 H^{1+\eta}_{comp}(\Omega). \tag{9.71}$$

Taking the lemma for granted, we can complete the proof of Proposition 9.4: using Lemmas 9.8, 9.9, the fact that the linear flow is in $L_t^\infty H_0^1(\Omega)\cap L_t^2 H^{3/2}_{comp}(\Omega)$ and Duhamel formula (9.58), estimate (9.57) follows immediately.

*Proof. (of Lemma 9.9):* The a-priori information (9.70) gives

$$u \in L_t^4 \dot{B}_4^{\sigma,2}(\Omega) \subset L_t^4 L^q(\Omega) \quad \text{for} \quad \frac{1}{q} = \frac{1}{4}-\frac{\sigma}{3},$$

and consequently $u^{2(1+\eta)} \in L_t^{2/(1+\eta)} L^{3/(1-\eta)}(\Omega)$. On the other hand, interpolating between $L_t^2 H^1_{comp}(\Omega)$ and $L_t^\infty H_0^1(\Omega)$ gives $\chi u \in L_t^r H^1_{comp}(\Omega)$ for every $r\in[2,\infty]$. Therefore, with $\chi_1^p = \chi$, we can estimate

$$\|\chi |u|^{2+2\eta} u\|_{L_t^{4/3}\dot{B}^{1,2}_M} \lesssim \|\chi_1 u\|_{L_t^{4/(1-2\eta)} H^1_{comp}(\Omega)} \|u^{2+2\eta}\|_{L_t^{2/(1+\eta)} L^{3/(1-\eta)}(\Omega)}, \tag{9.72}$$

where $\frac{1}{M} = \frac{1}{2}+\frac{1-\eta}{3} = \frac{5}{6}-\frac{\eta}{3}$. It remains to notice that for $M$ defined above, the embedding $\dot{B}^{1,2}_M(\Omega) \subset \dot{B}^{3/4+\eta,2}_{4/3}(\Omega)$ holds (indeed, $1 > 3/4+\eta$ and $1-3/M = 3/4+\eta-9/4$) and to use again Lemmas 9.7, 9.1. Another application of Lemma 9.7 with $L^q(B_q) := L_t^2 H^{1+\eta}_{comp}(\Omega)$, $H := H^{1/2+\eta}_{comp}(\Omega)$ and $L^r(B_r) := L_t^{4/3}\dot{B}^{3/4+\eta,2}_{4/3}(\Omega)$ achieves the proof of (9.71) and Lemma 9.9. □

*End of the proof of Proposition 9.4:* In order to complete the proof of Proposition 9.4 it remains to prove that (9.70) holds indeed, since we have used it to deduce (9.57). Let

$0 < T < \infty$ be small enough, so that by the local existence theory (see [87]) the $L^4_T \dot{B}^{\sigma,2}_4(\Omega)$ norm of $u$ is finite; in fact, the same can be said with $\sigma$ replaced by $\eta + \frac{1}{4}$. We shall prove that $T = \infty$ is allowed. For this, we interpolate between $L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)$ and $L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)$ with interpolation exponent $\theta = \frac{\eta}{2(1+\eta)}$ to obtain an estimate on the $L^4_T \dot{B}^{\sigma,2}_4(\Omega)$ norm, where $\sigma = 1/4 + \eta/(1+\eta)$:

$$\|u\|_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)} \le \|u\|^{\theta}_{L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)} \|u\|^{1-\theta}_{L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)}. \tag{9.73}$$

Recall that from Proposition 9.3 we have now a uniform bound,

$$\|u\|_{L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)} \lesssim C(E, M), \tag{9.74}$$

and from Lemma 9.8 we consequently also have a uniform bound on the Duhamel part coming from $g_2$, see (9.60). Finally, using (9.71) for $g_1$ and the uniform bounds we already have for the linear part and the $g_2$ part,

$$\|u\|_{L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)} \lesssim C_1(E, M) + C_2(E, M) \|\chi u\|^{1/2-\eta}_{L^2_t H^1_{comp}(\Omega)} \|u\|^{2(1+\eta)}_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)}. \tag{9.75}$$

Plugging (9.74), (9.75) in (9.73) yields

$$\|u\|_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)} \le C_3(E, M) + C_4(E, M) \|\chi u\|^{\gamma}_{L^2_t H^1_{comp}(\Omega)} \|u\|^{\rho}_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)}, \tag{9.76}$$

where $\rho, \gamma > 0$. The coefficients are uniformly bounded, and a splitting time argument performed on the $L^2_t H^1_{comp}(\Omega)$ norm which is finite provides global in time control of $u$ in $L^4_t \dot{B}^{\sigma,2}_4(\Omega)$. This finally completes the proof of Proposition 9.4. □

*Remark* 9.9. The space $L^4_t(\dot{B}^{\sigma,2}_4(\Omega))$ with $\sigma = \frac{1}{4} + \frac{\eta}{1+\eta}$ does not show up by accident: rather, it is a scale invariant space with respect to the critical regularity $s_p$. As such, it makes sense that it plays a pivotal role in the argument. Having reached (and in fact, gone beyond) critical scaling in our a priori estimates, the remaining part of the argument is somewhat less involved.

At this point of the proof, we could establish scattering in the scale-invariant Sobolev space; however we want to reach $H^1_0$. Recall that we may write

$$\|u(t,x) - e^{it\Delta_D}(u_0 + \int_0^{+\infty} e^{-is\Delta_D} |u|^{p-1} u(s) ds)\|_{H^1_0} = \| \int_t^{+\infty} e^{i(t-s)\Delta_D} |u|^{p-1} u(s) ds\|_{H^1_0},$$

from which we wish to use Duhamel to get

$$\| \int_t^{+\infty} e^{i(t-s)\Delta_D} |u|^{p-1} u(s) ds\|_{H^1_0} \lesssim \|g_1\|_{L^{4/3}(t,+\infty; \dot{B}^{5/4,2}_{4/3}(\Omega))} + \|g_2\|_{L^1(t,+\infty; H^1_0(\Omega))}, \tag{9.77}$$

from which scattering easily follows (the same argument applies at $t = -\infty$ as well).

Therefore we focus on the right handside and start with the easiest part, which is $g_2$.

**Lemma 9.10.** *We have* $g_2 = (1-\chi)u^p \in L^1_t H^1_0(\Omega)$.

*Proof.* We start by proving that

$$v = (1-\chi_1)u \in L_t^{2(1+\eta)} L^\infty(\Omega). \tag{9.78}$$

*Remark* 9.10. Notice that if we have (9.78) the proof is finished since then

$$\|v|v|^{2+2\eta}\|_{L^1_t H^1_0(\Omega)} \leq \||v|^{2(1+\eta)}\|_{L^1_t L^\infty(\Omega)} \|v\|_{L^\infty_t H^1_0(\Omega)}. \tag{9.79}$$

We proceed with (9.78). From Lemma 9.8 we know that $g_2 \in L^1_t H^{1-\eta}(\Omega)$ and $[\chi, \Delta_D]u \in L^2_t H^\eta_{comp}(\Omega)$, so using again the equation for $(1-\chi)u$ and Lemma 9.7,

$$(1-\chi)u \in L^2_t \dot{B}^{1-\eta,2}_6(\Omega)\big(\cap L^\infty_t H^1_0(\Omega)\big). \tag{9.80}$$

Recall that from Lemma 9.8 we also have $v \in L^2_t \dot{B}^{1/2,2}_6 \cap L^\infty_t H^{1/2}(\Omega)$. The Lemma now follows by interpolation and the Gagliardo-Nirenberg inequality (a similar key step exists in [87]). □

**Lemma 9.11.** *We have* $g_1 = \chi u^p \in L^{4/3}_t \dot{B}^{5/4,2}_{4/3}(\Omega)$.

*Proof.* We first prove

$$u \in L_t^{8(1+\eta)} L^{8(1+\eta)}(\Omega). \tag{9.81}$$

Indeed, from Propositions 9.3, 9.4 and interpolation, we get $u \in L^4_t \dot{B}^{1/4+\eta/2,2}_4(\Omega)$. Interpolating again between this bound and the energy bound $u \in L^\infty_t H^1_0(\Omega)$, followed by Sobolev embedding yields (9.81). Now we write

$$\|g_1\|_{L^{4/3}_t \dot{B}^{5/4,2}_{4/3}(\Omega)} \lesssim \|\chi u\|_{L^2_t H^{5/4}_{comp}(\Omega)} \|u^{2+2\eta}\|_{L^4_t L^4(\Omega)}, \tag{9.82}$$

and also by the Duhamel formula and the local smoothing estimate on the domain,

$$\|u\|_{L^2_t H^{5/4}_{comp}(\Omega)} \leq \|u_0\|_{H^{3/4}(\Omega)} + \|g_1\|_{L^{4/3}_t \dot{B}^{1,2}_{4/3}(\Omega)} + \|g_2\|_{L^1_t H^{3/4}(\Omega)}. \tag{9.83}$$

Certainly, using Lemma 9.10, the $g_2$ term is bounded. For $g_1$, we may write

$$\|g_1\|_{L^{4/3}_t \dot{B}^{1,2}_{4/3}(\Omega)} \lesssim \|\chi u\|_{L^2_t H^1_{comp}(\Omega)} \|u^{2+2\eta}\|_{L^4_t L^4(\Omega)}; \tag{9.84}$$

and we have reached a point where our right handside is uniformly bounded. Consequently the Lemma is proved, and this concludes the proof of Theorem 9.3. □

## 9.5 Appendix

In order to perform the various product estimates, we need a couple of useful lemma. Observe that with the spectral localization one cannot take advantage of convolution of Fourier supports. As a first step and in order to avoid cumbersome notations, we only consider functions and Besov spaces which do not depend on time. We will then explain how to re-instate the time dependance in the nonlinear estimates.

It is worth noting at this stage, however, that both $\Delta_j$ and $S_j$ operators are well-defined on $L^p_t L^q_x$ and $L^q_x L^p_t$ for all the pairs $(p, q)$ to be considered: this follows from [63] for the case $L^p_t L^q_x$ where the time norm is harmless. In the case $L^q_x L^2_t$, the arguments from [63] apply as well (heat estimates are proved for data in $L^p_x(H)$ where $H$ is an abstract Hilbert space, and when $H = L^2_t$, the heat kernel is diagonal and therefore Gaussian as well). By interpolation and duality we recover all pairs $(p, q)$.

*Remark* 9.11. In $\mathbb{R}^n$, one may perform product estimates in an easier way because of the convolution of Fourier supports. However, when dealing with non integer power-like nonlinearities, one cannot proceed so easily: the usual route is to use a characterization of Besov spaces via finite differences; here, because of the Banach valued Besov spaces, we perform a direct argument which is directly inspired by computations in [86], where the same sort of time-valued Besov spaces were unavoidable.

**Lemma 9.12.** *Let $f_j$ be such that $S_j f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s > 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot{B}^{s,q}_p$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k<j} S_j f_j.$$

Now,

$$\|\Delta_k(\sum_{k<j} S_j f_j)\|_p \lesssim 2^{-ks} \sum_{k<j} 2^{-s(j-k)}\eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

**Lemma 9.13.** *Let $f_j$ be such that $(I - S_j)f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s < 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot{B}^{s,q}_p$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k>j} (I - S_j) f_j.$$

Now,

$$\|\Delta_k(\sum_{k>j}(I - S_j) f_j)\|_p \lesssim 2^{-ks} \sum_{k<j} 2^{-s(j-k)}\eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

**Lemma 9.14.** *Consider $\alpha = 1$ or $\alpha \geq 2$, $f \in \dot{B}^{s,q}_p$ and $g \in L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{\alpha}{r} + \frac{1}{p}$: let*

$$T_g^\alpha f = \sum_j (S_j g)^\alpha \Delta_j f.$$

*Then*

$$T_g^\alpha f \in \dot{B}^{s,q}_m.$$

We split the "paraproduct" $T_g^\alpha f$:

$$T_g^\alpha f = \sum_j S_j((S_j g)^\alpha \Delta_j f) + \sum_j (I - S_j)((S_j g)^\alpha \Delta_j f);$$

the first part is easily dealt with by Lemma 9.12. For the second one, $K_g f$, taking once again advantage of the spectral supports

$$\Delta_k K_g f = \Delta_k \sum_{j<k} (I - S_j)((S_j g)^\alpha \Delta_j f).$$

Notice the situation is close to the one in Lemma 9.13, but we don't have a negative regularity for summing. We therefore derive

$$\begin{aligned}
\Delta_D K_g f &= \sum_{j<k} (I - S_j)\Delta_D((S_j g)^\alpha \Delta_j f) \\
&= \sum_{j<k} (I - S_j) \left(\Delta_D (S_j g)^\alpha \Delta_j f + (\Delta_D \Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1} \nabla S_j g \cdot \nabla \Delta_j f\right) \\
&= \sum_{j<k} (I - S_j) \big(\alpha \Delta_D S_j g (S_j g)^{\alpha-1} \Delta_j f + \alpha(\alpha-1)|\nabla S_j g|^2 (S_j g)^{\alpha-2} \Delta_j f \\
&\quad + (\Delta_D \Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1} \nabla S_j g \cdot \nabla \Delta_j f\big).
\end{aligned}$$

The first two pieces are again easily dealt with with Lemma 9.13, and the resulting function is in $\dot{B}^{s-2,q}_m$. The remaining cross term is handled with some help from [63]:

$$\nabla \Delta_j f = \nabla \exp(4^{-j} \Delta_D) \tilde{\Delta}_j f,$$

where the new dyadic block $\tilde{\Delta}_j$ is built on the function $\tilde{\psi}(\xi) = \exp(|\xi|^2)\psi(\xi)$. From the continuity properties of $\sqrt{s}\nabla \exp(s\Delta_D)$ on $L^p$, $1 < p < +\infty$, we immediatly deduce

$$\|\nabla \Delta_j f\|_p \lesssim 2^j \|\tilde{\Delta}_j f\|_p, \tag{9.85}$$

and we can easily sum and conclude. This will be enough to deal with the critical case, but for differences of nonlinear power-like mappings, we need

**Lemma 9.15.** *Consider $\alpha \geq 3$, $f, g \in X = \dot{B}^{s,q}_p \cap L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{\alpha-1}{r} + \frac{1}{p}$: Then, if $F(x) = |x|^{\alpha-1}x$ or $F(x) = |x|^\alpha$,*

$$\|F(u) - F(v)\|_{\dot{B}^{s,q}_m} \lesssim \|u - v\|_X (\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1}).$$

In order to obtain a factor $u-v$, we write

$$F(u)-F(v)=(u-v)\int_0^1 F'(\theta u+(1-\theta)v)d\theta. \tag{9.86}$$

We need to efficiently split this difference into two paraproducts involving $u-v$ and $F'(w)$ with $w=\theta u+(1-\theta)v$, and this requires an estimate on $F'(w)$: write another telescopic series

$$\begin{aligned}F'(w)=&\sum_j F'(S_{j+1}w)-F'(S_jw)\\=&\sum_j S_j(F'(S_{j+1}w)-F'(S_jw))+\sum_j(I-S_j)(F'(S_{j+1}w)-F'(S_jw))\\=&S_1+S_2.\end{aligned}$$

Exactly as before, the first sum $S_1$ is easily disposed of with Lemma 9.12, as

$$|F'(S_{j+1}w)-F'(S_jw)|\lesssim|\Delta_jw|(|S_{j+1}w|^{\alpha-2}+|S_jw|^{\alpha-2}).$$

The second sum $S_2$ requires again a trick; to avoid uncessary cluttering, we set $F(x)=x^\alpha$, ignoring the sign issue (recall that $\alpha\geq3$, hence $F'''(x)$ is well-defined as a function): we apply $\Delta_D$, let $\beta=\alpha-1\geq2$

$$\begin{aligned}\Delta_DS_2=&\sum_j(I-S_j)\Delta_D((S_{j+1}w)^{\alpha-1}-(S_jw)^{\alpha-1})\\=&\sum_j(I-S_j)\left(\beta(S_{j+1}w)^{\beta-1}\Delta_DS_{j+1}w-\beta(S_jw)^{\beta-1}\Delta_DS_jw\right.\\&\left.+\beta(\beta-1)(S_{j+1}w)^{\beta-2}(\nabla S_{j+1}w)^2-\beta(\beta-1)(S_jw)^{\beta-2}(\nabla S_jw)^2\right).\end{aligned}$$

We now apply Lemma 9.13 after inserting the right factors: we have four types of differences,

$$\begin{aligned}|((S_{j+1}w)^{\beta-1}-(S_jw)^{\beta-1})\Delta_DS_{j+1}w|&\lesssim C_\beta|\Delta_jw||\Delta_DS_{j+1}|(|S_{j+1}w|^{\beta-2}+|S_jw|^{\beta-2})\\|(S_{j+1}w)^{\beta-1}\Delta_D\Delta_jw|&\leq|\Delta_D\Delta_jw||S_{j+1}w|^{\beta-2}\\|((S_{j+1}w)^{\beta-2}-(S_jw)^{\beta-2})(\nabla S_{j+1}w)^2|&\lesssim\tilde C_\beta|\Delta_jw|^{\beta-2}|\nabla S_{j+1}w|^2\\|(S_{j+1}w)^{\beta-2}((\nabla S_jw)^2-(\nabla S_{j+1}w)^2)|&\leq|\nabla\Delta_jw|(|\nabla S_jw|+|\nabla S_{j+1}w||S_{j+1}w|^{\beta-2}\end{aligned}$$

where on the third line we wrote the worst case, namely $2\leq\beta<3$ (otherwise the power of $\Delta_jw$ in the third bound will be replaced by $|\Delta_jw|(|S_jw|^{\beta-3}+|S_{j+1}w|^{\beta-3})$).

By integrating, applying Hölder and using (9.85) to eliminate the $\nabla$ operator, we obtain as an intermediary result

$$F'(w)\in\dot B^{s,q}_\lambda,\quad\text{with }\frac1\lambda=\frac{\alpha-2}{r}+\frac1p.$$

We may now go back to the difference $F(u) - F(v)$ as expressed in (9.86) and perform a simple paraproduct decomposition in two terms to which Lemma 9.14 may be applied. Observe that there is no difficulty in estimating $F'(w)$ in $L^{m/(\alpha-1)}$, and that the integration in $\theta$ is irrelevant. This completes the proof.

We now go back to the first nonlinear estimate, namely (9.44). We write a telescopic series for the product five factors $u_1, u_2, u_3, u_4, u_5 \in X_T$,

$$u_1u_2u_3u_4u_5 = \sum_j S_{j+1}u_1S_{j+1}u_2S_{j+1}u_3S_{j+1}u_4S_{j+1}u_5 - S_ju_1S_ju_2S_ju_3S_ju_4S_ju_5$$

and we are reduced to studying five sums of the same type, of which the following is generic

$$S_1 = \sum_j \Delta_j u_1 S_ju_2S_ju_3S_ju_4S_ju_5,$$

and we intend to apply Lemma 9.14, which is trivially extended to a product of several factors. In principle,

$$u_k \in \dot{B}_5^{1,2}(L_T^{\frac{20}{11}}) \cap L_x^{\frac{20}{3}} L_T^{40}$$

is enough, using the first space of the $\Delta_j$ factor and the second one for all remaining $S_j$ factors, except for the use of (9.85) in the proof. Consider, from $u \in X_T$,

$$2^{\frac{11}{10}j}\|\Delta_j u\|_{L_x^5 L_T^2} + 2^{-\frac{3}{2}j}\|\partial_t \Delta_j u\|_{L_T^5 L_x^5} = \mu_j^0 \in l_j^2.$$

We will have, using [63],

$$2^{\frac{11}{10}j}\|\nabla\Delta_j u\|_{L_x^5 L_T^2} + 2^{-\frac{3}{2}j}\|\partial_t \nabla\Delta_j u\|_{L_T^5 L_x^5} = \mu_j^1 \in l_j^2, \text{ with } \|\mu^1\|_{l^2} \lesssim \|\mu^0\|_{l^2}.$$

By Gagliardo-Nirenberg in time, we have the correct estimate for $\Delta_j u$, for $k = 0, 1$

$$2^{(1-k)j}\|\nabla^k \Delta_j u\|_{L_x^5 L_T^{\frac{20}{11}}} \lesssim \mu_j^k.$$

We proceed with the low frequencies by proving a suitable Sobolev embedding.

**Lemma 9.16.** *Let $u \in \dot{B}_5^{\frac{1}{2},5}(L_T^5)$ and $\partial_t u \in \dot{B}_5^{-\frac{3}{2},5}(L_T^5)$. Then $u \in L_x^{\frac{20}{3}} L_T^{40}$.*

Let

$$2^{(\frac{1}{2}-k)j}\|\nabla^k\Delta_j u\|_{L_x^5 L_T^5} + 2^{-(k+\frac{3}{2})j}\|\partial_t\nabla^k\Delta_j u\|_{L_T^5 L_x^5} = \mu_j^k \in l_j^5,$$

notice we can easily switch time and space Lebesgue norms. Using Gagliardo-Nirenberg in time, we have

$$2^{(\frac{1}{6}-k)j}\|\nabla^k\Delta_j u\|_{L_x^5 L_T^{30}} \lesssim \mu_j^3 \in l_j^5. \tag{9.87}$$

Using now Gagliardo-Nirenberg in space, we also have

$$2^{-\frac{j}{10}}\|\Delta_j u\|_{L_x^\infty L_T^5} \lesssim 2^{-\frac{j}{10}}\|\Delta_j u\|_{L_T^5 L_x^\infty} \lesssim \mu_j^5$$

and the same thing for $2^{-2j}\partial_t\Delta_j u$ (or with an additional $2^j\nabla$). Now another Gagliardo-Nirenberg in time provides

$$2^{-(k+\frac{1}{2})j}\|\nabla^k\Delta_j u\|_{L^\infty_{T,x}} \lesssim \mu_j^6. \tag{9.88}$$

Finally, we take advantage of a discrete embedding between $l^1$ and weighted $l^\infty$ sequences:

$$\begin{aligned}
|u| &\leq \sum_{j<J}|\Delta_j u| + \sum_{j\geq J}|\Delta_j u| \\
&\leq \sum_{j<J} 2^{\frac{j}{2}} \sup_j 2^{-\frac{j}{2}}|\Delta_j u| + \sum_{j\geq J} 2^{-\frac{j}{6}} \sup_j 2^{\frac{j}{6}}|\Delta_j u| \\
&\lesssim 2^{\frac{J}{2}} \sup_j 2^{-\frac{j}{2}}|\Delta_j u| + 2^{-\frac{J}{6}} \sup_j 2^{\frac{j}{6}}|\Delta_j u| \\
|u|^4 &\lesssim \sup_j 2^{-\frac{j}{2}}|\Delta_j u| \left(\sup_j 2^{\frac{j}{6}}|\Delta_j u|\right)^3 \\
\||u|^4\|_{L_x^{\frac{5}{3}} L_T^{10}} &\lesssim \|\sup_j 2^{-\frac{j}{2}}|\Delta_j u|\|_{L^\infty_{T,x}} \|\sup_j 2^{\frac{j}{6}}|\Delta_j u|\|^3_{L^5_x L^{30}_T} \\
\|u\|_{L_x^{\frac{20}{3}} L_T^{40}} &\lesssim \|u\|\|^{\frac{1}{4}}_{\dot{B}^{\frac{1}{2},\infty}_\infty (L^\infty_t)} \|u\|\|^{\frac{3}{4}}_{\dot{B}^{\frac{1}{6},5}_5 (L^{30}_t)}
\end{aligned}$$

Notice that the estimate with a gradient is much easier: just interpolate between (9.87) and (9.88) with $k = 1$ to obtain

$$2^{-j}\|\nabla\Delta_j u\|_{L_x^{\frac{20}{3}} L_T^{40}} \lesssim \mu_j^7,$$

which we can now sum over $k < j$ to obtain control of $S_j u$.

The case $p < 5$ is handled in an similar way, and we leave the details to the reader, sparing him the complete set of exponents (depending on $p$ !) that would appear in the proof. For scaling reasons there is actually no need to perform the computation: the previous one on the critical case simply illustrates that we can sidestep issues related to the usual Littlewood-Paley theory by using direct arguments.

# References


[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions, with formulas, graphs, and mathematical tables.* Edited by Milton Abramowitz and Irene A. Stegun. Dover Publications Inc., New York, 1966.

[2] T. Akiyama, H. Kasai, Y. Shibata, and M. Tsutsumi. On a resolvent estimate of a system of Laplace operators with perfect wall condition. *Funkcial. Ekvac.*, 47(3):361–394, 2004.

[3] Georgios K. Alexopoulos. $L^p$ bounds for spectral multipliers from Gaussian estimates of the heat kernel. unpublished manuscript, 1999.

[4] Georgios K. Alexopoulos. Spectral multipliers for Markov chains. *J. Math. Soc. Japan*, 56(3):833–852, 2004.

[5] K. G. Andersson and R. Melrose. Propagation of singularities along gliding rays. In *Séminaire Goulaouic-Schwartz (1976/1977), Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 1*, page 5. Centre Math., École Polytech., Palaiseau, 1977.

[6] Ramona Anton. Global existence for defocusing cubic NLS and Gross-Pitaevskii equations in three dimensional exterior domains. *J. Math. Pures Appl. (9)*, 89(4):335–354, 2008.

[7] Ramona Anton. Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains. *Bull. Soc. Math. France*, 136(1):27–65, 2008.

[8] Pascal Auscher. On necessary and sufficient conditions for $L^p$-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. *Mem. Amer. Math. Soc.*, 186(871):xviii+75, 2007.

[9] Pascal Auscher, Alan McIntosh, and Philippe Tchamitchian. Heat kernels of second order complex elliptic operators and applications. *J. Funct. Anal.*, 152(1):22–73, 1998.

[10] Pascal Auscher and Philippe Tchamitchian. Square root problem for divergence operators and related topics. *Astérisque*, (249):viii+172, 1998.

[11] Hajer Bahouri and Jean-Yves Chemin. Équations d'ondes quasilinéaires et effet dispersif. *Internat. Math. Res. Notices*, (21):1141–1178, 1999.

[12] M. Ben-Artzi and A. Devinatz. Local smoothing and convergence properties of Schrödinger type equations. *J. Funct. Anal.*, 101(2):231–254, 1991.

[13] Matania Ben-Artzi and Sergiu Klainerman. Decay and regularity for the Schrödinger equation. *J. Anal. Math.*, 58:25–37, 1992. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.

[14] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[15] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. Strichartz estimates for the wave equation on manifolds with boundary. *to appear in Ann.Inst.H.Poincaré, Anal.Non Liréaire.*

[16] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. On Strichartz estimates for Schrödinger operators in compact manifolds with boundary. *Proc. Amer. Math. Soc.*, 136(1):247–256 (electronic), 2008.

[17] Jean-Marc Bouclet and Nikolay Tzvetkov. On global Strichartz estimates for non-trapping metrics. *J. Funct. Anal.*, 254(6):1661–1682, 2008.

[18] Jean Bourgain. *Global solutions of nonlinear Schrödinger equations*, volume 46 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

[19] Jean Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applcations to nonlinear evolution equations. *Geom. Funct. Anal.*, 3(2):107–156, 2003.

[20] N. Burq, P. Gérard, and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. *Invent. Math.*, 159(1):187–223, 2005.

[21] Nicolas Burq. Contrôle de l'équation des plaques en présence d'obstacles strictement convexes. *Mém. Soc. Math. France (N.S.)*, (55):126, 1993.

[22] Nicolas Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.

[23] Nicolas Burq. Estimations de Strichartz pour des perturbations à longue portée de l'opèrateur de schrödinger. In *Séminaire: Équations aux Dérivées Partielles. 2001–2002*, Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau, 2002.

[24] Nicolas Burq. Semi-classical estimates for the resolvent in nontrapping geometries. *Int. Math. Res. Not.*, (5):221–241, 2002.

[25] Nicolas Burq. Smoothing effect for Schrödinger boundary value problems. *Duke Math. J.*, 123(2):403–427, 2004.

[26] Nicolas Burq, Patrick Gérard, and Nicolay Tzvetkov. On nonlinear Schrödinger equations in exterior domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(3):295–318, 2004.

[27] Nicolas Burq, Patrick Gérard, and Nicolay Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[28] Nicolas Burq, Gilles Lebeau, and Fabrice Planchon. Global existence for energy critical waves in 3-D domains. *J. Amer. Math. Soc.*, 21(3):831–845, 2008.

[29] Nicolas Burq and Fabrice Planchon. Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications. *J. Funct. Anal.*, 236(1):265–298, 2006.

[30] Nicolas Burq and Fabrice Planchon. Global existence for energy critical waves in 3-d domains : Neumann boundary conditions, 2007. to appear in Amer. J. of Math.

[31] Gilles Carron, Thierry Coulhon, and El-Maati Ouhabaz. Gaussian estimates and $L^p$-boundedness of Riesz means. *J. Evol. Equ.*, 2(3):299–317, 2002.

[32] Thierry Cazenave. *An introduction to nonlinear Schrödinger equations*, volume 22 of *Textos de Metodos Matematicos*.

[33] Thierry Cazenave and Fred B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$. *Nonlinear Anal.*, 14(10):807–836, 1990.

[34] C. Chester, B. Friedman, and F. Ursell. An extension of the method of steepest descents. *Proc. Cambridge Philos. Soc.*, 53:599–611, 1957.

[35] Michael Christ and Alexander Kiselev. Maximal functions associated to filtrations. *J. Funct. Anal.*, 179(2):409–425, 2001.

[36] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865, 2008.

[37] P. Constantin and J.-C. Saut. Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.*, 1(2):413–439, 1988.

[38] Peter Constantin and Jean-Claude Saut. Local smoothing properties of Schrödinger equations. *Indiana Univ. Math. J.*, 38(3):791–810, 1989.

[39] Walter Craig, Thomas Kappeler, and Walter Strauss. Microlocal dispersive smoothing for the Schrödinger equation. *Comm. Pure Appl. Math.*, 48(8):769–860, 1995.

[40] Daniel Daners. Heat kernel estimates for operators with boundary conditions. *Math. Nachr.*, 217:13–41, 2000.

[41] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[42] E. B. Davies. Pointwise bounds on the space and time derivatives of heat kernels. *J. Operator Theory*, 21(2):367–378, 1989.

[43] E. B. Davies. The functional calculus. *J. London Math. Soc. (2)*, 52(1):166–176, 1995.

[44] E. B. Davies. Non-Gaussian aspects of heat kernel behaviour. *J. London Math. Soc. (2)*, 55(1):105–125, 1997.

[45] Shin-ichi Doi. Smoothing effects of Schrödinger evolution groups on Riemannian manifolds. *Duke Math. J.*, 82(3):679–706, 1996.

[46] Shin-ichi Doi. Smoothing effects of Schrödinger evolution groups on Riemannian manifolds. *Duke Math. J.*, 82(3):679–706, 1996.

[47] E. M. Dyn′kin. An operator calculus based on the Cauchy-Green formula. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 30:33–39, 1972. Investigations on linear operators and the theory of functions, III.

[48] Gregory Eskin. Parametrix and propagation of singularities for the interior mixed hyperbolic problem. *J. Analyse Math.*, 32:17–62, 1977.

[49] J. Ginibre and G. Velo. The global Cauchy problem for the nonlinear Schrödinger equation revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(4):309–327, 1985.

[50] J. Ginibre and G. Velo. Smoothing properties and retarded estimates for some dispersive evolution equations. *Comm. Math. Phys.*, 144(1):163–188, 1992.

[51] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. In *Partial differential operators and mathematical physics (Holzhau, 1994)*, volume 78 of *Oper. Theory Adv. Appl.*, pages 153–160. Birkhäuser, Basel, 1995.

[52] Jean Ginibre and Giorgio Velo. Quadratic Morawetz inequalities and asymptotic completeness in the energy space for the nonlinear Schrödinger and Hartree equations, 2008. `http://www.arxiv.org/abs/0807.0367`.

[53] Daniel Grieser. $l^p$ bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries. Thesis, UCLA, 1992.

[54] Andrew Hassell, Terence Tao, and Jared Wunsch. Sharp Strichartz estimates on nontrapping asymptotically conic manifolds. *Amer. J. Math.*, 128(4):963–1024, 2006.

[55] B. Helffer and J. Sjöstrand. Équation de Schrödinger avec champ magnétique et équation de Harper. In *Schrödinger operators (Sønderborg, 1988)*, volume 345 of *Lecture Notes in Phys.*, pages 118–197. Springer, Berlin, 1989.

[56] Lars Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[57] Mitsuru Ikawa. Decay of solutions of the wave equation in the exterior of two convex obstacles. *Osaka J. Math.*, 19(3):459–509, 1982.

[58] Mitsuru Ikawa. Decay of solutions of the wave equation in the exterior of several convex bodies. *Ann. Inst. Fourier (Grenoble)*, 38(2):113–146, 1988.

[59] Oana Ivanovici. Precise smoothing effect in the exterior of balls. *Asymptot. Anal.*, 53(4):189–208, 2007.

[60] Oana Ivanovici. Counter example to Strichartz estimates for the wave equation in domains, 2008. to appear in Math. Annalen, `arXiv:math/0805.2901`.

[61] Oana Ivanovici. On the Schrödinger equation outside strictly convex obstacles, 2008. to appear in Analysis and PDE, `arXiv:math/0809.1060`.

[62] Oana Ivanovici. Counterexamples to the Strichartz estimates for the wave equation in domains II, 2009. `arXiv:math/0903.0048`.

[63] Oana Ivanovici and Fabrice Planchon. Square function and heat flow estimates on domains, 2008. `arXiv:math/0812.2733`.

[64] Oana Ivanovici and Fabrice Planchon. On the energy critical Schrödinger equation in $3d$ non-trapping domains, 2009. `arXiv:math/0904.4749`.

[65] Lev Kapitanski. Global and unique weak solutions of nonlinear wave equations. *Math. Res. Lett.*, 1(2):211–223, 1994.

[66] L. V. Kapitanskiĭ. Estimates for norms in Besov and Lizorkin-Triebel spaces for solutions of second-order linear hyperbolic equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 171(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20):106–162, 185–186, 1989.

[67] L. V. Kapitanskiĭ. Some generalizations of the Strichartz-Brenner inequality. *Algebra i Analiz*, 1(3):127–159, 1989.

[68] Tosio Kato. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.*, 46(1):113–129, 1987.

[69] Tosio Kato and Kenji Yajima. Some examples of smooth operators and the associated smoothing effect. *Rev. Math. Phys.*, 1(4):481–496, 1989.

[70] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[71] Herbert Koch, Hart F. Smith, and Daniel Tataru. Subcritical $L^p$ bounds on spectral clusters for Lipschitz metrics. *Math. Res. Lett.*, 15(5):993–1002, 2008.

[72] Peter D. Lax and Ralph S. Phillips. *Scattering theory*, volume 26 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, second edition, 1989. With appendices by Cathleen S. Morawetz and Georg Schmidt.

[73] G. Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl. (9)*, 71(3):267–291, 1992.

[74] Gilles Lebeau. Estimation de dispersion pour les ondes dans un convexe. In *Journées "Équations aux Dérivées Partielles" (Evian, 2006)*. 2006. see http://www.numdam.org/numdam-bin/fitem?id=JEDP_2006____A7_0.

[75] Hans Lindblad and Christopher D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[76] André Martinez. *An introduction to semiclassical and microlocal analysis.* Universitext. Springer-Verlag, New York, 2002.

[77] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.

[78] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2):129–168, 1982.

[79] Richard B. Melrose and Michael E. Taylor. Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Adv. in Math.*, 55(3):242–315, 1985.

[80] Richard B. Melrose and Michael E. Taylor. The radiation pattern of a diffracted wave near the shadow boundary. *Comm. Partial Differential Equations*, 11(6):599–672, 1986.

[81] Richard B. Melrose and Michael E. Taylor. Boundary problems for the wave equations with grazing and gliding rays, 1987.

[82] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.

[83] Francis Nier. A variational formulation of Schrödinger-Poisson systems in dimension $d \leq 3$. *Comm. Partial Differential Equations*, 18(7-8):1125–1147, 1993.

[84] F. W. J. Olver. *Asymptotics and special functions.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Computer Science and Applied Mathematics.

[85] El-Maati Ouhabaz. Gaussian estimates and holomorphy of semigroups. *Proc. Amer. Math. Soc.*, 123(5):1465–1474, 1995.

[86] Fabrice Planchon. Dispersive estimates and the 2D cubic NLS equation. *J. Anal. Math.*, 86:319–334, 2002.

[87] Fabrice Planchon and Luis Vega. Bilinear virial identities and applications. *Ann. Scient. Ec. Norm. Sup.*, 42:263–292, 2009. `arXiv:math/0712.4076`.

[88] Luc Robbiano and Claude Zuily. Microlocal analytic smoothing effect for the Schrödinger equation. *Duke Math. J.*, 100(1):93–129, 1999.

[89] Luc Robbiano and Claude Zuily. Effet régularisant microlocal analytique pour l'équation de Schrödinger: le cas des données oscillantes. *Comm. Partial Differential Equations*, 25(9-10):1891–1906, 2000.

[90] A. Seeger and C. D. Sogge. On the boundedness of functions of (pseudo-) differential operators on compact manifolds. *Duke Math. J.*, 59(3):709–736, 1989.

[91] Per Sjölin. Regularity of solutions to the Schrödinger equation. *Duke Math. J.*, 55(3):699–715, 1987.

[92] Johannes Sjöstrand. Singularités analytiques microlocales. In *Astérisque, 95*, volume 95 of *Astérisque*, pages 1–166. Soc. Math. France, Paris, 1982.

[93] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[94] Hart F. Smith and Christopher D. Sogge. $L^p$ regularity for the wave equation with strictly convex obstacles. *Duke Math. J.*, 73(1):97–153, 1994.

[95] Hart F. Smith and Christopher D. Sogge. On the critical semilinear wave equation outside convex obstacles. *J. Amer. Math. Soc.*, 8(4):879–916, 1995.

[96] Hart F. Smith and Christopher D. Sogge. On the $L^p$ norm of spectral clusters for compact manifolds with boundary. *Acta Math.*, 198(1):107–153, 2007.

[97] Christopher D. Sogge. *Fourier integrals in classical analysis*, volume 105 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1993.

[98] Gigliola Staffilani and Daniel Tataru. Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. *Comm. Partial Differential Equations*, 27(7-8):1337–1372, 2002.

[99] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[100] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[101] Catherine Sulem and Pierre-Louis Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[102] Siu-Hung Tang and Maciej Zworski. Resonance expansions of scattered waves. *Comm. Pure Appl. Math.*, 53(10):1305–1334, 2000.

[103] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with non-smooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.

[104] Michael E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.

[105] Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196(2):443–485, 2002.

[106] François Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*. Plenum Press, New York, 1980. Pseudodifferential operators, The University Series in Mathematics.

[107] Hans Triebel. *Interpolation theory, function spaces, differential operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[108] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[109] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[110] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1988.

[111] Maciej Zworski. High frequency scattering by a convex obstacle. *Duke Math. J.*, 61(2):545–634, 1990.