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DISPERSIVE ESTIMATES FOR THE WAVE AND THE KLEIN-GORDON EQUATIONS IN LARGE TIME INSIDE THE FRIEDLANDER DOMAIN

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Abstract. We prove global in time dispersion for the wave and the Klein-Gordon equation inside the Friedlander domain by taking full advantage of the space-time localization of caustics and a precise estimate of the number of waves that may cross at a given, large time. Moreover, we uncover a significant difference between Klein-Gordon and the wave equation in the low frequency, large time regime, where Klein-Gordon exhibits a worse decay than the wave, unlike in the flat space.

1. **Introduction and main results.** We consider the following equation on a domain Ω with smooth boundary,

$$\begin{cases} (\partial_t^2 - \Delta + m^2)u(t,x) = 0, \ \ x\in\Omega \\ u|_{t=0} = u_0 \ \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0\,. \end{cases} \tag{1}$$

Here Δ stands for the Laplace-Beltrami operator on Ω. If $\partial\Omega \neq \emptyset$, our boundary condition is the Dirichlet one. Here we will deal with $m \in \{0,1\}$: when $m = 0$ we deal with the wave equation, while when $m = 1$ we consider the Klein Gordon equation.

If $\Omega = \mathbb{R}^d$ with the flat metric, the solution $u^m_{\mathbb{R}^d}(t,x)$ to (1) with data $u_0 = \delta_{x_0}, u_1 = 0$, $x_0 \in \mathbb{R}^d$ has an explicit representation formula

$$u^m_{\mathbb{R}^d}(t,x) = \frac{1}{(2\pi)^d}\int e^{i(x-x_0)\xi}\cos(t\sqrt{|\xi|^2+m^2})\,d\xi\,.$$

Fixed time dispersive estimates may be explicitly obtained for any $t\in\mathbb{R}$, with χ being a smooth cut-off function localizing around 1, $h\in(0,1)$: for the wave flow

$$\|\chi(h\sqrt{-\Delta})u^{m=0}_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\left\{1,(h/|t|)^{\frac{d-1}{2}}\right\}, \tag{2}$$

and for the Klein-Gordon flow (see [7], [8] and the references therein)

$$\|\chi(h\sqrt{-\Delta+1})u^{m=1}_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\left\{1,(h/|t|)^{\frac{d-1}{2}},(h/t)^{\frac{d-1}{2}}\frac{1}{\sqrt{h|t|}}\right\}. \tag{3}$$

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Alternatively, one may state these estimates for $h \in (0,1)$ (high frequency regime) and state similar estimates with $h = 1$ and χ replaced by χ_0, localized around 0 (low frequency regime). On any boundaryless Riemannian manifold (Ω, g) one may follow the same path, replacing the exact formula by a parametrix (which may be constructed locally within a small ball, thanks to finite speed of propagation). However, these techniques usually restrict results to be local in time: depending on other (global) geometrical properties of the underlying manifold, such local in time estimates may be combined with local energy decay estimates to produce global in time Strichartz estimates, provided there is no trapping (or some weak form of it). On a manifold with boundary, picturing light rays becomes much more complicated, and one may no longer think that one is slightly bending flat trajectories. There may be gliding rays (along a convex boundary) or grazing rays (tangential to a convex obstacle) or combinations of both, and as such, even constructing local in time parametrices becomes a difficult task.

Obtaining results for the case of manifolds *with* boundary has been surprisingly elusive and the problem had received considerable interest in recent years. Besides harmonic analysis tools, the starting point for these estimates is the knowledge of a parametrix for the linear flow, which turns out to be closely connected to propagation of singularities. It should be noted that parametrices have been available for the boundary value problem for a long time (see [10], [11], [9], [12], [13], [2]) as a crucial tool to establish propagation of singularities for the wave equation on domains. However, while efficient at proving that singularities travel along the (generalized) bicharacteristic flow, they do not seem strong enough to obtain dispersion, as they are not precise enough to capture separation of wave packets traveling with different initial directions.

Inside a strictly convex domain, a parametrix for the wave equation has been constructed in [5] (and recently refined in [4]) ; it provides optimal decay estimates, with a $(|t|/h)^{1/4}$ loss compared to (2), uniformly with respect to the distance of the source to the boundary, over a time length of constant size. This involves dealing with an arbitrarily large number of caustics and retain control of their order. In [4], one considers (1) with $m = 0$ locally inside the Friedlander domain in dimension $d \geq 2$, defined as the half-space, $\Omega_d = \{(x, y)|x > 0, y \in \mathbb{R}^{d-1}\}$ with the metric g_F inherited from the Laplace operator $\Delta_F = \partial_x^2 + (1 + x)\Delta_y$. The domain (Ω_d, g_F) is easily seen to model, locally, a strict convex, as a first order approximation of the unit disk $D(0, 1)$ in polar coordinates : set $r = 1 - x/2$, $\theta = y$. The Friedlander domain is unbounded as we consider $y \in \mathbb{R}^{d-1}$: trajectories escape to spatial infinity, except for those that are shot vertically (no tangential component), and we may expect some sort of long-time estimates. Moreover, in the construction of [4], the size of the time interval on which the parametrix is constructed does not seem to play a crucial role, unlike in [5] where a restriction appears, related to how many wave reflections one may explicitly construct.

In the present work, we indeed obtain a global in time parametrix for waves (following the approach of [4] for large frequencies) and prove global in time optimal dispersive bounds for the linear flow. We then do the same for the Klein-Gordon equation (for $m = 1$), for which, at least in large frequency case, sharp dispersive bounds are equally obtained. The parametrix is obtained (for both $m \in \{0, 1\}$) as a sum of wave packets corresponding to the successive reflections on the boundary : as the number of such waves that interfere at a given moment is time depending, for t larger than a certain power of the frequency, the sum of their L^∞ norms yields an important loss. In this case, we use the spectral properties of Δ_F to express the parametrix in terms of its eigenfunctions. This representation of the solution turns out to be particularly useful also in the low frequency case: that situation had not been dealt with in our previous works on the wave equation and has its own difficulties, including some surprising effects in the Klein-Gordon case.

Theorem 1.1. *Let $\psi_1 \in C_0^\infty(\mathbb{R}_+^*)$. There exists $C > 0$ such that, uniformly in $a > 0$, $h \in (0,1)$, $t \in \mathbb{R}$, the solution $u^m(t,x,a,y)$ to (1) with $m \in \{0,1\}$, with Δ replaced by Δ_F, Ω by Ω_d and with data $(u_0,u_1) = (\delta_{(a,0)}, 0)$, $\delta_{(a,0)}$ being any Dirac mass at distance a from $\partial\Omega_d$, is such that*

$$|\psi_1(h\sqrt{-\Delta_F}) u^m(t,x,a,y)| \leq \frac{C}{h^d} \min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-2}{2}+\frac{1}{4}}\right\}. \tag{4}$$

Let $\phi \in C_0^\infty((-2,2))$ equal to 1 on $[0,\frac{3}{2}]$. There exists C_0, such that, uniformly in $a > 0$, $t \in \mathbb{R}^*$

$$|\phi(\sqrt{-\Delta_F}) u^{m=0}(t,x,a,y)| \leq C_0 \min\left\{1, \frac{1}{|t|^{\frac{d-1}{2}}}\right\}. \tag{5}$$

There exist a constant C_1, such that uniformly in $a > 0$, $t \in \mathbb{R}^*$

$$|\phi(\sqrt{-\Delta_F}) u^{m=1}(t,x,a,y)| \leq C_1 \min\left\{1, \frac{1}{|t|^{\frac{d-2}{2}+\frac{1}{3}}}\right\}. \tag{6}$$

Let us comment on these estimates : as far as dispersion is concerned, (4) is the extension to large times of the main dispersion estimate in [5], and we know it to be optimal (already for small times, due to the presence of swallowtail singularities in the wave front). Note that, unlike in $\mathbb{R}^d$ when the case $m = 1$ improves for $|t| > \frac{1}{h}$, here we obtain the same decay regardless of the presence of a mass. This relates to how we compensate for overlapping waves for large time : even for the wave equation, one has to switch to the gallery modes and use the spectral sum, taking advantage of the fact that gallery modes satisfy a suitably modified wave equation in $\mathbb{R}^{d-1}$, where $-\Delta_y$ would be replaced by $-\Delta_y + c|\Delta_y|^{2/3}$. In estimating time decay, the nonlocal operator plays the role of the mass (it provides more curvature for the characteristic set) and one gets the Klein-Gordon decay in $\mathbb{R}^{d-1}$. Hence, when adding a mass term, one gets $-\Delta_y + c|\Delta_y|^{2/3} + 1$ and one does not get additional decay.

For low frequencies, (5) tells us that the long time dispersion for the wave equation is the same as in $\mathbb{R}^d$. This actually holds on more generic non trapping exterior domains. However, (6) exhibits a loss compared to (5), and this new effect appears to be strongly connected to our domain. In fact, both this loss and the absence of gain for $|t| > 1/h$ in the large frequency range may be informally related to the same operator that was introduced above: $-\Delta_y + c|\Delta_y|^{2/3} + 1$. If one restricts to the 2D case, e.g. $y \in \mathbb{R}$, then one has to deal with the following oscillatory integral, where $c > 0$,

$$v(t,z) = \int e^{it(z\eta - \sqrt{|\eta|^2 + c|\eta|^{4/3} + m^2})} \psi_1(\eta)\, d\eta\,,$$

for which one may check that there is at least a degenerate critical point of order two if $m = 1$ while there is none if $m = 0$. In fact, the second derivative of the phase does not depend on z, it does have a zero η_0 and then one may chose z so that the first derivative vanishes as well for that η_0. Interestingly enough, this effect occurs around $\eta \sim 1$ (and with $c \sim \omega_1$, the first zero of $\mathrm{Ai}(-\cdot)$), e.g. for a moderately transverse direction, and not because of a very vertical direction. As such, one may see this effect arise in more realistic wave guide type domains and we believe it to be of interest.

Finally, one can immediately deduce a suitable set of (long time) Strichartz estimates in our setting, which are exactly the long time version of the ones stated in [5], for both wave and Klein-Gordon equations.

In the remaining of the paper, $A \lesssim B$ means that there exists a constant C such that $A \leq CB$ and this constant may change from line to line but is independent of all parameters. It will be explicit when (very occasionally) needed. Similarly, $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

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2. **The half-wave propagator: spectral analysis and parametrix construction.** We recall a few notations, where Ai denotes the standard Airy function (see e.g. [14] for well-known properties of the Airy function), $Ai(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\frac{\sigma^3}{3}+\sigma x)}\, d\sigma$. Define

$$A_\pm(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3} z) = -e^{\pm 2i\pi/3} Ai(e^{\pm 2i\pi/3}(-z))\,, \quad \text{for } z \in \mathbb{C}\,, \tag{7}$$

then one checks that $Ai(-z) = A_+(z) + A_-(z)$ (see [14, (2.3)]). The next Lemma is proved in [6, Lemma 1] and requires the classical notion of asymptotic expansion: a function $f(w)$ admits an asymptotic expansion for $w \to 0$ when there exists a (unique) sequence $(c_n)_n$ such that, for any n, $\lim_{w\to 0} w^{-(n+1)}(f(w) - \sum_0^n c_n w^n) = c_{n+1}$. We will denote $f(w) \sim_w \sum_n c_n w^n$.

Lemma 1. *Define* $L(\omega) = \pi + i\log\frac{A_-(\omega)}{A_+(\omega)}\,,$ *for* $\omega \in \mathbb{R}\,,$ *then L is real analytic and strictly increasing. We also have*

$$L(0) = \pi/3\,,\ \lim_{\omega\to-\infty} L(\omega) = 0\,,\ L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} + \frac{\pi}{2} - B(\omega^{\frac{3}{2}})\,, \quad \textit{for } \omega > 1\,, \tag{8}$$

with $B(u) \sim_{1/u} \sum_{k=1}^\infty b_k u^{-k}$, $b_k \in \mathbb{R}$ *and* $b_1 > 0$ *. Moreover,* $L'(\omega_k) = 2\pi\int_0^\infty Ai^2(x-\omega_k)\, dx$ *, where here and thereafter,* $\{-\omega_k\}_{k\geq 1}$ *denote the zeros of the Airy function in decreasing order.*

We briefly recall a Poisson summation formula that will be crucial to construct a parametrix :

Lemma 2. *(see [6]) In* $\mathcal{D}'(\mathbb{R}_\omega)$*, one has*

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi\sum_{k\in\mathbb{N}^*}\frac{1}{L'(\omega_k)}\delta(\omega-\omega_k)\,.$$

Hence, for $\phi(\omega) \in C_0^\infty$*,*

$$\sum_{N\in\mathbb{Z}}\int e^{-iNL(\omega)}\phi(\omega)\, d\omega = 2\pi\sum_{k\in\mathbb{N}^*}\frac{1}{L'(\omega_k)}\phi(\omega_k)\,. \tag{9}$$

2.1. **Spectral analysis of the Friedlander model.** Let Ω_d be the half-space $\{(x,y)\in\mathbb{R}^d|, x>0, y\in\mathbb{R}^{d-1}\}$ and consider the operator $\Delta_F=\partial_x^2+(1+x)\Delta_y$ on Ω_d with Dirichlet boundary condition. After a Fourier transform in the y variable, the operator $-\Delta_F$ becomes $-\partial_x^2+(1+x)|\theta|^2$. For any $|\theta|\neq 0$, this is a positive self-adjoint operator on $L^2(\mathbb{R}_+)$, with compact resolvent.

Lemma 3. *([6, Lemma 2]) There exist orthonormal eigenfunctions $\{e_k(x,\theta)\}_{k\geq 0}$ with their corresponding eigenvalues $\lambda_k(\theta)=|\theta|^2+\omega_k|\theta|^{4/3}$, which form a Hilbert basis of $L^2(\mathbb{R}_+)$. These eigenfunctions have an explicit form*

$$e_k(x,\theta)=\frac{\sqrt{2\pi}|\theta|^{1/3}}{\sqrt{L'(\omega_k)}}Ai\Big(|\theta|^{2/3}x-\omega_k\Big), \tag{10}$$

where $L'(\omega_k)$ is given in Lemma 1, which yields $\|e_k(.,\theta)\|_{L^2(\mathbb{R}_+)}=1$.

In a classical way, for $x_0>0$, the Dirac distribution $\delta_{x=x_0}$ on $\mathbb{R}_+$ may be decomposed in terms of eigenfunctions $\{e_k\}_{k\geq 1}$ as follows

$$\delta_{x=x_0}=\sum_{k\geq 1}e_k(x,\theta)e_k(x_0,\theta). \tag{11}$$

This allows to obtain (at least formally) the Green function associated to the half-wave propagator for (1) in $(0,\infty)\times\Omega_d$:

$$G^{m,\pm}(t,x,y,t_0,x_0,y_0)=\sum_{k\geq 1}\int_{\mathbb{R}^{d-1}}e^{\pm i(t-t_0)\sqrt{m^2+\lambda_k(\theta)}}e^{i<(y-y_0),\theta>}e_k(x,\theta)e_k(x_0,\theta)d\theta\,. \tag{12}$$

In the following we fix + sign and write G^0 for the wave and G^1 for Klein-Gordon Green function. By symmetry of G^m, we may assume $x\leq x_0$.

We will deal separately with the following situations:

- The high frequency case "$\sqrt{-\Delta_F}\sim 2^j$" for $j\in\mathbb{N}$, $j\geq 1$; this corresponds to $\sqrt{\lambda_k(\theta)}\sim 2^j$. The main situation is the "tangent" case $|\theta|\sim 2^j$, which corresponds to tangent directions, when the number of reflections on the boundary is at its highest. Indeed, with $\tau=\frac{h}{i}\partial_t=hD_t$, $\xi=\frac{h}{i}\partial_x=hD_x$, $\theta=\frac{h}{i}\nabla_y=hD_y$, the characteristic set of $\partial_t^2-\Delta_F$ is $\tau^2=\xi^2+|\theta|^2+x|\theta|^2$. Using $\tau^2=\lambda_k(hD_y)$, one obtains (at the symbolic level) that on the micro-support of any eigenfunction e_k associated to λ_k (and hence to ω_k) we have
$$\lambda_k(\theta)=|\theta|^2+\omega_k|\theta|^{4/3},\quad \xi^2+x|\theta|^2=\omega_k|\theta|^{4/3}. \tag{13}$$
When $\lambda_k(\theta)\sim 2^{2j}$ and $|\theta|^2\sim 2^{2j}$ then $(\xi/|\theta|)^2+x=\omega_k|\theta|^{-2/3}$ may be small and we deal with a large number of reflecting rays on the boundary and their limits, the gliding rays (the case when $(\xi/|\theta|)^2$ is bounded from below by a fixed constant corresponds to transverse rays, which may also reflect many times on the boundary but provide better estimates). This is the (only) case dealt with in [5], [4] for the *local in time* wave equation in the Friedlander domain. Here we start by recalling the main steps to the parametrix construction from [4] and its main properties, and extend it to all t; we further adapt this construction to the Klein-Gordon case and proceed with fixed time decay bounds (first for this so called "tangential" part corresponding to directions with small initial angles $|\xi|/|\theta|$).

  Let $\psi_1\in C_0^\infty([\frac{3}{4},\frac{5}{4}])$ be a smooth function valued in $[0,1]$ and equal to 1 near 1. As $-\Delta_F(e^{i<y,\theta>}e_k(x,\theta))=\lambda_k(\theta)e^{i<y,\theta>}e_k(x,\theta)$, we introduce the spectral cut-off $\psi_1(h\sqrt{-\Delta_F})$ in G^m, where $h\in(0,1/2]$ is a small parameter.

Let $\psi \in C_0^\infty([\frac{3}{8},\frac{5}{4}])$ such that $\psi_1(h|\theta|)+\sum_{j\geq 1}\psi(2^j h|\theta|)=1$. We define the "tangential" part of the Green function G_h^m with $(x_0,y_0)=(a,0)$ as follows

$$G_h^{\#,m}(t,x,a,y) := \sum_{k\geq 1}\int_{\mathbb{R}^{d-1}} e^{it\sqrt{m^2+\lambda_k(\theta)}}e^{i<y,\theta>}\psi_1(h|\theta|) \times\psi_1(h\sqrt{\lambda_k(\theta)})e_k(x,\theta)e_k(a,\theta)d\theta\,. \tag{14}$$

As remarked in [4], the significant part of the sum over k in (14) becomes then a finite sum over $k \lesssim 1/h$, considering the asymptotic expansion of $\omega_k \sim k^{2/3}$ (and corresponds to initial angles $(\xi/|\theta|)^2 \lesssim (\omega_k|\theta|^{-2/3}) \sim \omega_k h^{2/3} \lesssim 1$, where the last inequality is due to (13) and the spectral cut-offs). Reducing the sum to $k \leq 2/h$ is equivalent to adding a spectral cut-off $\phi_2(x+h^2D_x^2/|\theta|^2)$ in $G_h^{\#,m}$, where $\phi_\gamma(\cdot)=\phi(\cdot/\gamma)$ for some smooth cut-off $\phi \in C_0^\infty((-2,2))$ equal to 1 on $[-\frac{3}{2},\frac{3}{2}]$: as the operator $-\partial_x^2+x|\theta|^2$ has the same eigenfunctions $e_k(x,\theta)$ associated to the eigenvalues $\lambda_k(\theta)-|\theta|^2=\omega_k|\theta|^{4/3}$, then $(x+h^2D_x^2/|\theta|^2)e_k(x,\theta)=(\omega_k|\theta|^{4/3}/|\theta|^2)e_k(x,\theta)$ and this new localization operator is exactly associated by symbolic calculus to the cut-off $\phi_2(\omega_k/|\theta|^{2/3})=\phi(\frac{1}{2}\omega_k/|\theta|^{2/3})$, that we can introduce in (14) without changing the contribution of the integral due to support considerations as $\psi_1(h|\theta|)\psi_1(h\sqrt{\lambda_k(\theta)})(1-\phi(\frac{1}{2}\omega_k/|\theta|^{2/3}))=0$. Indeed, on the support of $\psi_1(h|\theta|)\psi_1(h\sqrt{\lambda_k(\theta)})$ we have $|h\theta| \in [\frac{3}{4},\frac{5}{4}]$ and

$$\sqrt{|h\theta|^2+h^{2/3}\omega_k|h\theta|^{4/3}} = |h\theta|\sqrt{1+h^{2/3}\omega_k/|h\theta|^{2/3}} \in \Big[\frac{3}{4},\frac{5}{4}\Big],$$

while on the support of $(1-\phi(\frac{1}{2}\omega_k/|\theta|^{2/3}))$ we have $\frac{1}{2}h^{2/3}\omega_k/|h\theta|^{2/3} > \frac{3}{2}$: hence, if $|\eta|=|h\theta| \in [\frac{3}{4},\frac{5}{4}]$ belongs to the support of $\psi_1(\eta)$, then

$$|\eta|\sqrt{1+h^{2/3}\omega_k/|\eta|^{2/3}} \geq 2|\eta| \geq \frac{3}{2} > \frac{5}{4}$$

on the support of $1-\phi(\frac{1}{2}h^{2/3}\omega_k/|\eta|^{2/3})$, and therefore $\psi_1(h\sqrt{\lambda_k(\eta/h)})=0$.

It will be convenient to introduce a new, small parameter γ satisfying $\sup(a,h^{2/3}) \lesssim \gamma \lesssim 1$ and then split the (tangential part $G_h^{\#,m}$ of the) Green function into a dyadic sum $G_{h,\gamma}^m$ corresponding to a dyadic partition of unity supported for $\omega_k/|\theta|^{2/3} \sim \gamma \sim 2^j \sup(a,h^{2/3}) \lesssim 1$. Let $\psi_2(\rho) := \phi_2(\rho)-\phi_2(2\rho)$, then $\psi_2 \in C_0^\infty([\frac{3}{2},4])$ is equal to 1 on $[2,3]$ and decompose $\phi_2(\cdot)=\phi(\cdot/2)$ as follows

$$\phi_2(\cdot) = \phi_2\Big(\frac{\cdot}{\sup(a,h^{2/3})}\Big) + \sum_{\gamma=2^j\sup(a,h^{2/3}),1\leq j\leq \log_2(1/\sup(a,h^{2/3}))} \psi_2\Big(\frac{\cdot}{\gamma}\Big), \tag{15}$$

which allows to write $G_h^{\#,m}=\sum_{\sup(a,h^{2/3})\leq\gamma\lesssim 1} G_{h,\gamma}^m$ where (rescaling the θ variable for later convenience) $G_{h,\gamma}^m$ takes the form :

$$G_{h,\sup(a,h^{2/3})}^m(t,x,a,y) = \sum_{k\geq 1}\frac{1}{h^{d-1}}\int_{\mathbb{R}^{d-1}} e^{it\sqrt{m^2+\lambda_k(\eta/h)}}e^{\frac{i}{h}<y,\eta>}\psi_1(|\eta|) \times\psi_1(h\sqrt{\lambda_k(\eta/h)})\phi_2\Big(\frac{h^{2/3}\omega_k}{|\eta|^{2/3}\gamma}\Big)e_k(x,\eta/h)e_k(a,\eta/h)d\eta\,, \tag{16}$$

and for $\gamma = 2^j \sup(a, h^{2/3})$, $1 \leq j \leq \log_2(1/\sup(a, h^{2/3}))$,

$$G^m_{h,\gamma}(t,x,a,y) = \sum_{k\geq 1} \frac{1}{h^{d-1}} \int_{\mathbb{R}^{d-1}} e^{it\sqrt{m^2+\lambda_k(\eta/h)}} e^{\frac{i}{h}<y,\eta>} \psi_1(|\eta|) \times \psi_1(h\sqrt{\lambda_k(\eta/h)}) \psi_2\Big(\frac{h^{2/3}\omega_k}{|\eta|^{2/3}\gamma}\Big) e_k(x,\eta/h) e_k(a,\eta/h) d\eta\,. \quad (17)$$

Remark 1. Notice that, when $\gamma \sim \sup(a, h^{2/3})$ (for $j = 0$), the support of the cut-off $\phi_2\Big(h^{2/3}\omega_k/(|\eta|^{2/3}\sup(a, h^{2/3}))\Big)$ is restricted to values $h^{2/3}\omega_k \leq 4|\eta|^{2/3}\sup(a, h^{2/3})$ with $|\eta| \in [\frac{3}{4}, \frac{5}{4}]$ on the support of $\psi_1(\eta)$. However, for values $h^{2/3}\omega_k \leq \frac{1}{2}|\eta|^{2/3}\sup(a, h^{2/3})$, the corresponding Airy factors $e_k(a, \eta/h)$ are exponentially decreasing (see [14, Section 2.1.4.3]) and yield an irrelevant contribution: therefore writing $\phi_2(\frac{\cdot}{\sup(a,h^{2/3})})$ or $\phi_2(\frac{\cdot}{\sup(a,h^{2/3})}) - \phi_{\frac{1}{8}}(\frac{\cdot}{\sup(a,h^{2/3})})$ yields the same contribution in $G^m_{h,\sup(a,h^{2/3})}$ modulo $O(h^\infty)$. This remark will be useful when dealing with values $t \lesssim \frac{1}{h^2}$ when $O(h^\infty) = O((h/t)^\infty)$.

- When "$\sqrt{-\Delta_F} \sim 2^j$" and $0 < |\theta| \leq 2^{j-1}$ then $\omega_k|\theta|^{4/3} \sim 2^{2j}$. According to (13), this case corresponds to directions satisfying $x + (\xi/|\theta|)^2 \sim \omega_k/|\theta|^{2/3} \sim 2^{2j}/|\theta|^2 \geq 4$ and hadn't been dealt with in the previous works [5], [4] (as, in bounded time, the situation $x + \xi^2/|\theta|^2 \gtrsim 1$ corresponds to a bounded number of reflections and follows from [1]). For $\psi \in C_0^\infty([\frac{3}{8}, \frac{5}{4}])$ such that $\psi_1(h|\theta|) + \sum_{j\geq 1} \psi(2^j h|\theta|) = 1$, we define the corresponding part of the Green function G^m as follows $G^{\flat,m}_h := \sum_{j\geq 1} G^m_{h,2^j h}$, where, for $\tilde{h} = 2^j h$, we have

$$G^m_{(h,\tilde{h})}(t,x,a,y) := \sum_{k\geq 1} \int_{\mathbb{R}^{d-1}} e^{it\sqrt{m^2+\lambda_k(\theta)}} e^{i<y,\theta>} \psi(\tilde{h}|\theta|) \psi_1(h\sqrt{\lambda_k(\theta)}) e_k(x,\theta) e_k(a,\theta) d\theta\,. \quad (18)$$

  Notice that in (18) the sum over k is finite as on the support of ψ_1 we have $\omega_k \lesssim (\tilde{h}^2/h^3)^{2/3}$. In this case, at least as long as the time is not too large (at least for $t/h \lesssim (\tilde{h}^2/h^3)^3$), we may consider large initial values $1 \lesssim a \leq 4(\frac{\tilde{h}}{h})^2$ (as, for $a \geq 4(\frac{\tilde{h}}{h})^2$ the factor $e_k(a,\theta)$ yields $G^m_{(h,\tilde{h})}(t,x,a,y) = O((\tilde{h}^2/h^3)^{-\infty}) = O((h/t)^\infty)$. After several suitable changes of variable, obtaining dispersive bounds for $G^m_{(h,\tilde{h})}$ will reduce to uniform bounds for $G^m_{h^3/\tilde{h}^2,\gamma=1}$ (as in (17) with $h^3/\tilde{h}^2$ instead of h and $\gamma = 1$). In some suitable sense, we reduce this case to the previous one by rescaling, which is consistent with the geometric picture of light rays: transverse rays for long time look the same (after dezooming) as tangential rays on a short time interval.
- In the case of small frequencies "$|\sqrt{-\Delta_F}| \leq 2$ ", "$|\sqrt{-\Delta_y}| \leq 2$" we let $\phi \in C_0^\infty((-2,2))$ such that $\phi = 1$ on $[-\frac{3}{2}, \frac{3}{2}]$ and introduce the spectral cut-off $\phi(\sqrt{-\Delta_F})$ in G^m. We have to consider all possible situations $|\theta| \sim 2^{-j}$ for $j \in \mathbb{N}$. As $(1-\phi(|\theta|))$ is supported for $|\theta| \geq 3/2$, then $(1-\phi(|\theta|))\phi(\sqrt{\lambda_k(\theta)}) = 0$, and we can add $\phi(|\theta|)$ into the symbol of $\phi(\sqrt{-\Delta_F})G^m$. Let $\psi_2 \in C_0^\infty([\frac{3}{4}, 2])$ such that $\psi_2(\rho) := \phi(\rho) - \phi(2\rho)$ then $\sum_{j\in\mathbb{N}} \psi_2(2^j|\theta|) = \phi(|\theta|)$ and on the support of $\psi_2(2^j|\theta|)$ we have $|\theta| \sim 2^{-j}$ which allows to use again Lemma 3

and (12). This situation hasn't been encountered in our previous works. Let

$$G^m_j(t,x,a,y) := \sum_{k\geq 1} \int e^{i\left(<y,\theta>+t\sqrt{m^2+\lambda_k(\theta)}\right)} \psi_2(2^j|\theta|)\phi(\sqrt{\lambda_k(\theta)})e_k(x,\theta)e_k(a,\theta)d\theta, \tag{19}$$

then $\sum_j G^m_j$ is the solution to (1) with data $(\phi(\sqrt{-\Delta_F})\delta_{(a,0)},0)$, $a>0$. In this case we will see new that effects arise in the case of the Klein-Gordon equation, as mentioned in the introduction.

The spectral sum $G^{\#,m}_h$ in terms of reflected waves. Using the Airy-Poisson formula (9), we obtain a parametrix, both as a "spectral" sum and its counterpart after Poisson summation. Let $G^m_{h,\gamma}$ as in (17), then, using (9), its Fourier transform in y, that we denote $\hat{G}^m_{h,\gamma}$, equals

$$\begin{aligned}\hat{G}^m_{h,\gamma}(t,x,a,\eta/h) = \frac{1}{2\pi}\sum_{N\in\mathbb{Z}}\int_{\mathbb{R}} & e^{-iNL(\omega)}e^{i\frac{t}{h}|\eta|\sqrt{1+\omega(h/|\eta|)^{2/3}+m^2(h/|\eta|)^2}}\chi_1(\omega)\\ &\times \psi_1(|\eta|)\psi_1(|\eta|\sqrt{1+(h/|\eta|)^{2/3}\omega})\psi_2((h/|\eta|)^{2/3}\omega/\gamma)\\ &\times \frac{|\eta|^{2/3}}{h^{2/3}}Ai(x|\eta|^{2/3}/h^{2/3}-\omega)Ai(a|\eta|^{2/3}/h^{2/3}-\omega)d\omega\,.\end{aligned} \tag{20}$$

Here, $\chi_1(\omega)=1$ for $\omega>2$ and $\chi_1(\omega)=0$ for $\omega<1$, and obviously $\chi_1(\omega_k)=1$ for all k, as $\omega_1>2$. At this point, as $\eta\in[\frac12,\frac32]$, we may drop also the $\psi_1(|\eta|\sqrt{1+(h/|\eta|)^{2/3}\omega})$ localization by support considerations (slightly changing any cut-off support if necessary). Recall that

$$Ai(x|\eta|^{2/3}/h^{2/3}-\omega) = \frac{(|\eta|/h)^{1/3}}{2\pi}\int e^{\frac{i}{h}|\eta|(\frac{\sigma^3}{3}+\sigma(x-(h/|\eta|)^{2/3}\omega))}\,d\sigma\,. \tag{21}$$

Rescaling $\alpha=(h/|\eta|)^{2/3}\omega$ yields

$$\begin{aligned}G^m_{h,\gamma}(t,x,a,y) := \frac{1}{(2\pi)^3h^{d+1}}\sum_{N\in\mathbb{Z}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} & e^{\frac{i}{h}\Phi^m_{N,h,a}(t,x,y,\sigma,s,\alpha,\eta)}\\ &\times|\eta|^2\psi_1(|\eta|)\psi_2(\alpha/\gamma)\,dsd\sigma d\alpha d\eta\,,\end{aligned} \tag{22}$$

where we have set $\Phi^m_{N,a,h}=\Phi^m_{N,a,h}(t,x,y,\sigma,s,\alpha,\eta)$ with

$$\begin{aligned}\Phi^m_{N,h,a} = <y,\eta>+|\eta|\Big(&t\sqrt{1+\alpha+m^2(h/|\eta|)^2}\\ &+\frac{\sigma^3}{3}+\sigma(x-\alpha)+\frac{s^3}{3}+s(a-\alpha)-N\frac{h}{|\eta|}L(|\eta|^{2/3}\alpha/h^{2/3})\Big)\,.\end{aligned} \tag{23}$$

Remark 2. The critical points with respect to s,σ of (23) satisfy $\sigma^2=\alpha-x$, $s^2=\alpha-a$ and on the support of ψ_2 we have $\alpha\sim\gamma$. Making the change of coordinates $\alpha=\gamma A$, $s=\sqrt{\gamma}S$, $\sigma=\sqrt{\gamma}\Upsilon$ (see Section 3.2 for $\gamma\sim a$ and Section 3.3 for $\gamma>8a$) transforms $G^m_{h,\gamma}$ into an integral with parameter $\lambda_\gamma:=\gamma^{3/2}/h$: in order to apply stationary phase arguments, this parameter needs to be larger than a power $h^{-\varepsilon}$ for some $\varepsilon>0$ and therefore the parametrix (22) is useful only when $\gamma\gtrsim h^{2(1-\varepsilon)/3}$. If $a>h^{2(1-\varepsilon)/3}$ for some $\varepsilon>0$ this will always be the case, but when $a\leq h^{2(1-\varepsilon)/3}$ and $\max\{a,h^{2/3}\}\lesssim\gamma\leq h^{2(1-\varepsilon)/3}$ we cannot use (22) anymore.

Before starting the proof of Theorem 1.1 in the high frequency case, we show that this reduces to the two dimensional case. We first need to establish a propagation of singularities type result for G_h^m, that will be necessary in order to obtain dispersion estimates in the $d-2$ tangential variables.

Lemma 4. *Let $G_h^m(t,x,a,y)=\sum_{\max\{h^{2/3},a\}\leq\gamma<1}G_{h,\gamma}^m(t,x,a,y)$. There exists c_0 such that*

$$\sup_{x,y,t\in\mathcal{B}}|G_h^m(t,x,a,y)|\leq Ch^{-d}O((h/|t|)^\infty)\,,\quad\forall|t|>h,\tag{24}$$

where $\mathcal{B}=\{0\leq x\lesssim a,|y|\leq c_0t,0<h\leq|t|\}$.

Proof. A proof of this Lemma has been given in [3, Lemma 3.2] in the case of the wave equation on a generic strictly convex domain, based on propagation of singularities type results. In [3], the time is restricted to a bounded interval $h\leq|t|\leq T_0$ for some small T_0, which is the time interval considered in that paper. Same arguments apply in the case of the wave of Klein-Gordon equation in the Friedlander model domain, where the time can be taken large. □

Using Lemma 4 and the fact that our model domain is isotropic, we can integrate in the $d-2$ tangential variables $\eta/|\eta|$ and reduce the analysis to the two-dimensional case (by rotational invariance). As such, in the rest of the paper, as long as we deal with the high frequency case, we consider $d=2$. In the next two sections we consider only high frequencies.

3. **The parametrix regime in 2D. The high frequency case.**

3.1. **Localizing waves for $h^{2/3(1-\epsilon)}\lesssim a\lesssim\gamma\lesssim1$ for some small $\epsilon>0$, when $\sup(a,h^{2/3})=a$.** In [5], it has been shown that, as long as $a\gtrsim h^{4/7}=h^{2(1-1/7)/3}$, only a finite number of integrals in $G_{h,a}^{m=0}$ may overlap ; for such a and for $\gamma\gtrsim a$, it follows that at a fixed time t, the supremum of the sum in (22) is essentially bounded by the supremum of a finite number of waves that live at time t. Later on, in [4], it has been shown that, when $h^{2/3(1-\epsilon)}\leq a\lesssim\gamma\leq h^{4/7}$, the number of waves that cross each other at a given t in the sum (22) becomes unbounded even for small $t\lesssim1$. Moreover, the number of N with "significant contributions" had been (sharply) estimated which allowed to obtain refined bounds in this regime (better than in [5], where only the "spectral" version (17) of $G_{h,a}^{m=0}$ was then available for small a).

We claim that, in this regime, although an important contribution comes from $m=1$, it doesn't exceed the one already obtained for $m=0$: as a consequence, for both $m\in\{0,1\}$, we have to sum-up exactly the same number of terms in (22). Assume (without loss of generality) $t>0$. Let $m\in\{0,1\}$, $d=2$, $\eta\in\mathbb{R}$, then $G_{h,\gamma}^m=\sum_N V_{N,\gamma}^m$ where we have set

$$V_{N,\gamma}^m(t,x,a,y):=\frac{1}{(2\pi)^3h^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\frac{i}{h}\Phi_{N,a,h}^m(t,x,y,\sigma,s,\alpha,\eta)}\eta^2\psi(\eta)\psi_2(\alpha/\gamma)\,dsd\sigma d\alpha d\eta\,.\tag{25}$$

In the same way we define, for $\phi_2(\cdot)=\phi(\cdot/2)$ supported in $2\times[-2,2]$ as before,

$$V_{N,a}^m(t,x,a,y):=\frac{1}{(2\pi)^3h^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\frac{i}{h}\Phi_{N,a,h}^m(t,x,y,\sigma,s,\alpha,\eta)}\eta^2\psi(\eta)\phi_2(\alpha/a)\,dsd\sigma d\alpha d\eta\,.\tag{26}$$

The next lemma, which holds for all t, will be useful only for values $t\lesssim1/h^2$.

Lemma 5. *(see [4, Lemma 4]) Let $a \leq \gamma \lesssim 1$. At fixed $t > \sqrt{\gamma}$, the sum defining $G^m_{h,\gamma}$ is only significant for $N \lesssim t/\sqrt{\gamma}$ and $0 \leq x \leq 2\gamma$, e.g.*

$$\sum_{\{N \geq 16t/\sqrt{\gamma}\} \cup \{x > 2\gamma\}} V^m_{N,\gamma}(t, x, a, \cdot) = O(h^\infty).$$

Notice that, for $t \lesssim 1/h^2$, we have $O(h^\infty) = O((h/t)^\infty)$.

Remark 3. The proof for $m = 0$ had been given in [4] and is based on non-stationary phase arguments for phase functions whose large parameter is $\lambda_\gamma := \gamma^{3/2}/h$; for values of N such that $N > 16t/\sqrt{\gamma}$, the phase is non-stationary in at least one variable and provides enough decay to sum in N and to give an $O(\lambda_\gamma^{-\infty})$ contribution. As $\lambda_\gamma \geq \lambda_a = a^{3/2}/h \geq h^{-\epsilon}$, all these contributions are $O(h^\infty)$ (notice that the hypothesis $h^{2/3(1-\epsilon)} \lesssim a \leq \gamma$ from the beginning of this section is necessary in order to have $\lambda_\gamma > h^{-\epsilon}$). If, moreover, $t \lesssim 1/h^2$, then $O(h^\infty) = O((h/t)^\infty)$. The case $m = 1$ is dealt with in the same way (as m^2 comes with a very small factor h^2).

Remark 4. The proof of Lemma 5 only uses the variable $T := t/\sqrt{\gamma}$, and not the size of t which can be large. However, it turns out that when $|t|$ is much larger than $1/h^2$, as the number of terms in the sum over N becomes too important, the parametrix written under the form of reflected waves doesn't provide suitable estimates anymore. In this case we need to work with (16) and (17). Therefore Lemma 5 will be useful only for values $t \lesssim \frac{1}{h^2}$ and in this case we have $h^3 \lesssim h/t$ which yields $O(h^\infty) = O((h/t)^\infty)$.

Next, we estimate the number of overlapping waves when $m = 1$ and prove that there is no significant difference with respect to $m = 0$. In order to do that, we have to introduce some notations. For a given space-time location (t, x, y), let $\mathcal{N}^m_\gamma(t, x, y)$ be the set of N with significant contributions in (22) (e.g. for which there exists a stationary point for the phase in all variables), and let $\mathcal{N}^m_{1,\gamma}(t, x, y)$ be the set of N belonging to $\mathcal{N}^m_\gamma(t', x', y')$ for some (t', x', y') sufficiently close to (t, x, y) such that $|t' - t| \leq \sqrt{\gamma}$, $|x - x'| < \gamma$ and $|y' + t'\sqrt{1+\gamma} - y - t\sqrt{1+\gamma}| < \gamma^{3/2}$,

$$\mathcal{N}^m_\gamma(t, x, y) = \{N \in \mathbb{Z}, (\exists)(\sigma, s, \alpha, \eta) \text{ such that } \nabla_{(\sigma,s,\alpha,\eta)} \Phi^m_{N,a,h}(t, x, y, \sigma, s, \alpha, \eta) = 0\},$$

$$\mathcal{N}^m_{1,\gamma}(t, x, y) = \cup_{\{(t',x',y') \mid |t'-t| \leq \sqrt{\gamma}, |x-x'| < \gamma, |y'+t'\sqrt{1+\gamma}-y-t\sqrt{1+\gamma}| < \gamma^{3/2}\}} \mathcal{N}^m_\gamma(t', x', y').$$

The next result will be useful as long as $t \lesssim 1/h^2$.

Proposition 1. *Let $t > h$ and $1 \gtrsim \gamma \geq a \gtrsim h^{2(1-\epsilon)/3}$. The following estimates hold true:*

- *We control the cardinal of $\mathcal{N}^m_{1,\gamma}(t, x, y)$,*

$$\left|\mathcal{N}^m_{1,\gamma}(t, x, y)\right| \lesssim O(1) + \frac{t}{\gamma^{1/2}(\gamma^3/h^2)} + m^2 \frac{th^2}{\gamma^{3/2}}\,, \tag{27}$$

  and this bound is optimal.
- *The contribution of the sum over $N \notin \mathcal{N}^m_{1,\gamma}(t, x, y)$ in (22) is $O(h^\infty)$.*

Remark 5. In the present work we adapt the proof from [4] to all time. For $m = 0$, the sharp bounds (27) are obtained in [4, Prop.1] by estimating the distance between any two points $N_1, N_2 \in \mathcal{N}^{m=0}_{1,\gamma}(t, x, y)$ and the proof holds for any t. When $m = 1$, the difference comes from the term $m^2(h/\eta)^2$ in the coefficient of t which may contribute significantly only when $th \gg 1$, a case that hadn't be dealt with in our previous works. Following closely the approach in [4], we notice that the terms involving $m^2(h/\eta)^2$ give rise to the last addendum $m^2\frac{|t|h^2}{\gamma^{3/2}}$ in (27). However, as $|t|h^2/\gamma^{3/2} \leq th^2/\gamma^{7/2}$, for $\gamma \lesssim 1$ there is no real difference between the cases $m = 0$ and $m = 1$.

Remark 6. The proof of the second statement of Proposition 1 relies on non-stationary phase arguments for phase functions with large parameter $\lambda_\gamma = \gamma^{3/2}/h > h^{-\epsilon}$. When $N \notin \mathcal{N}^m_{1,\gamma}(t,x,y)$, we obtain an uniform lower bound for the absolute value of the gradient of all these phase functions (with respect to all variables of integrations suitably rescaled as in (28) below). Therefore there is always one variable for which the non-stationary phase applies and each $V^m_{N,\gamma}(t,x,a,y)$ with $N \notin \mathcal{N}^m_{1,\gamma}(t,x,y)$ provides a $O(\lambda_\gamma^{-M})$ contribution for any $M \geq 1$ and we conclude using that we only need to sum over $N \lesssim t/\sqrt{\gamma}$ by Lemma 5. As we will use Proposition 1 only for values of time satisfying $t \lesssim 1/h^2$, we have $O(h^\infty) = O((h/t)^\infty)$.

Proof of Proposition 1. Let $\gamma \in \{2^j a, j \geq 1\}$, in which case $V^m_{h,\gamma}$ given in (25) has cut-off $\psi_2(\alpha/\gamma)$ with ψ_2 supported in $[3/4,4]$ or $\gamma = a$, in which case $V^m_{h,a}$ given in (26) has cut-off $\phi_2(\alpha/a)$ supported in $[-4,4]$. As Proposition 1 will be applied only for values $t \lesssim 1/h^2$ when $O(h^\infty) = O((h/t)^\infty)$, we can use Remark 1 to replace $\phi_2(\cdot/a)$ by $\phi_2(\cdot/a) - \phi_{\frac{1}{8}}(\cdot/a)$ without changing the contribution in $V^m_{h,a}$ modulo $O((h/t)^\infty)$ terms. As ϕ_2 is supported on $[-4,4]$ and equal to 1 on $[-3,3]$, it follows that $\phi_2 - \phi_{\frac{1}{8}}$ is supported in $[3/16,4]$ therefore, we can assume that the symbol of $V^m_{h,a}$ is supported in $[3/16,4]$. We sketch the proof of (27) in the case $m = 1$. We rescale variables as follows :

$$x = \gamma X\,, \alpha = \gamma A\,, t = \sqrt{\gamma}T\,, s = \sqrt{\gamma}S\,, \sigma = \sqrt{\gamma}\Upsilon\,, y + t\sqrt{1+\gamma} = \gamma^{3/2}Y\,, \tag{28}$$

$$\Psi^m_{N,a,\gamma,h}(T,X,Y,\Upsilon,S,A,\eta) := \eta\left(Y + \Upsilon^3/3 + \Upsilon(X-A) + S^3/3 + S(\frac{a}{\gamma} - A)\right.$$
$$\left. + T\frac{\sqrt{1+\gamma A + m^2h^2/\eta^2} - \sqrt{1+\gamma}}{\gamma} - \frac{4}{3}NA^{3/2}\right) + \frac{N}{\lambda_\gamma}B(\eta\lambda_\gamma A^{3/2}). \tag{29}$$

then $\Phi^m_{N,a,h}(\sqrt{\gamma}T, \gamma X, \gamma^{3/2}Y - \sqrt{\gamma}\sqrt{1+\gamma}T, \sqrt{\gamma}\Upsilon, \sqrt{\gamma}S, \gamma A, \eta)$ becomes $\gamma^{3/2}\Psi^m_{N,a,\gamma,h}$ $(T,X,Y,\Upsilon,S,A,\eta)$ and, in the new variables, the phase function in (25) becomes $\lambda_\gamma\Psi^m_{N,a,\gamma,h}$ where $\lambda_\gamma = \gamma^{3/2}/h$. The relevance of the $\gamma^{3/2}$ factor in rescaling will make itself clear later. The critical points of $\Psi^m_{N,a,\gamma,h}$ with respect to Υ, S, A, η satisfy

$$\Upsilon^2 + X = A, \quad S^2 + a/\gamma = A, \tag{30}$$

$$T = 2\sqrt{1+\gamma A + m^2h^2/\eta^2}\Big(\Upsilon + S + 2N\sqrt{A}(1 - \frac{3}{4}B'(\eta\lambda_\gamma A^{3/2}))\Big), \tag{31}$$

$$Y + T\Big(\frac{\sqrt{1+\gamma A + m^2h^2/\eta^2} - \sqrt{1+\gamma}}{\gamma} - \frac{m^2h^2/\eta^2}{\gamma\sqrt{1+\gamma A + m^2h^2/\eta^2}}\Big) \tag{32}$$

$$+\Upsilon^3/3 + \Upsilon(X-A) + S^3/3 + S(\frac{a}{\gamma} - A) \tag{33}$$
$$= \frac{4}{3}NA^{3/2}(1 - \frac{3}{4}B'(\eta\lambda_\gamma A^{3/2})).$$

Introducing the term $2N\sqrt{A}(1-\frac{3}{4}B'(\eta\lambda_\gamma A^{3/2}))$ from (31) in the second line of (32) provides a relation between Y and T that doesn't involve N nor B' as follows:

$$Y+\frac{T}{\sqrt{1+\gamma A+m^2h^2/\eta^2}+\sqrt{1+\gamma}}\Big((A-1)-\frac{m^2h^2/\eta^2}{\gamma\sqrt{1+\gamma A+m^2h^2/\eta^2}}\Big)$$
$$+\Upsilon^3/3+\Upsilon(X-A)+S^3/3+S(\frac{a}{\gamma}-A)=\frac{2}{3}A\Big(\frac{T}{2\sqrt{1+\gamma A+m^2h^2/\eta^2}}-(\Upsilon+S)\Big). \tag{34}$$

Let $t>h$ and let $N_j\in\mathcal{N}^m_{1,\gamma}(t,x,y)$, with $j\in\{1,2\}$ be any two elements of $\mathcal{N}^m_{1,\gamma}(t,x,y)$. Then there exists (t_j,x_j,y_j) such $N_j\in\mathcal{N}^m_\gamma(t_j,x_j,y_j)$; writing $t_j=\sqrt{\gamma}T_j$, $x_j=\gamma X_j$, $y_j+t_j\sqrt{1+\gamma}=\gamma^{3/2}Y_j$ and rescaling (t,x,y) as in (28), the condition below holds true

$$|T_j-T|\le 1,\quad |X_j-X|\le 1,\quad |Y_j-Y|\le 1.$$

We prove that $|N_1-N_2|$ is bounded by $O(1)+m^2|T|h^2/\gamma+|T|/\lambda_\gamma^2$, which will achieve the first part of Proposition 1. Since $N_j\in\mathcal{N}^m(t_j,x_j,y_j)$, it means that there exists $\Upsilon_j,A_j,\eta_j,S_j$ such that (30), (31), (32) holds with T,X,Y,Υ,S,A,η replaced by $T_j,X_j,Y_j,\Upsilon_j,S_j,A_j,\eta_j$, respectively. We re-write (31) as follows

$$2N_j\sqrt{A_j}(1-\frac{3}{4}B'(\eta_j\lambda_\gamma A_j^{3/2}))=\frac{T_j}{2\sqrt{1+\gamma A_j+m^2h^2/\eta_j^2}}-(\Sigma_j+S_j). \tag{35}$$

Multiplying (35) by $\sqrt{A_{j'}}$, for $j,j'\in\{1,2\}$, $j'\neq j$, taking the difference and dividing by $\sqrt{A_1A_2}$ yields

$$2(N_1-N_2)=\frac{3}{2}\Big(N_1B'(\eta_1\lambda_\gamma A_1^{3/2})-N_2B'(\eta_2\lambda_\gamma A_2^{3/2})\Big)-\frac{\Sigma_1+S_1}{\sqrt{A_1}}+\frac{\Sigma_2+S_2}{\sqrt{A_2}}$$
$$+\frac{T_1}{2\sqrt{A_1}\sqrt{1+\gamma A_1+m^2h^2/\eta_1^2}}-\frac{T_2}{2\sqrt{A_2}\sqrt{1+\gamma A_2+m^2h^2/\eta_2^2}}. \tag{36}$$

Using that $\Sigma_j,S_j\lesssim A_j$, $A_j\sim 1$, it follows that $\frac{\Sigma_j+S_j}{\sqrt{A_j}}=O(1)$, for $j\in\{1,2\}$. The first term, involving B', in the right hand side of (36) behaves like $(N_1+N_2)/\lambda_\gamma^2$, which follows using $B'(\eta\lambda A^{3/2})\sim -\frac{b_1}{\eta^2\lambda^2A^3}$ and $\eta,A\sim 1$. Notice that we cannot take any advantage of the fact that we estimate a difference of two terms, since each $N_jB'(\eta_j\lambda_\gamma A_j^{3/2})$ corresponds to some η_j,A_j (close to 1) and the difference $\frac{1}{\eta_1A_1^{3/2}}-\frac{1}{\eta_2A_2^{3/2}}$ is bounded by a constant that has no reason to be small (the difference between A_j turns out to be $O(1/T)$, but we don't have any information about the difference between η_j which is simply bounded by a small constant on the support of ψ). Therefore the bound $(N_1+N_2)/\lambda_\gamma^2$ for the terms involving B' in (36) is sharp. Since $N_j\sim T_j$, and $|T_j-T|\le 1$, it follows that this contribution is $\sim T/\lambda_\gamma^2$. We are reduced to prove that the difference of the last two terms in the second line of (36) is $O(1)+m^2Th^2/\gamma$. Write

$$\frac{T_j}{2\sqrt{A_j}\sqrt{1+\gamma A_j+m^2h^2/\eta_j^2}}=\frac{T_j}{2\sqrt{A_j}\sqrt{1+\gamma A_j}}\Big(1-\frac{m^2h^2}{2\eta_j^2\sqrt{1+\gamma A_j}}+O(m^4h^4)\Big).$$

The difference for $j=1,2$ of the contributions involving η_j^{-2} cannot be estimated better than by m^2h^2T. We now proceed, as in [4] in the case $m=0$, with the difference of the main terms :

$$\begin{aligned}\Big|\frac{T_1}{\sqrt{A_1}\sqrt{1+\gamma A_1}}-\frac{T_2}{\sqrt{A_2}\sqrt{1+\gamma A_2}}\Big| &\leq \frac{T_2}{\sqrt{A_1A_2}}\Big|\frac{\sqrt{A_2}}{\sqrt{1+\gamma A_1}}-\frac{\sqrt{A_1}}{\sqrt{1+\gamma A_2}}\Big| \\ &\qquad +\frac{|T_1-T_2|}{\sqrt{A_1}\sqrt{1+\gamma A_1}} \\ &\leq \frac{T_2|A_2-A_1|(1+\gamma(A_1+A_2))}{\sqrt{A_1A_2(1+\gamma A_1)(1+\gamma A_2)}(\sqrt{A_1(1+\gamma A_1)}+\sqrt{A_2(1+\gamma A_2)})} \\ &\qquad +\frac{2}{\sqrt{A_1}\sqrt{1+\gamma A_1}} \leq C(1+T_2|A_2-A_1|),\end{aligned} \tag{37}$$

where $C>0$ is some constant depending only on the size of the support of ψ_2 (for $j\geq 1$) contained in $[3/4,4]$ (or on the size of the support of $\phi_2-\phi_{\frac{1}{8}}$ contained in $[3/16,4]$ when $\gamma=a$). The only thing that really matters here is the fact that all A_j stay outside a fixed neighborhood of 0.

We have only used $|T_2-T_1|\leq 2$ and $A_j\sim 1$. Notice that for T bounded we can conclude since $|T_2-T|\leq 1$. We are therefore reduced to bound $T_2|A_2-A_1|$ when T_2 is sufficiently large. For that, we need to take into account the Y variable. We use (34) with T,X,Y,Σ,S,A,η replaced by $T_j,X_j,Y_j,\Sigma_j,S_j,A_j,\eta_j$, $j\in\{1,2\}$ to eliminate the terms containing N and B' as follows:

$$\begin{aligned}Y_j+\Sigma_j^3/3+\Sigma_j(X_j-A_j)+S_j^3/3+S_j(\frac{a}{\gamma}-A_j)+\frac{2}{3}A_j(\Sigma_j+S_j)\\ =T_j\Big(\frac{A_j}{3\sqrt{1+\gamma A_j}}-\frac{(A_j-1)}{\sqrt{1+\gamma A_j}+\sqrt{1+\gamma}}+O(m^2h^2/\gamma)\Big).\end{aligned} \tag{38}$$

If T is sufficiently large then so is T_j, and we divide the last equation by T_j in order to estimate the difference A_1-A_2 in terms of $Y_1/T_1-Y_2/T_2$ as follows

$$\frac{Y_j}{T_j}+O(\frac{A_j^{3/2}}{T_j})=F_\gamma(A_j),\quad F_\gamma(A)=\frac{A}{3\sqrt{1+\gamma A}}-\frac{(A-1)}{\sqrt{1+\gamma A}+\sqrt{1+\gamma}}+O(m^2h^2/\gamma). \tag{39}$$

Taking the difference of (39) written for $j=1,2$ gives

$$(\frac{Y_2}{T_2}-\frac{Y_1}{T_1})+O(\frac{1}{T_1})+O(\frac{1}{T_2})+O(m^2h^2/\gamma)=(A_2-A_1)\int_0^1\partial_AF_\gamma(A_1+o(A_2-A_1))do.$$

Using that $\partial_AF_\gamma(1)=-\frac{(1+2\gamma)}{6(1+\gamma)^{3/2}}$ and that A_j are close to 1, T_j are large, it follows that we can express A_2-A_1 as a smooth function of $(\frac{Y_2}{T_2}-\frac{Y_1}{T_1})$ and $O(\frac{1}{T_1})+O(\frac{1}{T_2})$, with coefficients depending on A_j and γ. Write

$$A_2-A_1=H_\gamma\Big(\frac{Y_2}{T_2}-\frac{Y_1}{T_1},\frac{1}{T_1},\frac{1}{T_2}\Big)+O(m^2h^2/\gamma)=-6(\frac{Y_2}{T_2}-\frac{Y_1}{T_1})+O(\frac{1}{T_1},\frac{1}{T_2})+O(m^2h^2/\gamma),$$

and replacing the last expression in the last line of (37) yields

$$T_2|A_2-A_1|\lesssim T_2\Big|\frac{Y_2}{T_2}-\frac{Y_1}{T_1}\Big|+O(T_2/T_1)+O(1)+|T|O(m^2h^2/\gamma) \tag{40}$$

$$\lesssim |Y_2-Y_1|+Y_1|1-T_2/T_1|+O(T_2/T_1)+O(1)+|T|m^2h^2/\gamma \tag{41}$$

$$=O(1)+m^2|T|h^2/\gamma,$$

where we have used $|Y_1-Y_2|\leq 2$, $|T_1-T_2|\leq 2$ and that Y_1/T_1 is bounded (which can easily be seen from (39)). This ends the proof of (27). The second statement of Proposition 1 follows exactly like in [4, Proposition 1]. □

In the next three subsections we give an overview of dispersive estimates obtained for the wave equation (for $m = 0$, see [4] for small t) and we generalize them to large time and for the Klein-Gordon flow (when $m = 1$). Although the results in Sections 3.2 and 3.3 are available for all t, we will only use them for $t \lesssim 1/h^2$, in which case all the remainders satisfy $O(h^\infty) = O((h/t)^\infty)$. Section 3.4 is devoted to the case of larger values of $t \gtrsim 1/h^2$ when we work with spectral sums.

3.2. **Tangential waves for $\gamma = a \sim h^{2(1-\epsilon)/3}$ for some $\epsilon > 0$.** This corresponds to initial angles $|\xi|/|\theta| \lesssim \sqrt{a}$. We rescale variables as follows:

$$x = aX\,, \alpha = aA\,, t = \sqrt{a}\sqrt{1+a}T\,, s = \sqrt{a}S\,, \sigma = \sqrt{a}\Upsilon\,, y + t\sqrt{1+a} = a^{3/2}Y\,. \tag{42}$$

Define $\lambda = a^{3/2}/h > h^{-\epsilon}$ to be our large parameter, then $V^m_{N,a}$ from (26) becomes

$$V^m_{N,a}(t,x,a,y) = \frac{a^2}{(2\pi h)^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i\lambda\Psi^m_{N,a,a,h}}\eta^2\psi_1(\eta)(\phi_2 - \phi_{\frac{1}{8}})(A)\,dSd\Upsilon dAd\eta\,, \tag{43}$$

where $\Psi^m_{N,a,a,h}$ is given by (29) with γ replaced by a due to the change of variables. As the support of $\phi_2 - \phi_{\frac{1}{8}}$ is the compact set $[3/16, 4]$, we have $A \in [3/16, 8]$. Since the critical points satisfy $S^2 = A - 1$, $\Upsilon^2 = A - X$, it follows that we can restrict to $|S|, |\Upsilon| < 3$ and $A \in [9/10, 8]$ without changing the contribution of the integrals modulo $O(h^\infty)$ (since for $A \le 9/10$ the phase is non-stationary in S); we insert suitable cut-offs, $\chi_2(S)\chi_2(\Upsilon)$ supported in $[-3, 3]$ and $\psi_3(A)$ supported in $[9/10, 8]$ and obtain $V^m_{N,a}(t,x,a,y) = W^m_{N,a}(T,X,Y) + O(h^\infty)$, where we abuse notations with respect to γ, replaced by a :

$$W^m_{N,a}(T,X,Y) = \frac{a^2}{(2\pi h)^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i\lambda\Psi^m_{N,a,a,h}}\eta^2\psi_1(\eta)\chi_2(S)\chi_2(\Upsilon)\psi_3(A)\,dSd\Upsilon dAd\eta\,. \tag{44}$$

Proposition 2. *(see [4, Prop.2] for $m = 0$) Let $m \in \{0, 1\}$. Let $|N| \lesssim \lambda$ and let $W^m_{N,a}(T,X,Y)$ be defined in (44). Then the stationary phase theorem applies in A and yields, modulo $O(h^\infty)$ terms*

$$\begin{aligned} W^m_{N,a}(T,X,Y) = \frac{a^2}{h^3(N\lambda)^{\frac{1}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} & e^{i\lambda\Psi^m_{N,a,a,h}(T,X,\Upsilon,S,A_c,\eta)}\eta^2 \\ & \times \psi_1(\eta)\chi_3(S,\Upsilon,a,1/N,h,\eta)\,dSd\Upsilon d\eta\,, \end{aligned} \tag{45}$$

where χ_3 has compact support in (S,Υ) and harmless dependency on the parameters $a, h, 1/N, \eta$.

Remark 7. For $m = 0$, Proposition 2 had been proved in [4, Prop.2]. Let $m = 1$ and $|N| \lesssim \lambda$: we state that the factor $e^{\frac{i}{h}t\eta(\sqrt{1+aA+m^2h^2/\eta^2}-\sqrt{1+aA})}$ can be brought into the symbol, in which case the phase of $W^{m=1}_{N,a}(T,X,Y)$ can be taken to be $\Psi^{m=0}_{N,a,a,h}$. As $\sqrt{1+aA+m^2h^2/\eta^2} - \sqrt{1+aA} = O(h^2)$ and $\frac{t}{\sqrt{a}} \sim N \lesssim \lambda = a^{3/2}/h$ (using Lemma 5), we find $th \lesssim a^2$, which yields $\frac{t}{h} \times O(h^2) = th \lesssim 1$.

Proposition 3. *(see [4, Prop.3] for $m = 0$) Let $m \in \{0, 1\}$, $N \gg \lambda$ and $W^m_{N,a}(T,X,Y)$ be defined in (44), then the stationary phase applies in both A and η and yields*

$$W^m_{N,a}(T,X,Y) = \frac{a^2}{h^3 N}\int_{\mathbb{R}^2} e^{i\lambda\Psi^m_{N,a,a,h}(T,X,Y,\Upsilon,S,A_c,\eta_c)}\chi_3(S,\Upsilon,a,1/N,h)\,dSd\Upsilon + O(h^\infty)\,, \tag{46}$$

where χ_3 has compact support in (S,Υ) and harmless dependency on the parameters $a, h, 1/N$.

Proof of Proposition 3. The only new situation is the case $N \gtrsim \lambda$ when $m=1$, when we cannot eliminate the terms depending on m^2h^2/η^2 from the phase. In order to prove Proposition 3 for $m=1$, we notice that the main contribution of the determinant of the Hessian matrix (with respect to A and η) comes from $NB(\eta\lambda A^{3/2})$ and not from the terms involving m^2h^2/η^2. The second order derivatives of phase $\Psi^{m=1}_{N,a,h}$ of $W^{m=1}_{N,a}(T,X,Y)$, defined in (29) are given by

$$\partial^2_{\eta,\eta}(\Psi^{m=1}_{N,a,a,h}) = N\lambda A^3 B''(\eta\lambda A^{3/2}) + O(m^2Th^2/a) \sim \frac{N}{\lambda^2} + m^2 O(Th^2/a) \sim \frac{N}{\lambda^2},$$
$$\partial^2_{A,A}(\Psi^{m=1}_{N,a,a,h}) = -\eta\frac{N}{A^{1/2}},$$
$$\partial^2_{\eta,A}(\Psi^{m=1}_{N,a,a,h}) = \eta^{-1}\partial_A\Psi^{m=1}_{N,a,a,h} + \frac{3}{2}\eta\lambda N A^2 B''(\eta\lambda A^{3/2})) + O(m^2Th^2). \tag{47}$$

Since $B''(\eta\lambda A^{3/2}) = O(\lambda^{-3})$, then $\partial_{\eta,A}\Psi^{m=1}_{N,a,a,h} \sim N/\lambda^2$. At the critical points, the Hessian of $\Psi^{m=1}_{N,a,a,h}$ satisfies

$$\det \operatorname{Hess} \Psi^{m=1}_{N,a,a,h}|_{\partial_A\Psi^{m=1}_{N,a,a,h}=\partial_\eta\Psi^{m=1}_{N,a,a,h}=0} \sim \frac{N^2}{\lambda^2}, \quad N \gtrsim \lambda.$$

□

We are left with the integration over (S,Υ). In [4, Prop.4,5,6 & Corollary 1], sharp bounds for $W^{m=0}_{N,a}$ had been obtained: we recall them as they will be crucial in order to prove Theorem 1.1 for all t. As noticed in Remark 7, when $N \lesssim \lambda$ the factor $e^{\frac{i}{h}a^{3/2}(\Psi^{m=1}_{N,a,a,h}-\Psi^{m=0}_{N,a,a,h})}$ may be brought into the symbol, hence the critical value of the phase $\Psi^m_{N,a,a,h}$ in Proposition 2 for $m=1$ is the same as the one for $m=0$. Therefore, the following results hold as in [4, Prop.4,5,6,& 7]:

Proposition 4. *Let $m \in \{0,1\}$. For $T \le \frac{5}{2}$ we have*

$$\sum_N |W^m_{N,a}(T,X,Y)| = |W^m_{0,a}(T,X,Y)| + \sum_{\pm 1}|W^m_{\pm 1,a}(T,X,Y)| + O(h^\infty)$$
$$\lesssim \frac{1}{h^2}\inf\Big(1, \Big(\frac{h}{t}\Big)^{1/2}\Big). \tag{48}$$

Proposition 5. *Let $m \in \{0,1\}$. For $1 \le N < \lambda^{1/3}$ and $|T-4N| \lesssim 1/N$, we have*

$$\big|W^m_{N,a}(T,X,Y)\big| \lesssim \frac{1}{h^2}\frac{h^{1/3}}{((N/\lambda^{1/3})^{1/4} + |N(T-4N)|^{1/6})}\,. \tag{49}$$

Proof of Proposition 5. (49) is obtained from the following bounds obtained for $m \in \{0,1\}$ as in [4, Prop.6]

$$\Big|\int e^{i\lambda\Psi^m_{N,a,a,h}(T,X,\Upsilon,S,A_c,\eta)}\eta^2\psi_1(\eta)\chi_3(S,\Upsilon,a,1/N,h,\eta)\,dSd\Upsilon d\eta\Big|$$
$$\lesssim \frac{1}{N^2}\frac{(\lambda/N^3)^{-5/6}}{(\lambda/N^3)^{-1/12} + (N^2|\frac{T}{4N}-1|)^{1/6}}. \tag{50}$$

□

Proposition 6. *Let $m \in \{0,1\}$. For $1 \le N < \lambda^{1/3}$, $|T-4N| \gtrsim 1/N$, $X \le 1$, we have*

$$\big|W^m_{N,a}(T,X,Y)\big| \lesssim \frac{1}{h^2}\frac{h^{1/3}}{(1+|N(T-4N)|^{1/2})}\,. \tag{51}$$

Proof of Proposition 6. (51) is obtained from the following bounds obtained for $m \in \{0,1\}$ as in [4, Prop.6]

$$\Big|\int e^{i\lambda\Psi^m_{N,a,a,h}(T,X,\Upsilon,S,A_c,\eta)}\eta^2\psi_1(\eta)\chi_3(S,\Upsilon,a,1/N,h,\eta)\,dSd\Upsilon d\eta\Big| \lesssim \frac{1}{N^2}\Big(\frac{\lambda}{N^3}\Big)^{-5/6}\frac{1}{1+(N^2|\frac{T}{4N}-1|)^{1/2}}. \tag{52}$$

□

Proposition 7. *Let $m \in \{0,1\}$. Let $\lambda^{1/3} \lesssim N$, $X \le 1$, then the following estimates hold true*

1. *When $\lambda^{1/3} \lesssim N \lesssim \lambda$,*

$$\big|W^m_{N,a}(T,X,Y)\big| \lesssim \frac{1}{h^2}\frac{h^{1/3}}{((N/\lambda^{1/3})^{1/2}+\lambda^{1/6}|T-4N|^{1/2})}\,. \tag{53}$$

2. *When $\lambda \lesssim N$,*

$$\big|W^m_{N,a}(T,X,Y)\big| \lesssim \frac{1}{h^2}\frac{h^{1/3}\sqrt{\lambda/N}}{(N/\lambda^{1/3})^{1/2}}\,. \tag{54}$$

Proof of Proposition 7. For $\lambda^{1/3} \lesssim N \lesssim \lambda$, $X \le 1$, (53) follows by showing that the integral in (45) satisfies

$$\Big|\int e^{i\lambda\Psi^m_{N,a,a,h}(T,X,\Upsilon,S,A_c,\eta)}\eta^2\psi_1(\eta)\chi_3(S,\Upsilon,a,1/N,h,\eta)\,dSd\Upsilon d\eta\Big| \lesssim \frac{\lambda^{-2/3}}{1+\lambda^{1/3}|\frac{T}{4N}-1|^{1/2}}, \tag{55}$$

which, in turn, follows for both $m \in \{0,1\}$ as in the proof of [4, Prop.7]. From the discussion above, the only new situation is the case $N > \lambda^2$ when $m = 1$, when we cannot eliminate the terms depending on m^2h^2/η^2 from the phase. In order to prove the last statement of Proposition 7, we notice that, after applying the stationary phase in both A and η, we are essentially left with a product of two Airy functions (corresponding to the remaining integrals in Υ, S) whose worst decay is $\lambda^{-2/3}$. Using (46), we obtain

$$|W^m_{N,a}(T,X,Y)| \lesssim \frac{a^2}{h^3N}\times\lambda^{-2/3} \sim \frac{1}{h^2}\frac{1}{N}\frac{a^2}{h}\frac{h^{2/3}}{a} \sim \frac{1}{h^2}\frac{1}{N}\frac{a}{h^{1/3}}$$
$$= \frac{1}{h^2}\frac{1}{N}h^{1/3}\lambda^{2/3} = \frac{1}{h^2}\frac{h^{1/3}\sqrt{\lambda/N}}{(N/\lambda^{1/3})^{1/2}}.$$

□

It should be clear from the previous estimates that there are different sub-regimes when studying decay, especially when t is large. We need to stress that the proofs of Propositions 5, 6 and 7 in [4] only use the variable with $T = \frac{t}{\sqrt{a}}$ (which is compared to different powers of λ) and not t, which can therefore be taken as large as needed. Nonetheless, as mentioned in the end of Section 3.1, the parametrix under the form of reflected waves will only be used to obtain dispersive bounds for $t \lesssim 1/h^2$, when $O(h^\infty) = O((h/t)^\infty)$.

3.3. **"Almost transverse" waves.** Let $\max\{4a, h^{2/3(1-\epsilon)}\} \leq \gamma \lesssim 1$ and

$$x = \gamma X\,, \quad \alpha = \gamma A\,, \quad t = \sqrt{\gamma}T\,, \quad s = \sqrt{\gamma}S\,, \quad \sigma = \sqrt{\gamma}\Upsilon\,, \quad y + t\sqrt{1+\gamma} = \gamma^{3/2}Y\,. \tag{56}$$

Let $\lambda_\gamma = \gamma^{3/2}/h$ be the large parameter and $\Psi^m_{N,a,\gamma,h}$ be as in (29). As in the case $\gamma = a$, we find $V^m_{h,\gamma}(t,x,a,y) = W^m_{N,\gamma}(T,X,Y) + O(h^\infty)$, where

$$W^m_{N,\gamma}(T,X,Y) := \frac{\gamma^2}{(2\pi h)^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i\lambda_\gamma \Psi^m_{N,a,\gamma,h}}\eta^2\psi_1(\eta)\chi_2(S)\chi_2(\Upsilon)\psi_2(A)\,dSd\Upsilon dAd\eta\,. \tag{57}$$

The following results hold for both $m \in \{0,1\}$ as in [4, Prop.8,9,10 & 11]:

Proposition 8. *Let $m \in \{0,1\}$. For $0 < T \leq \frac{5}{2}$, $t = \sqrt{\gamma}T$, we have*

$$\begin{aligned}\sum_N |W^m_{N,\gamma}(T,X,Y)| &= |W^m_{0,\gamma}(T,X,Y)| + \sum_{\pm 1}|W^m_{\pm 1,\gamma}(T,X,Y)| + O(h^\infty)\\ &\lesssim \frac{1}{h^2}\inf\Big(1, \Big(\frac{h}{t}\Big)^{1/2}\Big).\end{aligned} \tag{58}$$

Proposition 9. *(see [4, Prop. 9, 10] for $m = 0$) For $1 \leq T \sim N \lesssim \lambda_\gamma^2$ we have*

$$|W^m_{N,\gamma}(T,X,Y)| \lesssim \frac{\gamma^2}{h^3}\frac{1}{\sqrt{\lambda_\gamma N}} \times \lambda_\gamma^{-1/2-1/3} = \frac{h^{1/3}}{h^2}\frac{1}{\sqrt{N}}\,.$$

As a consequence, for $\sqrt{\gamma} \leq t \lesssim \sqrt{\gamma}\lambda_\gamma^2$ we obtain

$$|G^m_{h,\gamma}(t,x,a,y)| \lesssim \sum_{N \in \mathcal{N}^m_{1,\gamma}(t,x,y)} |W^m_{N,\gamma}(T,X,Y)| \lesssim \frac{1}{h^2}(\frac{h}{t})^{1/2}\lambda_\gamma^{1/6},$$

since $\#\mathcal{N}^m_1(t,x,y) = O(1)$ for such values of t.

Proposition 10. *(see [4, Prop.9] for $m = 0$) For $T \gtrsim \lambda_\gamma^2$ we have*

$$|W^m_{N,\gamma}(T,X,Y)| \lesssim \frac{\gamma^2}{h^3}\frac{1}{N}\frac{1}{\lambda_\gamma^{5/6}}.$$

Remark 8. When t is very large (which was not the case in [4] !), Proposition 10 is not helpful anymore as, due to the overlapping (and the fact that $\#\mathcal{N}^m_{1,\gamma}(t,\cdot)$ is proportional to t), summing-up over N yields an important loss. In the next section we show that using the parametrix in the form of a spectral sum provides a sharp bound for $G^m_{h,\gamma}(t,x,a,y)$ in the "almost transverse" case.

Remark 9. Notice that we obtained the exact same bounds for both $m = 0$ and $m = 1$. Using Remark 7, it becomes clear that this holds for all $T \sim N \lesssim \lambda_\gamma$; for $T \sim N \gg \lambda_\gamma$, the stationary phase applies in both A and η: for $m = 1$ we act as in the proof of Proposition 3 and use the fact that the main contribution of the second derivative with respect to η of $\Psi^{m=1}_{N,\gamma}$ comes from $NB(\eta\lambda_\gamma A^{3/2})$ and not from the terms involving m^2h^2/η^2 (and both have factors N or T, with $N \sim T$). This is the precise statement related to the heuristics described in the introduction.

3.4. **The case $h^{2/3} \lesssim \gamma \lesssim h^{2(1-\varepsilon)/3}$ and the case $\#\mathcal{N}^m_{1,\gamma}(t,\cdot)$ unbounded.** When $\gamma \lesssim h^{2(1-\varepsilon)/3}$ writing a parametrix under the form over reflected waves (25) is not useful anymore, since the parameter $\lambda_\gamma = \gamma^{3/2}/h$ is small and stationary phase arguments do not apply. In this case we can only work with the parametrix in the form (17). Notice that (17) can also be used (with surprisingly good results) when the time is very large : in fact, since the number of waves that overlap is proportional with t, summing over N when $t > 1/h^2$ will eventually provide dispersive bounds that are (much) worse than those announced in Theorem 1.1. In particular, for $t/\sqrt{\gamma} \geq \lambda_\gamma^2$ we have to work with (17) to prove Theorem 1.1. Recall that in (17) (for $d = 2$) the sum is taken for $k \lesssim 1/h$ (as $\gamma \lesssim 1$ and $k \sim \gamma^{3/2}/h$ on the support of ψ_2 and $k \lesssim \sup(a, h^{2/3})^{3/2}/h$ on the support of ϕ_2). We intend to apply the stationary phase for each integral and keep the Airy factors as part of the symbol. Let

$$\phi_k = y\eta + t\eta\sqrt{1 + \omega_k h^{2/3}\eta^{-2/3} + m^2h^2\eta^{-2}},$$

then

$$\partial_\eta \phi_k = y + t\frac{1 + \frac{2}{3}\omega_k h^{2/3}\eta^{-2/3}}{\sqrt{1 + \omega_k h^{2/3}\eta^{-2/3} + m^2h^2\eta^{-2}}},$$

$$\begin{aligned}\partial^2_{\eta,\eta}\phi_k = \frac{t}{\sqrt{1 + \omega_k h^{2/3}\eta^{-2/3} + m^2h^2\eta^{-2}}^3}\Big[m^2\Big(h^2/\eta^3 + \frac{2}{9}\omega_k h^{2/3}\eta^{-5/3}\Big) \\ -\frac{1}{9}\omega_k h^{2/3}\eta^{-5/3}\Big(1 + 2\omega_k(h/\eta)^{2/3}\Big)\Big] \sim -\frac{t}{9}\omega_k h^{2/3}\eta^{-5/3}.\end{aligned} \tag{59}$$

As $\eta \sim 1$ on the support of ψ_1, for t large enough the large parameter is $t\omega_k/h^{1/3}$. One needs to check that one has, for some $\nu > 0$, $|\partial_\eta^j Ai((\eta/h)^{2/3}x - \omega_k)| \leq C_j\mu^{j(1/2-\nu)}$. Since one has

$$\sup_{b>0}|b^l Ai^{(l)}(b - \omega_k)| \lesssim \omega_k^{3l/2}, \quad \forall l \geq 0, \tag{60}$$

it will be enough to check that there exists $\nu > 0$ such that, for every $k \lesssim 1/h$ and every t sufficiently large, the following holds

$$\max\Big\{(t/h)^\nu, \omega_k^{3/2}\Big\} \lesssim (t\omega_k/h^{1/3})^{1/2-\nu}. \tag{61}$$

Remark 10. The additional condition $(t\omega_k/h^{1/3})^{1/2-\nu} \geq (t/h)^\nu$ has been added in order to obtain remainders satisfying $O((t\omega_k/h^{1/3})^{-\infty}) = O((h/t)^\infty)$. This condition holds as soon as $t \gtrsim h^{\frac{1}{3}(1-\frac{4\nu}{1-4\nu})}$ for some $\nu > 0$, therefore only values $t \lesssim h^{1/3}$ don't meet this requirement.

As $\omega_k \sim \gamma/h^{2/3} = \lambda_\gamma^{2/3}$ on the support of ψ_2 (and $\omega_k \lesssim \sup(a, h^{2/3})^{3/2}/h$ on the support of ϕ_2), proving (61) for some $\nu > 0$ is equivalent to showing that $\max\Big\{(t/h)^\nu, \lambda_\gamma\Big\} \lesssim (t\lambda_\gamma^{2/3}/h^{1/3})^{1/2-\nu}$ which in turn is equivalent to $\gamma^3/h^2 \lesssim (t\gamma/h)^{1-2\nu}$, which we further write

$$\gamma(\gamma/h)^{1+2\nu} \lesssim t^{1-2\nu} \quad \text{ and } t \gtrsim h^{\frac{1}{3}(1-\frac{4\nu}{1-4\nu})}. \tag{62}$$

Let $t(h,\gamma,\nu) := \Big(\gamma(\gamma/h)^{1+2\nu}\Big)^{1/(1-2\nu)}$ for small $\nu > 0$, then the last inequality holds true for any $t \geq \max\{h^{\frac{1}{3}(1-\frac{4\nu}{1-4\nu})}, t(h,\gamma,\nu)\}$ and the stationary phase applies.

Remark 11. In particular, when the cardinal of $\mathcal{N}^m_{1,\gamma}(t,\cdot)$ is unbounded, we have $t \geq \gamma^{1/2}\frac{\gamma^3}{h^2}$ and therefore the first inequality in (62) holds for $\nu > 0$. If moreover $t \gtrsim h^{\frac{1}{3}(1-\frac{4\nu}{1-4\nu})}$ for some $\nu > 0$, the stationary phase applies with the Airy factors as part of the symbol as soon as the number of overlapping waves is large and the remainders are $O((h/t)^\infty)$.

Lemma 6. *([5, Lemma 3.5]) There exists C such that for $L \geq 1$ the following holds*

$$\sup_{b\in\mathbb{R}}\Big(\sum_{1\leq k\leq L}\omega_k^{-1/2}Ai^2(b-\omega_k)\Big)\leq CL^{1/3},$$
$$\sup_{b\in\mathbb{R}_+}\Big(\sum_{1\leq k\leq L}\omega_k^{-1/2}h^{2/3}Ai'^2(b-\omega_k)\Big)\leq Ch^{2/3}L. \tag{63}$$

When (62) holds, we can apply the stationary phase in η (with η on the support $[\frac12,\frac32]$ of ψ). Let $\eta_c(\omega_k)$ denote the critical point that satisfies $-6(\frac{y}{t}+1)=\omega_k h^{2/3}/\eta^{2/3}+O((\omega_k h^{2/3}/\eta^{2/3})^2)$.

- The "tangential" case $\gamma\sim a\gtrsim h^{2/3}$. As $h^{2/3}\omega_k\lesssim a$ on the support of the symbol ϕ_2, then $|\eta_c(\omega_k)/h)^{2/3}a-\omega_k|\lesssim 1$ and $Ai((\eta_c/h)^{2/3}x-\omega_k)$ may stay close to $Ai(0)$. For all $t\geq t(h,a,\nu)$, $\nu>0$ such that $t\geq h^{\frac13(1-\frac{4\nu}{1-4\nu})}$, the stationary phase with respect to η yields, with $\lambda=a^{3/2}/h$ and $k\lesssim\lambda$ and $L\sim\lambda$ in (63), and modulo $O((h/t)^\infty)$ terms

$$\begin{aligned}|G^m_{h,a}(t,x,a,y)|\lesssim\frac{h^{1/3}}{h^2}\Big|\sum_{k\lesssim\lambda}\frac{\psi_2(h^{2/3}\omega_k/(\eta_c^{2/3}a))}{L'(\omega_k)}\sqrt{\frac{h^{1/3}}{t\omega_k}}\\ \times Ai((\eta_c/h)^{2/3}x-\omega_k)Ai((\eta_c/h)^{2/3}a-\omega_k)\Big|\\ \lesssim\frac{1}{h^2}(\frac{h}{t})^{1/2}\Big|\sum_{k\lesssim\lambda}\omega_k^{-1}\Big|\sim\frac{1}{h^2}(\frac{h}{t})^{1/2}\lambda^{1/3}.\end{aligned}\tag{64}$$

Remark 12. Notice that in this case, and for t very large, the cut-off ϕ_2 cannot be replaced anymore by $\phi_2-\phi_{\frac18}$ because introducing $\phi_{\frac18}(h^{2/3}\omega_k/(|\eta|^{2/3}a))$ in the integrals defining $G^m_{h,a}$ yields only $O(h^\infty)$ terms (using the exponential decay of the Airy factors $e_k(a,\eta/h)$) but we do not necessarily have $O(h^\infty)=O((h/t)^\infty)$. However, this is not a problem when working with the spectral sums : the condition on the support of ϕ_2 yields $h^{2/3}\omega_k\lesssim a$, which translates into $\omega_k\lesssim\lambda^{2/3}$ (instead of $\omega_k\sim\lambda_\gamma$ on the support of ψ_2). As Lemma 6 holds for the sum over all $k\lesssim\lambda$, we obtain the same estimates as if we had $\phi_2-\phi_{\frac18}$ instead of ϕ_2. Notice that this wasn't the case when working with the sum over reflections and $t\lesssim 1/h^2$, as in that case the condition that the support of the symbol should not contain a fixed neighborhood of 0 was crucial.

- The "almost transverse" case $\gamma\geq\max\{4a,h^{2/3}\}$: while for $x\sim\gamma$ the factor $Ai((\eta_c/h)^{2/3}x-\omega_k)$ may stay close to $Ai(0)$, for $\omega_k\sim(\eta_c/h)^{2/3}\gamma\geq 4(\eta_c/h)^{2/3}a$, we obtain better decay from $|Ai((\eta_c/h)^{2/3}a-\omega_k)|\lesssim\frac{1}{1+\omega_k^{1/4}}$. For $4\lambda\lesssim k\lesssim\frac1h$ on the support of $\psi_2(\frac{h^{2/3}\omega_k}{\eta^{2/3}\gamma})$ and for $t\geq\max\{h^{\frac13(1-\frac{4\nu}{1-4\nu})},t(h,\gamma,\nu)\}$, the Cauchy-Schwarz inequality together with Lemma 6 with $L\sim\lambda_\gamma$ yield, modulo $O((h/t)^\infty)$ terms

$$\begin{aligned}|G^m_{h,\gamma}(t,x,a,y)|\leq\frac{1}{h^2}(\frac{h}{t})^{1/2}\Big|\sum_{k\sim\lambda_\gamma}\omega_k^{-1}Ai^2((\eta_c/h)^{2/3}x-\omega_k)\Big|^{1/2}\\ \times\Big|\sum_{k\sim\lambda_\gamma}\omega_k^{-1}Ai^2((\eta_c/h)^{2/3}a-\omega_k)\Big|^{1/2}\lesssim\frac{1}{h^2}(\frac{h}{t})^{1/2}\lambda_\gamma^{1/6},\end{aligned}\tag{65}$$

where we have used $\left|\sum_{k\sim\lambda_\gamma}\omega_k^{-1}Ai^2((\eta_c/h)^{2/3}a-\omega_k)\right|\leq\left|\sum_{k\sim\lambda_\gamma}\omega_k^{-3/2}\right|\lesssim 1$, as $\omega_k/\lambda_\gamma^{2/3}\in[3/4,4]$ on the support of ψ_2 and $\lambda_\gamma>2\lambda$.

4. Dispersive estimates for the wave and Klein Gordon flow in large time. Proof of Theorem 1.1 in the high frequency case $\sqrt{-\Delta_F}\sim\frac{1}{h}$ and $|\theta|\sim\frac{1}{h}$.

4.1. "Almost transverse" waves.

Let $\max\{4a,h^{2/3(1-\epsilon)}\}\leq\gamma\lesssim 1$.

Proposition 11. *For all $t>\sqrt{\gamma}$ and $m\in\{0,1\}$, we have*

$$\Big|\sum_{\max\{4a,h^{2/3(1-\epsilon)}\}\leq\gamma\lesssim 1}G^m_{h,\gamma}(t,\cdot)\Big|\lesssim\frac{1}{h^2}(\frac{h}{t})^{1/3},$$

where the sum is taken over dyadic γ as in (15).

Proof. Let $t\gtrsim\frac{1}{h^2}$, then $t>t(h,\gamma,\nu)$ for all $h^{2/3}\lesssim\gamma\lesssim 1$ (and, obviously, $t\geq h^{\frac{1}{3}(1-\frac{4\nu}{1-4\nu})}$) : applying the stationary phase in η and using Lemma 6 with $L\sim\frac{1}{h}$ yields a parametrix in terms of a spectral sum modulo $O((h/t)^\infty)$ terms and, as in (65)

$$|\sum_{4a\leq\gamma\lesssim 1}G^m_{h,\gamma}(t,\cdot)|\lesssim\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\frac{1}{h^{1/6}}.\tag{66}$$

As $(\frac{h}{t})^{\frac{1}{18}}\leq h^{1/6}$ for all $t>\frac{1}{h^2}$, we find $|\sum_{4a\leq\gamma\lesssim 1}G^m_{h,\gamma}(t,\cdot)|\leq\frac{1}{h^2}(\frac{h}{t})^{\frac{1}{2}-\frac{1}{18}}$. Notice that for such large values of t these kind of bounds are much better then those obtained using Proposition 10 and summing up over $N\in\mathcal{N}^m_{1,\gamma}(t,\cdot)$ (and then over $\cup_{\gamma\lesssim 1}$), because the cardinal of $\mathcal{N}^m_{1,\gamma}(t,\cdot)$ is proportional to $t/\sqrt{\gamma}$ which provides a bound independent of t for the sum of $W^m_{N,\gamma}(t,\cdot)$.

We are left with $h^{1/3}\lesssim\sqrt{\gamma}\leq t\lesssim\frac{1}{h^2}$ in which case we recall that $O(h^\infty)=O((h/t)^\infty)$. Let $t\sim\frac{\gamma_0^{7/2}}{h^2}$ for some $h^{2/3}<\gamma_0\lesssim 1$ (notice that $\gamma_0\sim h^{2/3}$ corresponds to $t\sim h^{1/3}$, while $\gamma_0\sim 1$ corresponds to $t\sim\frac{1}{h^2}$). Assume first $\gamma_0\geq h^{2/3(1-\varepsilon)}$ for some $\varepsilon>0$: if, moreover, $\gamma_0>4a$, we estimate separately the sums over $4a\leq\gamma\lesssim\gamma_0$ and $\gamma_0<\gamma\lesssim 1$:

1. When $\gamma\lesssim\gamma_0$, then, using Remark 11, $t\sim\frac{\gamma_0^{7/2}}{h^2}\gtrsim\frac{\gamma^{7/2}}{h^2}>t(h,\gamma,\nu)$. Moreover, as $\gamma_0\geq h^{2/3(1-\varepsilon)}$, then $t\geq h^{1/3(1-7\epsilon)}$ and therefore we can use again (65) with $\lambda_\gamma\leq\lambda_{\gamma_0}$ together with $\lambda_{\gamma_0}^{1/6}\sim\gamma_0^{1/4}/h^{1/6}\lesssim(h/t)^{-1/14}h^{1/21}\ll(h/t)^{-1/6}$.
2. For dyadic $\gamma_0<\gamma\lesssim 1$ we have $t\sim\frac{\gamma_0^{7/2}}{h^2}<\frac{\gamma^{7/2}}{h^2}$: in this case we use the form of $G^m_{h,\gamma}$ as a sum over reflected waves. According to Proposition 1 (for $m\in\{0,1\}$), $\#\mathcal{N}^m_{1,\gamma}(t,\cdot)=O(1)$, so only a bounded number of waves $W^m_{N,\gamma}(t,\cdot)$ overlap. Using Proposition 9 for $h^{2/3(1-\epsilon)}\lesssim\gamma_0<\gamma\leq\min\{1,t^2\}$ yields

$$|\sum_{\gamma_0<\gamma\leq\min\{1,t^2\}}G^m_{h,\gamma}(t,\cdot)|\leq\sum_{\gamma_0<\gamma\leq\min\{1,t^2\}}\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\lambda_\gamma^{1/6}.\tag{67}$$

- If $t\geq 1$, then $\gamma_0<\gamma\lesssim 1$ in the sum (67) and $\lambda_{\gamma\sim 1}^{1/6}\sim\frac{1}{h^{1/6}}\leq(\frac{t}{h})^{1/6}$, hence

$$|\sum_{\gamma_0<\gamma\lesssim 1}G^m_{h,\gamma}(t,\cdot)|\leq\sum_{\gamma_0<\gamma\lesssim 1}\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{\frac{1}{2}-\frac{1}{6}}.$$

- If $\sqrt{\gamma}\leq t<1$, the last sum in (67) is bounded by $\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\frac{t^{1/2}}{h^{1/6}}$ and $t^{1/2}\lesssim t^{1/6}$, hence
$$|\sum_{\gamma_0<\gamma\leq t^2} G^m_{h,\gamma}(t,\cdot)|\lesssim \frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\lambda^{1/6}_{\gamma=t^2}\lesssim \frac{1}{h^2}\Big(\frac{h}{t}\Big)^{\frac{1}{2}-\frac{1}{6}}.$$
When $t^2<\gamma<1$ it means that $t<\sqrt{\gamma}$ and we apply Proposition 8 which yields
$$|\sum_{t^2<\gamma\lesssim 1} G^m_{h,\gamma}(t,\cdot)|\lesssim \frac{1}{h^2}\inf\Big(1,\Big(\frac{h}{t}\Big)^{1/2}\Big)\log(1/h).$$

Let now $\max\{4a,h^{2/3}\}<\gamma_0<h^{2/3(1-\epsilon)}$, for some small $0<\epsilon<1/7$, then $1\leq\lambda_{\gamma_0}\leq h^{-\epsilon}$. For $1\gtrsim\gamma>\max\{h^{2/3(1-\epsilon)},t^2\}$ we obtain again, using Proposition 8,
$$|\sum_{\max\{h^{2/3(1-\epsilon)},t^2\}<\gamma\lesssim 1} G^m_{h,\gamma}(t,\cdot)|\leq \sum_{\gamma_0<\gamma\lesssim 1}\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\log(1/h).$$
We are left with $\sum_{\gamma\leq\max\{h^{2/3(1-\epsilon)},t^2\}} G^m_{h,\gamma}(t,\cdot)$ for $h^{1/3}\lesssim t\sim\frac{\gamma_0^{7/2}}{h^2}\leq h^{1/3(1-7\epsilon)}$. We use (63) (without the preliminary stationary phase in η) with $L\sim\frac{(h^{2/3(1-\epsilon)})^{3/2}}{h}=h^{-\epsilon}$, which yields, as $\frac{h}{t}\gtrsim h^{2/3(1+\epsilon/2)}$,
$$\Big|\sum_{\gamma<h^{2/3(1-\epsilon)}} G^m_{h,\gamma}(t,\cdot)\Big|\lesssim \frac{h^{1/3}}{h^2}L^{1/3}=\frac{h^{1/3(1-\epsilon)}}{h^2}\leq\frac{1}{h^2}\Big(\frac{h}{t}\Big)^{\frac{1}{2}(1-\epsilon)/(1+\epsilon/2)}.$$
Taking ϵ small enough achieves the proof. □

4.2. **Tangential waves $G^m_{h,a}$.** We focus on dispersive bounds for $G^m_{h,a}$, $m\in\{0,1\}$.

Proposition 12. *There exists $C>0$ such that for every $h^{2/3}\lesssim a\lesssim 1$, $h\in(0,1/2]$, $h<t\in\mathbb{R}_+$, $m\in\{0,1\}$, the spectrally localized Green function $G^m_{h,a}(t,x,a,y)$ satisfies*
$$|G^m_{h,a}(t,x,a,y)|\lesssim\frac{1}{h^2}\Big(a^{1/4}(\frac{h}{t})^{1/4}+(\frac{h}{t})^{1/3}\Big),\qquad \textit{for all } (x,y)\in\Omega_d. \tag{68}$$

Proof of Proposition 12. Recall from Proposition 1 that when $a>h^{2/3(1-\epsilon)}$ for some small $\epsilon>0$,
$$\big|\mathcal{N}^m_{1,a}(t,x,y)\big|=O(1)+m^2\frac{|t|h^2}{a^{3/2}}+\frac{|t|}{a^{1/2}\lambda^2},\quad \lambda=a^{3/2}/h.$$
When $m=1$, the term $\frac{|t|h^2}{a^{3/2}}$ is much smaller than $\frac{|t|}{a^{1/2}\lambda^2}$ and therefore it doesn't modify in any way the number of waves that overlap. Since all the estimates obtained in Sections 3.2 and 3.4 are exactly the same for $G^m_{h,a}$ for both $m=0$ and $m=1$, we do not make the difference between these two cases. In the following we describe all the situations according to the size of t and a. Notice first that if $a\ll h^{2/3}$ there is no contribution coming from tangent directions, so we consider only values for the initial distance from the boundary such that $a\gtrsim h^{2/3}$. Let $t>0$.

Let first $h^{2/3(1-\epsilon)}\lesssim a\lesssim 1$ for some small $\epsilon>0$ and consider $t\geq\frac{a^{7/2}}{h^2}$: then $t/\sqrt{a}\geq\lambda^2$ and in this case $\#\mathcal{N}^m_{1,a}(t,\cdot)$ is large and the number of waves $W^m_{N,a}(t,\cdot)$ that cross each other behaves like $\frac{t}{\sqrt{a}\lambda^2}$. As $t\geq t(h,a,\nu)$ and $t>h^{2/3(1-\epsilon)\times 7/2}/h^2=h^{1/3(1-7\epsilon)}$, then t clearly satisfies condition (62). The stationary phase with respect to η allows to obtain a parametrix (modulo $O((h/t)^\infty)$ terms) as a spectral sum and we use (64) to bound it as follows

$$|G^m_{h,a}(t,\cdot)| \lesssim \frac{1}{h^2}(\frac{h}{t})^{1/2}\lambda^{1/3} \ll \frac{1}{h^2}(\frac{h}{t})^{1/3},$$

where the last equality holds for all $t \gg h\lambda^2$; as $t \geq \sqrt{a}\lambda^2$ and $\sqrt{a} \gtrsim h^{1/3} \gg h$, this is obviously true. Consider now $a > h^{2/3(1-\epsilon)}$ and $t \leq \frac{a^{7/2}}{h^2}$ when only a finite number of $W^m_{N,a}(t,\cdot)$ can meet : sharp dispersive bounds are then provided by Propositions 5, 6 and 7. Moreover, in this case we recall that $O(h^\infty) = O((h/t)^\infty)$. If $t > \frac{a}{h^{1/3}} = \sqrt{a}\lambda^{1/3}$ then Proposition 7 applies and we obtain, as only a bounded number of N are involved in the sum

- If $t \in [\frac{a}{h^{1/3}}, \frac{a^2}{h})$, which corresponds to $\frac{t}{\sqrt{a}} \in (\lambda^{1/3}, \lambda)$, then, using (53), we find

$$\begin{aligned}|G^m_{h,a}(t,\cdot)| &= |\sum_{\frac{t}{\sqrt{a}}\sim N\in\mathcal{N}^m_{1,a}(t,\cdot)} W^m_{N,a}(\frac{t}{\sqrt{a}},\cdot)| \lesssim \frac{1}{h^2}\frac{h^{1/6}\sqrt{a}}{\sqrt{t}}\\ &\lesssim \frac{1}{h^2}(\frac{h}{t})^{1/4}a^{1/4},\end{aligned}\tag{69}$$

  where the last inequality holds for $t > \frac{a}{h^{1/3}}$. As $(\frac{h}{t})^{1/4}a^{1/4} \leq (\frac{h}{t})^{1/3}$ only if $t \leq \frac{h}{a^3}$, it follows that the (RHS) of (69) can be bounded by $\frac{1}{h^2}(\frac{h}{t})^{1/3}$ only if $\frac{a}{h^{1/3}} \leq \frac{h}{a^3}$, hence for $a \leq h^{1/3}$. In particular, when $h^{1/3} < a \lesssim 1$, then $1 \leq t \lesssim 1/h$ and the previous estimate is sharp.
- If $t \in [\frac{a^2}{h}, \frac{a^{7/2}}{h^2})$, which corresponds to $\frac{t}{\sqrt{a}} \in (\lambda, \lambda^2)$, then, using (54), we find

$$\begin{aligned}|G^m_{h,a}(t,\cdot)| &= |\sum_{\frac{t}{\sqrt{a}}\sim N\in\mathcal{N}^m_{1,a}(t,\cdot)} W^m_{N,a}(\frac{t}{\sqrt{a}},\cdot)| \lesssim \frac{1}{h^2}\frac{h^{1/3}\lambda^{1/2+1/6}}{t/\sqrt{a}}\\ &= \frac{1}{h^2}\frac{a^{3/2}}{th^{1/3}} \leq \frac{1}{h^2}(\frac{h}{t})^{1/3},\end{aligned}\tag{70}$$

  where the last inequality holds for all $t > \frac{a^{9/4}}{h}$, hence for all $t \geq \frac{a^2}{h} \gtrsim \frac{a^{9/4}}{h}$.

Let now $\sqrt{a} \lesssim t < \frac{a}{h^{1/3}}$, then $1 \leq \frac{t}{\sqrt{a}} \leq \lambda^{1/3}$ and Proposition 5 yields a sharp bound

$$|G^m_{h,a}(t,\cdot)| \leq \frac{1}{h^2}(\frac{h}{t})^{1/4}a^{1/4},\tag{71}$$

which is reached at $t = t_n = 4n\sqrt{a}\sqrt{1+a}$ for $1 \lesssim n \leq \frac{\sqrt{a}}{h^{1/3}}$. As $(\frac{h}{t})^{1/4}a^{1/4} \leq (\frac{h}{t})^{1/3}$ only if $t \leq \frac{h}{a^3}$, it follows that when $a \leq h^{1/3}$ we have $t < \frac{a}{h^{1/3}} \leq \frac{h}{a^3}$ and $|G^m_{h,a}(t,\cdot)| \leq \frac{1}{h^2}(\frac{h}{t})^{1/3}$ everywhere on $[\sqrt{a}, \frac{a}{h^{1/3}}]$, while for $a > h^{1/3}$, (71) is sharp on $(\frac{h}{a^{1/3}}, \frac{a}{h^{1/3}}]$ and becomes $|G^m_{h,a}(t,\cdot)| \leq \frac{1}{h^2}(\frac{h}{t})^{1/3}$ on $[\sqrt{a}, \frac{h}{a^3}]$. Remark that in the last case $a > h^{1/3}$ we must have $t \lesssim 1$, which is the situation considered in [5] (and [4]). Let $h < t \leq \sqrt{a}$, then $|G^m_{h,a}(t,\cdot)| \leq \frac{1}{h^2}(\frac{h}{t})^{1/2}$ using Proposition 4.

We are left with the case $h^{2/3} \lesssim a < h^{2/3(1-\epsilon)}$ for a small $1/2 > \epsilon > 0$ and with $t \leq \frac{a^{7/2}}{h^2}$. One can easily check that for $t > h^{1/3(1-2\epsilon)}$, the condition (62) is satisfied for $4\nu := \frac{2\epsilon}{1+2\epsilon}$. As for such ν, $(t\omega_k/h^{1/3})^{1/2-\nu} > (t/h^{1/3})^{1/2-\nu} \geq (t/h)^\nu$ is large, the stationary phase in η applies and (64) holds, yielding

$$|G^m_{h,a}(t,\cdot)| \lesssim \frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}\lambda^{1/3} \lesssim \frac{1}{h^2}\Big(\frac{h}{t}\Big)^{1/2}h^{-\epsilon},$$

where the sum in $G^m_{h,a}$ is restricted to $k \lesssim \lambda \leq h^{-\epsilon}$. As for $t > h^{1/3(1-2\epsilon)}$ we have $h^{-\epsilon} < (t/h)^{3\epsilon/(2(1+\epsilon))}$, we conclude taking ϵ small enough. Let now $t \leq h^{1/3(1-2\epsilon)}$. Using Lemma 6 with $L \sim h^{-\epsilon}$ together with the Cauchy-Schwarz inequality yields

$$|G^m_{h,a}(t,\cdot)| \lesssim h^{-2}h^{1/3}L^{1/3} = \frac{h^{1/3(1-\epsilon)}}{h^2}\,, \tag{72}$$

and as $t \leq h^{1/3(-2\epsilon)}$ we have $h^{1/3(1-\epsilon)} \lesssim (\frac{h}{t})^{\frac{1}{2}\frac{1-\epsilon}{1+\epsilon}}$, we conclude taking ϵ small. $\square$

5. **Dispersive estimates for the wave and Klein Gordon flow in large time. Proof of Theorem 1.1 in the high frequency case "$\sqrt{-\Delta_F} \sim \frac{1}{h}$", $|\theta| \sim \frac{1}{\tilde h}$, $\tilde h > 2h$.** In this section we consider again $d = 2$ as the higher dimensional case can be dealt with exactly like in (the end of) Section 2.1 and prove the following :

Proposition 13. *Let $G^m_{(h,\tilde h)}(t,x,a,y)$ be as in (18) and $m \in \{0,1\}$. There exists a uniform constant $\tilde C > 0$ such that for every $h \in (0,1/2]$, $2h \leq \tilde h$ and $t > h$,*

$$|G^m_{(h,\tilde h)}(t,x,a,y)| \leq \frac{\tilde C}{h^2}\Big(\frac{h}{t}\Big)^{1/4}\Big(\frac{h}{\tilde h}\Big), \tag{73}$$

Moreover, for $t > h$, $|G^{\flat,m}_h(t,x,a,y)| = |\sum_{j\geq 1} G^m_{h,2^j h}(t,x,a,y)| \leq \frac{1}{h^2}(\frac{h}{t})^{1/4}$.

In the remaining part of this section we prove Proposition 13. In (18), the symbols $\psi(\tilde h\theta)$ and $\psi_1(h\sqrt{\lambda_k(\theta)})$ are supported for $\tilde h\theta \sim 1$, hence $h^2\lambda_k(\theta) \sim 1$, which implies $\omega_k \sim \Big(\frac{\tilde h^2}{h^3}\Big)^{2/3}$ and $k \sim \frac{\tilde h^2}{h^3}$. Let $\theta = \eta/\tilde h$, with $\eta \sim 1$ on the support of ψ. The condition $\omega_k \sim \frac{\tilde h^{4/3}}{h^2}$ (which holds on the support of ψ_1) implies $a \lesssim (\tilde h/h)^2$ there where the Airy factors aren't exponentially small.

- When $a \leq \frac{1}{4}(\tilde h/h)^2$ and $x \leq a$, both Airy factors $e_k(x,\eta/\tilde h)$ and $e_k(a,\eta/\tilde h)$ have only non degenerate critical points.
- When $a \sim (\tilde h/h)^2$, the Airy factor $e_k(a,\eta/\tilde h)$ may have degenerate critical points of order 2 (and the same thing may happen to $e_k(x,\eta/\tilde h)$ for x near a).
- When $a > 4(\tilde h/h)^2$, $e_k(a,\eta/\tilde h)$ is exponentially small and yields a contribution $O((\tilde h^2/h^3)^{-\infty})$. Therefore, at least as long as $t/h \lesssim (\tilde h^2/h^3)^3$, we can ignore the contribution coming from initial data at distance $a > 4(\tilde h/h)^2$ as, for such t, $O((\tilde h^2/h^3)^{-\infty}) = O((h/t)^\infty)$.

We let $a := (\tilde h/h)^2\tilde a$, then $\tilde a \lesssim 1$ and $a\theta^{2/3} = \tilde a\Big(\tilde h^2/h^3\Big)^{2/3}\eta^{2/3}$. We define $\tilde\lambda := \frac{\tilde h^2}{h^3}$, which satisfies $\tilde\lambda > \frac{1}{h}$. In the two dimensional case, the phase functions of (18) becomes

$$\frac{1}{\tilde h}\tilde\phi_{(h,\tilde h)}(t,y,\omega_k,\eta) := \frac{1}{\tilde h}\Big(y\eta + t\eta\frac{\tilde h}{h}\sqrt{\omega_k/(\eta\tilde\lambda)^{2/3} + (h/\tilde h)^2 + h^2m^2/\eta^2}\Big) \tag{74}$$

and the Airy phase functions transform into

$$s^3/3 + s(\eta\tilde\lambda)^{2/3}\Big(\tilde a - \omega_k/(\eta\tilde\lambda)^{2/3}\Big) + \sigma^3/3 + \sigma(\eta\tilde\lambda)^{2/3}\Big(\frac{x}{a}\tilde a - \omega_k/(\eta\tilde\lambda)^{2/3}\Big). \tag{75}$$

Using $\frac{1}{2} > \frac{h}{\tilde h}$ and $\omega_k/(\eta\tilde\lambda)^{2/3} \sim 1$, the main term in the phase is $\frac{t\eta}{h}\sqrt{\omega_k/(\eta\tilde\lambda)^{2/3}}$. For $T \lesssim \tilde\lambda^2$, we show that we can reduce the analysis to the previous case (when

$\tilde h = h$). Applying the Airy-Poisson formula (9) and replacing the Airy factors by their integral formulas (see (21)) allow to write $G^m_{(h,\tilde h)}$ under the following form

$$\begin{aligned} G^m_{(h,\tilde h)}(t,x,a,y) = \frac{\tilde h^{1/3}}{(2\pi)^3\tilde h^2}\sum_{N\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{\frac{i}{h}\tilde\phi_{(h,\tilde h)}(t,y,\omega,\eta)-iNL(\omega)}\psi(\eta)\eta^{2/3} \\ \times e^{i\eta(s^3/3+s(a\eta^{2/3}/\tilde h^{2/3}-\omega)+\sigma^3/3+\sigma(x\eta^{2/3}/\tilde h^{2/3}-\omega))} \\ \times \psi_1\Big(\eta\sqrt{\omega/(\eta\tilde\lambda)^{2/3}+(h/\tilde h)^2+h^2m^2/\eta^2}\Big)dsd\sigma d\omega d\eta, \end{aligned} \tag{76}$$

where $\omega\sim(\eta\tilde\lambda)^{2/3}$ on the support of ψ_1. Set $T:=t(\frac{h}{\tilde h})^2$, $X:=x\frac{h^2}{\tilde h^2}$, $Y:=\frac{h^3}{\tilde h^3}(y+t)$ and rescale

$$\omega=(\eta\tilde\lambda)^{2/3}\tilde A,\quad s=/(\eta\tilde\lambda)^{1/3}S,\quad \sigma=(\eta\tilde\lambda)^{1/3}\Upsilon.$$

As $\eta\in[3/8,5/4]$ on the support of ψ and as ψ_1 is supported in $[3/4,5/4]$, we can re-write ψ_1 as a cutoff $\psi_2(\tilde A)$ for some $\psi_2\in\mathcal{C}^\infty_0([1/5,12])$ equal to 1 on $[9/25,100/9]$ in order to keep the same notations as in the previous section. In the new variables

$$\begin{aligned} G^m_{(h,\tilde h)}(t,x,a,y) = \frac{1}{(2\pi)^3\tilde h}\Big(\frac{\tilde h^2}{h^3}\Big)^2\sum_N\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i\frac{\tilde h^2}{h^3}\tilde\Psi^m_{N,\tilde a,(h,\tilde h)}(T,X,Y,\Upsilon,S,\tilde A,\eta)} \\ \times\psi(\eta)\psi_2(\tilde A)d\Upsilon dSd\tilde Ad\eta, \end{aligned} \tag{77}$$

where $\tilde A\sim 1$, $\tilde a\lesssim 1$ and where we have defined (compare with (29))

$$\begin{aligned} \tilde\Psi^m_{N,\tilde a,(h,\tilde h)}(T,X,Y,\Upsilon,S,\tilde A,\eta) := \eta\Big(Y+\Upsilon^3/3+\Upsilon(X-\tilde A)+S^3/3+S(\tilde a-\tilde A) \\ +T\Big(\sqrt{\tilde A+(h/\tilde h)^2+m^2h^2/\eta^2}-(h/\tilde h)\Big)-\frac{4}{3}N\tilde A^{3/2}\Big)+\frac{N}{\tilde\lambda}B(\eta\tilde\lambda\tilde A^{3/2}). \end{aligned} \tag{78}$$

In the new variables, $G^m_{(h,\tilde h)}$ has a form similar to $G^m_{h,\gamma=1}$ (with λ_γ replaced by $\tilde\lambda$). As in Section 3.2, using the equation satisfied by the critical points and the support of ψ_2, we can restrict ourselves to $|S|,|\Upsilon|\le 3$ and insert suitable cut-offs $\chi(S)\chi(\Upsilon)$ without changing the contribution of the integrals in (77) modulo $O(\tilde\lambda^{-\infty})$. The parametrix under the form (77) will be useful only for $T\ll\tilde\lambda^2$ which is equivalent to $t/h\ll(\tilde h^2/h^3)^3=\tilde\lambda^3$: in this regime we always have $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$.

We can write $G^m_{(h,\tilde h)}(t,x,a,y)=\sum\tilde W^m_N(T,X,Y)$, where $\tilde W^m_N(T,X,Y)$ has the same form as $W^m_{N,a}(T,X,Y)$ in (44) but where γ is replaced by 1, the factor a^2 by $(\frac{\tilde h}{h})^4$ (and is due to the change of variables in ω,s,σ), λ by $\tilde\lambda$ and $\Psi^m_{N,a,a,h}$ is replaced by $\tilde\Psi^m_{N,\tilde a,(h,\tilde h)}$ as follows

$$\tilde W^m_N(T,X,Y)=\frac{\tilde\lambda^2}{\tilde h}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i\tilde\lambda\tilde\Psi^m_{N,\tilde a,(h,\tilde h)}(T,X,Y,\Upsilon,S,\tilde A,\eta)}\psi(\eta)\psi_2(\tilde A)\chi(S)\chi(\Upsilon)d\Upsilon dSd\tilde Ad\eta. \tag{79}$$

We obtain the equivalent of Propositions 2 and 3 for $N\sim T\lesssim\tilde\lambda^2$:

Proposition 14. *Let $m\in\{0,1\}$. Let $N\lesssim\tilde\lambda$ and let $\tilde W^m_N(T,X,Y)$ be defined in (79). Then the stationary phase theorem applies in $\tilde A$ and yields, modulo $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$ terms*

$$\begin{aligned} \tilde W^m_N(T,X,Y)=\frac{\tilde\lambda^2}{\tilde h(N\tilde\lambda)^{\frac12}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i\lambda\Psi^m_{N,a,a,h}(T,X,\Upsilon,S,A_c,\eta)} \\ \times\eta^2\psi(\eta)\chi_3(S,\Upsilon,a,1/N,h,\eta)\,dSd\Upsilon d\eta\,, \end{aligned} \tag{80}$$

where χ_3 has compact support in (S,Υ) and harmless dependency on $a,h,1/N,\eta$.

Proposition 15. *Let $m\in\{0,1\}$, $\tilde\lambda\ll N\lesssim\tilde\lambda^2$ and $\tilde W_N^m(T,X,Y)$ be defined in (79), then the stationary phase applies in both $\tilde A$ and η and, yields modulo $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$ terms*

$$\tilde W_N^m(T,X,Y)=\frac{\tilde\lambda^2}{\tilde hN}\int_{\mathbb{R}^2}e^{i\lambda\Psi^m_{N,a,a,h}(T,X,Y,\Upsilon,S,A_c,\eta_c)}\chi_3(S,\Upsilon,a,1/N,h)\,dSd\Upsilon\,,\quad(81)$$

where χ_3 has compact support in (S,Υ) and harmless dependency on $a,h,1/N$.

Remark 13. Notice that Proposition 15 remains true for all N, but, as for $N>\tilde\lambda^2$ the number of overlapping wave becomes large, it won't allow to obtain sharp results. Moreover, for $T\sim N$ unbounded by a suitable power of $\tilde\lambda$, we don't have $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$ anymore.

Using (50), (52) and (55) with λ is replaced by $\tilde\lambda$ we obtain the equivalent of Propositions 5, 6 and 7 where the only difference is that h is replaced by $\tilde h$, a by $\tilde a$, λ by $\tilde\lambda$ and we have to distinguish three main situations according to whether $N<\tilde\lambda^{1/3}$, $\tilde\lambda^{1/3}\le N\le\tilde\lambda$ and $\tilde\lambda<N\lesssim\tilde\lambda^2$.

Proposition 16. *Let $m\in\{0,1\}$. For $1\le N<\tilde\lambda^{1/3}$, $|T-4N|\lesssim1/N$, $X\le1$ we have, when $\tilde a\sim1$,*

$$\left|\tilde W_N^m(T,X,Y)\right|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\frac{1}{((N/\tilde\lambda^{1/3})^{1/4}+|N(T-4N)|^{1/6})}\,.\quad(82)$$

Proposition 17. *Let $m\in\{0,1\}$. For $1\le N<\tilde\lambda^{1/3}$, $|T-4N|\gtrsim1/N$, $X\le1$ we have*

$$\left|\tilde W_N^m(T,X,Y)\right|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\frac{1}{(1+|N(T-4N)|^{1/2})}\,.\quad(83)$$

Proposition 18. *Let $m\in\{0,1\}$. Let $\tilde\lambda^{1/3}\lesssim N$, $X\le1$, then the following estimates hold true*

1. *When $\tilde\lambda^{1/3}\lesssim N\lesssim\tilde\lambda$, $\left|\tilde W_N^m(T,X,Y)\right|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\frac{1}{((N/\tilde\lambda^{1/3})^{1/2}+\tilde\lambda^{1/6}|T-4N|^{1/2})}$.*
2. *When $\tilde\lambda\lesssim N(\sim T\lesssim\tilde\lambda^2)$, $\left|\tilde W_N^m(T,X,Y)\right|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\frac{\sqrt{\tilde\lambda/N}}{(N/\tilde\lambda^{1/3})^{1/2}}$.*

Moreover, as in Lemme 5 from Section 3, for $N\ge16T$ the phase functions of $W_N^m(T,\cdot)$ are all non-stationary in at least one variable and provide enough decay to sum in N and to give an $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$ contribution.

The last statement of Proposition 18 follows from as in Lemme 5 from Section 3 : for $N\ge16T$, the phase functions of $W_N^m(T,\cdot)$ are all non-stationary in at least one variable and provide enough decay to sum in N and to give an $O(\tilde\lambda^{-\infty})=O((h/t)^\infty)$ contribution.

The case $\tilde\lambda\lesssim N\sim T\lesssim\tilde\lambda^2$ occurs when $\frac{\tilde h^4}{h^6}=\tilde\lambda^2\gtrsim t(\frac{h}{\tilde h})^2\gtrsim\frac{\tilde h^2}{h^3}=\tilde\lambda$, hence for $\tilde\lambda^3\gtrsim t/h\gtrsim\tilde\lambda^2$. Using Propositions 16, 17 and 18 we can now follow exactly the same approach as in Section 4 and obtain the corresponding estimates for each $\tilde h>2h$. We define a set $\tilde{\mathcal N}_1^m(T,X,Y)$ as follows

$$\tilde{\mathcal N}_1^m(T,X,Y)=\cup_{\{|(T',X',Y')-(T,X,Y)|\le1\}}\tilde{\mathcal N}^m(T,X,Y),$$
$$\tilde{\mathcal N}^m(T,X,Y)=\{N\in\mathbb{Z},\exists(\Sigma,\Upsilon,\tilde A,\eta),\nabla_{(\Sigma,\Upsilon,\tilde A,\eta)}\tilde\Psi^m_{N,\tilde a,(h,\tilde h)}(T,X,Y,\Upsilon,S,\tilde A,\eta)=0\}.$$

Proposition 19. *Let $t \in \mathbb{R}$, $t > h$, $\tilde{h} \geq 2h$ and $T = t(h/\tilde{h})^2$. The following holds true:*

- *We control the cardinal of $\tilde{\mathcal{N}}_1^m(T,X,Y)$,*
$$\left|\tilde{\mathcal{N}}_1^m(T,X,Y)\right| \lesssim O(1) + T/\tilde{\lambda}^2 + m^2h^2T\,, \tag{84}$$
and this bound is optimal.
- *The contribution of the sum over $N \notin \tilde{\mathcal{N}}_1^m(T,X,Y)$ in (22) is $O(\tilde{\lambda}^{-\infty})$.*

Remark 14. For $T \lesssim \tilde{\lambda}^2$, it follows from Proposition 19 that $\left|\tilde{\mathcal{N}}_1^m(T,X,Y)\right| \lesssim O(1) + m^2h^2T$. Moreover, for such $T \lesssim \tilde{\lambda}$ we have $t/h \lesssim \tilde{\lambda}^3$, therefore the contribution of the sum over $N \notin \tilde{\mathcal{N}}_1^m(T,X,Y)$ in (22) is $O(\tilde{\lambda}^{-\infty}) = O((h/t)^{\infty})$.

Remark 15. Notice that when $m = 1$ and t is large enough, the main contribution in the right hand side of (84) comes this time from the last term : in fact, as in this case $\tilde{\lambda}^2 = \frac{1}{h^2}(\tilde{h}/h)^2 \geq 1/h^2$, asking $T \lesssim \tilde{\lambda}^2$ doesn't imply anymore $h^2T = O(1)$ as in Section 3 when we had $\lambda = \gamma^{3/2}/h \lesssim 1/h$.

In order to prove Proposition 19 we use exactly the same approach as in the proof of Proposition 1 but with the phase function (78). The parameter γ is replaced by 1, and the difference of any two difference points $N_{1,2} \in \tilde{\mathcal{N}}_1^m(t,x,y)$ is estimated (as in (36)) as follows,

$$|N_1 - N_2| = O(1) + m^2h^2T + \frac{T}{\tilde{\lambda}^2}, \quad T = t\Big(\frac{h}{\tilde{h}}\Big)^2, \quad \tilde{\lambda} = \frac{\tilde{h}^2}{h^3}.$$

Let first $m = 0$: as long as $T \leq \tilde{\lambda}^2$ there is no overlap if $m = 0$ and we can use directly the (sharp) bounds from Propositions 16, 17 and 18. When $T > \tilde{\lambda}^2$, condition (61) holds and yields

$$|G^{m=0}_{(h,\tilde{h})}(t,x,a,y)| \lesssim \frac{\tilde{h}^{1/3}}{\tilde{h}^2}\Big(\frac{h}{t}\Big)^{1/2}\tilde{\lambda}^{2/3},$$

hence for $(t/h) > \tilde{\lambda}^3$ we obtain $|G^{m=0}_{(h,\tilde{h})}(t,x,a,y)| \lesssim \frac{\tilde{h}^{1/3}}{\tilde{h}^2}\Big(\frac{h}{t}\Big)^{\frac{1}{2}-\frac{2}{9}}$. The situation becomes more delicate when $m = 1$ as there may be overlap due to the presence of the term h^2T in (84).

- For $N \sim T \lesssim \tilde{\lambda}^{1/3}$, the set $\tilde{\mathcal{N}}_1^{m=1}(T,X,Y)$ is unbounded when $h^2 \times \tilde{h}^{2/3}/h = h\tilde{h}^{2/3} \gg 1$: notice that this may happen as we have to consider all values $\tilde{h} = 2^jh$, $j \geq 1$. However, if $T = 4N_T$ for some $N_T \in \mathbb{N}$, then $|T - 4N| \geq 1 > \frac{1}{N}$ for all $N \neq N_T$ and therefore Proposition 16 applies only for N_T while for all $N \neq N_T$, $N \sim N_T$, we use Proposition 17. In this case we obtain
$$\begin{aligned}\sum_{N\sim N_T, N\in\tilde{\mathcal{N}}_1^{m=1}(T,X,Y)} |W_N^{m=1}(T,X,Y)| &\leq |W^{m=1}_{N=N_T}(T,X,Y)| \\ &+ \sum_{N\neq N_T, N\in\tilde{\mathcal{N}}_1^{m=1}(T,X,Y)} |W_N^{m=1}(T,X,Y)|.\end{aligned} \tag{85}$$

The first integral $|W^{m=1}_{N=N_T}(T,X,Y)|$ is bounded by $\frac{\tilde h^{1/3}}{\tilde h^2}\frac{\tilde\lambda^{1/12}}{t^{1/4}(h/\tilde h)^{1/2}}=\frac{1}{h^2}(\frac{h}{t})^{1/4}(\frac{h}{\tilde h})$. The sum over $N\neq N_T$ is bounded using Proposition 17 as follows

$$\begin{aligned}\sum_{N\neq N_T, N\in\tilde{\mathcal N}_1^{m=1}(T,X,Y)}|W^{m=1}_N(T,\cdot)| &\lesssim \sum_{N\neq N_T, N\in\tilde{\mathcal N}_1^{m=1}(T,X,Y)}\frac{\tilde h^{1/3}}{\tilde h^2}\frac{1}{(1+|N(T-4N)|^{1/2})}\\ &\leq \frac{\tilde h^{1/3}}{\tilde h^2}\sum_{N=N_T+j,|j|\lesssim h^2N_T}\frac{1}{1+(N_T+j)^{1/2}|j|^{1/2}}\\ &\leq \frac{\tilde h^{1/3}}{\tilde h^2}\Big(\int_0^{1/2}\frac{dx}{x^{1/2}(1-x)^{1/2}+N_T^{-1}}+\int_0^{1/2}\frac{dx}{x^{1/2}(1+x)^{1/2}+N_T^{-1}}\Big),\end{aligned}\tag{86}$$

where the last two integrals correspond to the sum over $N<N_T$ and $N>N_T$, respectively. The second integral is obviously bounded; to see that the first one is also bounded, let $x=\sin^2 z$, $z\in[0,\pi/2)$, then $1-x=\cos^2 z$, $dx=2\sin z\cos z$ and conclude as $\frac{\tilde h^{1/3}}{\tilde h^2}\lesssim\frac{1}{h^2}(\frac{h}{t})^{1/4}(\frac{h}{\tilde h})h^{1/4}$ in the regime $T\lesssim\tilde\lambda^{1/3}$.

- Let $\tilde\lambda^{1/3}\lesssim N\sim T\lesssim\tilde\lambda^{5/3}$, when $\tilde\lambda^{4/3}\lesssim t/h\lesssim\tilde\lambda^{8/3}$. As the sum over reflected waves provides an important loss in this regime, we shall use $G^{m=1}_{(h,\tilde h)}$ as in (18), which becomes, after the change of coordinates $\theta=\eta/\tilde h$,

$$\sum_{k\sim\tilde\lambda}\frac{\tilde h^{1/3}}{\tilde h^2}\int_{\mathbb R}e^{\frac{i}{\tilde h}\tilde\phi^{m=1}_{(h,\tilde h)}(t,y,\omega_k,\eta)}\psi(\eta)\eta^{2/3}\psi_1\Big(h\sqrt{\lambda_k(\eta/\tilde h)}\Big)e_k(x,\eta/\tilde h)e_k(a,\eta/\tilde h)d\eta\,.\tag{87}$$

The phase functions $\frac{1}{\tilde h}\tilde\phi^{m=1}_{(h,\tilde h)}$ are given in (74). The condition (61) is not necessarily satisfied for such T (as $\omega_k^{3/2}\sim\tilde\lambda$, while $T\lesssim\tilde\lambda^{5/3}$ yields only $(t/h)\lesssim\tilde\lambda^{8/3}$). For $\omega_k\sim\tilde\lambda^{2/3}$ and $\eta\sim 1$ on the support of the symbol, we have, for $m\in\{0,1\}$

$$\frac{1}{\tilde h}|\partial^2_{\eta,\eta}\tilde\phi^{m=1}_{(h,\tilde h)}(t,y,\omega_k,\eta)|\sim\frac{2}{9}\frac{t}{h}\Big(\omega_k/(\eta\tilde\lambda)^{2/3}\Big)^{1/2}\Big(1+O((h/\tilde h)^2)+O(h^2)\Big).$$

The phase functions of the two Airy factors are given in (75) and one can easily check that their second order derivatives with respect to η behave at most like $\tilde\lambda$ there where (75) may be stationary with respect to s,σ (when the non-stationary theorem phase applies we obtain a contribution $O((h/t)^\infty)$ as the sum over k is finite). Therefore, the stationary phase with respect to η applies and yields a factor $(h/t)^{1/2}$. However, the terms in the remaining sum over k cannot be bounded better than by a uniform constant : as $L'(\omega_k)\sim\sqrt{2\omega_k}\sim k^{1/3}$, the sum over $k\sim\tilde\lambda$ is bounded by $\tilde\lambda^{2/3}$ and we eventually obtain

$$|G^{m=1}_{(h,\tilde h)}(t,x,a,y)|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\Big(\frac{h}{t}\Big)^{1/2}\Big|\sum_{k\sim\tilde\lambda}\frac{1}{L'(\omega_k)}\Big|\lesssim\frac{\tilde h^{1/3}}{\tilde h^2}\Big(\frac{h}{t}\Big)^{1/2}\tilde\lambda^{2/3},\tag{88}$$

We next prove that $\frac{\tilde h^{1/3}}{\tilde h^2}\Big(\frac{h}{t}\Big)^{1/2}\tilde\lambda^{2/3}\lesssim\frac{1}{h^2}\Big(\frac{h}{\tilde h}\Big)\Big(\frac{h}{t}\Big)^{1/4}$ for $(t/h)\gtrsim\tilde\lambda^{4/3}$. This is equivalent to proving that $(h/\tilde h)\tilde h^{1/3}(h/t)^{1/4}\tilde\lambda^{2/3}\lesssim 1$ for all such t, which is equivalent to $(h/\tilde h)\tilde h^{1/3}\tilde\lambda^{2/3}\leq\tilde\lambda^{1/3}(\leq(t/h)^{1/4})$. As $(\tilde h\tilde\lambda)^{1/3}=(\tilde h/h)$, the last inequality holds true.

- Let $T>\tilde\lambda^{5/3}$: we use the form of the parametrix as a sum over eigenmodes k and apply the stationary phase with respect to η with the phase function given in (74) and the Airy factors in the symbols (as we did in Section 3.4) :

as $\omega_k^{3/2} \sim \tilde{\lambda}$, the condition (61), necessary and sufficient in order to consider the Airy factors as part of the symbol, reads as $\tilde{\lambda} \lesssim (t/h)^{1/2-\nu}$ for some $\nu > 0$; as we have $(t/h) > \tilde{\lambda}^{8/3}$ and this condition is satisfied for all $0 < \nu \leq 1/4$. Applying the stationary phase in η yields again (88) (notice that we cannot take advantage of Lemma 6 to obtain $\tilde{\lambda}^{1/3}$ instead of $\tilde{\lambda}^{2/3}$ in the estimate (88) as for each k, the critical point $\eta_c = \eta_c(t, y, \omega_k)$ depends on k and may be such that $a\eta_c^{2/3}/\tilde{h}^{2/3} - \omega_k$ stays close to 0). As $t/h > \tilde{\lambda}^{8/3}$, then $\tilde{\lambda}^{2/3} \leq (t/h)^{1/4}$ and the last term in (88) is bounded as follows

$$\frac{\tilde{h}^{1/3}}{\tilde{h}^2}\Big(\frac{h}{t}\Big)^{1/2}\tilde{\lambda}^{2/3} \leq \frac{1}{\tilde{h}^2}\Big(\frac{h}{t}\Big)^{1/4} = \frac{1}{h^2}\Big(\frac{h}{\tilde{h}}\Big)^2\Big(\frac{h}{t}\Big)^{1/4}.$$

To obtain the last statement of Proposition 13 we sum up for all $\tilde{h} = 2^j h$.

6. **Dispersive estimates for the wave flow in large time and the Klein Gordon flow. Proof of Theorem 1.1 in the low frequency case.** We let $G^m_{SF}(t,x,a,y) := \sum_{j\in\mathbb{N}} G^m_j(t,x,a,y)$ where G^m_j is defined in (19), then

$$G^m_{SF}(t,x,a,y) := \sum_{k\geq 1}\int e^{i(y\theta + t\sqrt{\lambda_k(\theta)+m^2})}\phi(|\theta|)\phi(\sqrt{\lambda_k(\theta)})\frac{|\theta|^{2/3}}{L'(\omega_k)} \times Ai(x|\theta|^{2/3}-\omega_k)Ai(a|\theta|^{2/3}-\omega_k)d\theta\,, \tag{89}$$

where the integral is taken for $\theta \in \mathbb{R}^{d-1}\setminus\{0\}$. Let $\chi_0 \in C_0^\infty([-2,2])$ be equal to 1 on $[-3/2, 3/2]$, let $M > 1$ be sufficiently large and write $G^m_{SF} = G^m_{SF,\chi_0} + G^m_{SF,1-\chi_0}$ where for $\chi \in \{\chi_0, 1-\chi_0\}$ we have set

$$G^m_{SF,\chi}(t,x,a,y) := \sum_{k\geq 1}\int e^{i(y\theta + t\sqrt{\lambda_k(\theta)+m^2})}\phi(|\theta|)\phi(\sqrt{\lambda_k(\theta)})\chi\Big(\frac{t\lambda_k(\theta)}{M}\Big)e_k(x,\theta)e_k(a,\theta)d\theta.$$

Lemma 7. *Let $M > 1$ be large enough and $t > 0$. Then there exists $C(M,d) \sim M^{d/2}$ such that*

$$\|G^m_{SF,\chi_0}\|_{L^\infty(\Omega_d)} \leq C(M,d)\min\Big\{1, \frac{1}{t^{\frac{d}{2}}}\Big\}.$$

Proof of Lemma 7. Writing $\theta = \rho\Theta$ with $\rho = |\theta|$, we have

$$G^m_{SF,\chi_0}(t,x,a,y) = \sum_{k\geq 1}\int_0^\infty \int_{\Theta\in\mathbb{S}^{d-2}} e^{-i\rho|y|\Theta_1}\sin^{d-2}\Theta_1 d\Theta \rho^{d-2} e^{it\sqrt{\lambda_k(\rho)+m^2}} \times\phi(\rho)\phi(\sqrt{\lambda_k(\rho)})\chi_0\Big(\frac{t\lambda_k(\rho)}{M}\Big)\frac{\rho^{2/3}}{L'(\omega_k)}Ai(x\rho^{2/3}-\omega_k)Ai(a\rho^{2/3}-\omega_k)d\rho, \tag{90}$$

where $e_1 = (1,0,...,0) \in \mathbb{R}^{d-1}$. On the support of $\chi_0(\frac{t\lambda_k(\rho)}{M})(\sum_j \psi(2^j\rho))$ we have $\lambda_k(\rho) = \rho^2 + \omega_k\rho^{4/3} \leq M/t$ which implies $\rho \leq \sqrt{M/t}$ and $\omega_k \leq \Big(M/t - \rho^2\Big)/\rho^{4/3}$ (as on the support of $\psi(2^j\rho)$, $j \geq 1$, ρ doesn't vanish). Let $L = L(M/t,\rho) := \Big(M/t-\rho^2\Big)^{3/2}/\rho^2$; as $\omega_k \sim k^{2/3}$, it follows that on the support of the symbol of

G^m_{SF,χ_0} we must have $k \le L(M/t,\rho)$. We estimate G^m_{SF,χ_0} as follows

$$
\begin{aligned}
|G^m_{SF,\chi_0}(t,\cdot)| \le \int_{\rho\le\sqrt{M/t}} \Big|\int_{\mathbb{S}^{d-2}} d\Theta\Big| \rho^{d-2+2/3} \\
\times \Big|\sum_{1\le k\le L(M/t,\rho)} \frac{1}{L'(\omega_k)} Ai(x\rho^{2/3}-\omega_k)Ai(a\rho^{2/3}-\omega_k)\Big| d\rho \\
\lesssim \int_{\rho\le\sqrt{M/t}} \rho^{d-2+2/3}\Big(\sum_{1\le k\le L(M/t,\rho)} \frac{1}{L'(\omega_k)} Ai^2(x\rho^{2/3}-\omega_k)\Big)^{1/2} \\
\times \Big(\sum_{1\le k\le L(M/t,\rho)} \frac{1}{L'(\omega_k)} Ai^2(a\rho^{2/3}-\omega_k)\Big)^{1/2} d\rho \\
\le \int_{\rho\le\sqrt{M/t}} \rho^{d-2+2/3}\Big(M/t-\rho^2\Big)^{1/2}/\rho^{2/3} d\rho \\
= \int_{\rho\le\sqrt{M/t}} \rho^{d-2}\Big(M/t-\rho^2\Big)^{1/2} d\rho,
\end{aligned}
\tag{91}
$$

where in the second line we have applied the Cauchy-Schwarz inequality and then used (63) from Lemma 6. Taking $\rho = \sqrt{\frac{M}{t}}w$ gives

$$
|G^m_{SF,\chi_0}(t,\cdot)| \lesssim \Big(\frac{M}{t}\Big)^{\frac{d-2+1+1}{2}} \int_{w\le 1} (1-w^2)^{1/2} dw \le M^{d/2}/t^{d/2}.
$$

Let now $t<1$ and let $\rho = 2^{-j}\tilde{\rho}$ for some $j\ge 0$: as $\lambda_k(\theta) \le 4$ on the support of ϕ, then for fixed $j\ge 0$, the sum over k in G^m_j is finite as $\omega_k \le 4\times 2^{4j/3}$. We estimate each $G^m_j(t,\cdot)$ as follows

$$
\begin{aligned}
|G^m_j(t,x,a,y)| \lesssim 2^{-j(d-1+2/3)}\Big(\sum_{1\le k\lesssim 2^{2j}} \frac{1}{L'(\omega_k)} Ai^2(x\rho^{2/3}-\omega_k)\Big)^{1/2} \\
\times\Big(\sum_{1\le k\lesssim 2^{2j}} \frac{1}{L'(\omega_k)} Ai^2(a\rho^{2/3}-\omega_k)\Big)^{1/2} \lesssim 2^{-j(d-1)}.
\end{aligned}
\tag{92}
$$

As the sum is convergent, we obtain a uniform bound for $G^m_{SF,\chi_0}(t,\cdot)$ for $t<1$. □

We are left with $G^m_{SF,1-\chi_0}(t,\cdot) = \sum_{j\in\mathbb{N}} G^m_{j,1-\chi_0}(t,\cdot)$, where $G^m_{j,1-\chi_0}$ has the same form as (19) with the additional cut-off $(1-\chi_0)\Big(\frac{t\lambda_k(\theta)}{M}\Big)$ inserted into the symbol. Write, for some symbol $c(\Theta')$,

$$
\begin{aligned}
G^m_{j,1-\chi_0}(t,x,a,y) := \sum_{k\ge 1}\int_0^\infty \int_{\Theta'\in\mathbb{R}^{d-2}} e^{-i\rho|y|\sqrt{1-|\Theta'|^2}} c(\Theta') d\Theta' \rho^{d-2} e^{it\sqrt{\lambda_k(\rho)+m^2}} \\
\phi(\rho)\phi(\sqrt{\lambda_k(\rho)})(1-\chi_0)\Big(\frac{t\lambda_k(\rho)}{M}\Big)\psi_2(2^j\rho)\frac{\rho^{2/3}}{L'(\omega_k)} Ai(x\rho^{2/3}-\omega_k)Ai(a\rho^{2/3}-\omega_k)d\rho.
\end{aligned}
\tag{93}
$$

Make the change of variables $\rho = 2^{-j}\tilde{\rho}$, with $\tilde{\rho}\in[\frac{3}{4},2]$ on the support of $\psi_2(\tilde{\rho})$. Set

$$
\phi^m_{k,j}(t,x,a,y,\tilde{\rho},\Theta') := -2^{-j}\tilde{\rho}|y|\sqrt{1-|\Theta'|^2} + t\sqrt{m^2+2^{-2j}\tilde{\rho}^2+2^{-4j/3}\tilde{\rho}^{4/3}\omega_k}. \tag{94}
$$

The first order derivative of $\phi^m_{k,j}$ with respect to $\tilde\rho$ is given by

$$\partial_{\tilde\rho}\phi^m_{k,j}(t,x,a,y,\tilde\rho,\Theta') = -2^{-j}|y|\sqrt{1-|\Theta'|^2} + t\frac{2^{-2j}\tilde\rho + \frac{2}{3}2^{-4j/3}\tilde\rho^{1/3}\omega_k}{\sqrt{m^2+2^{-2j}\tilde\rho^2+2^{-4j/3}\tilde\rho^{4/3}\omega_k}}, \tag{95}$$

and the second order derivative with respect to $\tilde\rho$ is given by

$$\partial^2_{\tilde\rho}\phi^m_{k,j}(t,x,a,y,\tilde\rho,\Theta') = \frac{t}{\sqrt{m^2+2^{-2j}\tilde\rho^2+2^{-4j/3}\tilde\rho^{4/3}\omega_k}^3}\Big[m^2\Big(2^{-2j}+\frac{2}{9}\tilde\rho^{-2/3}2^{-4j/3}\omega_k\Big) - \frac{1}{9}(2^{-4j/3}\omega_k)2^{-2j}\tilde\rho^{4/3} - \frac{2}{9}\tilde\rho^{2/3}(2^{-4j/3}\omega_k)^2\Big]. \tag{96}$$

Notice that if $m=0$, the second order derivative of $\phi^{m=0}_{k,j}$ does not vanish anywhere (since the two terms in the second line of (96) have same sign). On the other hand, for $m=1$ and $j\geq 0$, the second derivative of $\phi^{m=1}_{k,j}$ may cancel at some $\tilde\rho$ on the support of ψ_2 : this may happen for an unique $k=k(j)\sim 2^{4j/3}$. Since the phase functions may behave differently according to whether $m=0$ or $m=1$, we deal separately with these cases. In the following, we consider the case $m=0$ and then explain how to deal with the degenerate critical points of $\phi^{m=1}_{k(j),j}$.

6.0.1. *The wave flow.* Let $m=0$. We start by noticing that on the support of $1-\chi_0$ we have $t\geq\frac{3M}{2\lambda_k(\rho)}$ and that on the support of $\phi(\sqrt{\lambda_k(\rho)})$ we have $\lambda_k(\rho)\leq 4$: this implies that on the support of the symbol we must have $t\geq\frac{3}{8}M$. We first deal with the case when $|\theta|$ is not too small.

Proposition 20. *Let M be sufficiently large and $t\gtrsim M$. There exists a constant $C>0$ independent of t such that for all $\epsilon>0$*

$$\Big|\sum_{2^j\leq t^{(1-\epsilon)/4}} G^{m=0}_{j,1-\chi_0}(t,\cdot)\Big| \leq \frac{C}{|t|^{(d-1)/2}}. \tag{97}$$

Proof of Proposition 20. We apply the stationary phase in $\rho=2^{-j}\tilde\rho$ as in Section 3.4 with phase function $\phi_{k,j}$ and with the Airy factors as part of the symbol. For $m=0$, the main contribution in brackets in (96) is $-\frac{2}{9}(2^{-4j/2}\omega_k)^2\tilde\rho^{2/3}$ (when $j\geq 2$ this is obvious; when $j\in\{0,1\}$ is small and when $2^{-2j}\sim 2^{-4j/3}\omega_k\sim 1$ this remains true as the last two terms in (96) have same sign) and the second derivative behaves like

$$\partial^2_{\tilde\rho}\phi^{m=0}_{k,j}\sim -\frac{2}{9}t(2^{-4j/3}\omega_k)^{1/2}.$$

Using (60) it follows that, in order to apply the stationary phase with respect to $\tilde\rho$ with the Airy factors in the symbol we must have, for some $\nu>0$,

$$(t(2^{-4j/3}\omega_k)^{1/2})^{1/2-\nu}\geq\omega_k^{3/2}. \tag{98}$$

The last inequality can be re-written as $(t(2^{-4j/3}\omega_k)^{1/2})^{1/2-\nu}\geq 2^{2j}(2^{-4j/3}\omega_k)^{3/2}$, which is equivalent to $t^{1/2-\nu}\geq 2^{2j}(2^{-4j/3}\omega_k)^{(5+2\nu)/4}$, and as $2^{-4j/3}\omega_k\leq 4$, we need to assume $2^{2j}\leq t^{1/2-\nu}$ for some $\nu>0$. For j such that $2^j\leq t^{(1-\epsilon)/4}$, (98) holds with $\nu=\epsilon/2$. The phase $\phi_{k,j}$ is stationary when $2^{-j}|y|\sim|t|\sqrt{2^{-4j/3}\omega_k}$ and on the support of the symbol we have $|t|\sqrt{2^{-4j/3}\omega_k}\geq\frac{1}{2}|t|2^{-4j/3}\omega_k\gtrsim M$: indeed, this follows using $\sqrt{\lambda_k(\rho)}\geq\frac{1}{2}\lambda_k(\rho)$ (as $\sqrt{\lambda_k(\rho)}\leq 2$) and $\lambda_k(\rho)=\rho^2+\rho^{4/3}\omega_k\sim\rho^{4/3}\omega_k$. Moreover, the phase is stationary with respect to Θ' at $\Theta'=0\in\mathbb{R}^{d-2}$, and the determinant of the Hessian matrix of second order derivative is $\sim(2^{-j}|y|)^{d-2}\sim$

$(|t|\sqrt{2^{-4j/3}\omega_k})^{d-2}$. We can therefore apply the stationary phase with respect to both $\tilde{\rho}$ and Θ', which yields

$$|G^{m=0}_{j,1-\chi_0}(t,\cdot)| \lesssim \sum_{k\lesssim 2^{2j}} 2^{-j}\frac{2^{-j(d-2)}}{(|t|\sqrt{2^{-4j/3}\omega_k})^{(d-2)/2}} \times \frac{2^{-2j/3}}{L'(\omega_k)}\frac{1}{(t\sqrt{2^{-4j/3}\omega_k})^{1/2}}. \quad (99)$$

Using that $L'(\omega_k) \sim \sqrt{2\omega_k}$, (99) reads as (for $d \geq 2$)

$$|G^{m=0,d=2}_{j,1-\chi_0}(t,\cdot)| \lesssim \frac{1}{|t|^{1/2}} \times 2^{-j(1+2/3-1/3)} \sum_{k\leq 2^{2j}} \frac{1}{\omega_k^{3/4}} \leq \frac{1}{|t|^{1/2}} 2^{-4j/3} \times 2^j = \frac{2^{-j/3}}{|t|^{1/2}},$$

$$|G^{m=0,d=3}_{j,1-\chi_0}(t,\cdot)| \lesssim \frac{1}{|t|} \times 2^{-2j}\log(2^{2j}), \quad |G^{m=0,d\geq 4}_{j,1-\chi_0}(t,\cdot)| \lesssim \frac{1}{|t|^{(d-1)/2}} \times 2^{-2j(2d-3)/3}.$$

Summing up over j such that $2^j \leq t^{(1-\epsilon)/4}$ allows to conclude. □

From now on we let $t \gtrsim M$ and $2^j \gtrsim |t|^{(1-\epsilon)/4}$, which corresponds to small initial angles θ. Fix $a \in 2^{2j_0/3}[\frac{1}{2}, 2]$ for some $j_0 \in \mathbb{Z}$. We first notice that for $a\rho^{2/3} > \omega_k$ the estimates become trivial using the exponential decay of the Airy function on the positive real line. Let $a\rho^{2/3} \leq \omega_k$. As $\rho = 2^{-j}\tilde{\rho}$, $\tilde{\rho} \in [\frac{3}{4}, 2]$ on the support of ψ_2 and $2^{-4j/3}\omega_k \leq \lambda_k(2^{-j}\tilde{\rho}) \leq 4$ on the support of ϕ, we obtain the condition $\frac{1}{2}(\frac{3}{4})^{2/3}2^{2(j_0-j)/3} \leq a2^{-2j/3}\tilde{\rho}^{2/3} \leq \omega_k \leq 4 \times 2^{4j/3}$ which further yields $2^{2(j_0-j)/3} \leq 2(4/3)^{2/3}4 \times 2^{4(j+1)/3}$, and as $(4/3)^{2/3} < 2^{1/3}$ we find $j_0 < 3(j+2)$. We start with the sum over $j > j_0$ as in this case the Airy factors can be dealt with using (7) and the decay of $A_\pm$.

Proposition 21. *There exists a constant $C = C(d)$ independent of j_0 or M, such that the following holds*

$$\Big|\sum_{j>j_0} G^{m=0}_{j,1-\chi_0}(t,\cdot)\Big| \leq \frac{C}{|t|^{(d-1)/2}}.$$

Proof of Proposition 21. Using Proposition 20 we are left with the case $\rho = 2^{-j}\tilde{\rho}$ with $2^j \geq t^{(1-\epsilon)/4}$, which corresponds to waves that propagate within directions of very small angles $\lesssim M^{-1/4}$. As $j_0 - j \leq -1$, $x \leq a$ (by symmetry of the Green function) and $a2^{-2j/3}\tilde{\rho}^{2/3} \leq 2 \times 2^{2(j_0-j)/3}2^{2/3} \leq 2 < \frac{4}{5}\omega_k$ for all $k \geq 1$, we can write both Airy factors in (93) using (7). We obtain four different phase functions, where $\pm_1$ and $\pm_2$ mean independent signs,

$$\phi^{m=0,\pm_1,\pm_2}_{k,j} := \phi^{m=0}_{k,j} \pm_1 \frac{2}{3}(\omega_k - a2^{-2j/3}\tilde{\rho}^{2/3})^{3/2} \pm_2 \frac{2}{3}(\omega_k - x2^{-2j/3}\tilde{\rho}^{2/3})^{3/2}, \quad (100)$$

whose derivatives are given by

$$\begin{aligned}\partial_{\tilde{\rho}}\phi^{m=0,\pm_1,\pm_2}_{k,j} = \partial_{\tilde{\rho}}\phi^{m=0}_{k,j} &\mp_1 \frac{2}{3}a2^{-2j/3}\tilde{\rho}^{-1/3}(\omega_k - a2^{-2j/3}\tilde{\rho}^{2/3})^{1/2} \\ &\mp_2 \frac{2}{3}x2^{-2j/3}\tilde{\rho}^{-1/3}(\omega_k - x2^{-2j/3}\tilde{\rho}^{2/3})^{1/2},\end{aligned} \quad (101)$$

$$\begin{aligned}\partial^2_{\tilde{\rho}}\phi^{m=0,\pm_1,\pm_2}_{k,j} = \partial^2_{\tilde{\rho}}\phi^{m=0}_{k,j} &\pm_1 \frac{2}{9}a2^{-2j/3}\tilde{\rho}^{-4/3}(\omega_k - a2^{-2j/3}\tilde{\rho}^{2/3})^{1/2}\Big(1 + \frac{a2^{-2j/3}\tilde{\rho}^{2/3}}{(\omega_k - a2^{-2j/3}\tilde{\rho}^{2/3})}\Big) \\ &\pm_2 \frac{2}{9}x2^{-2j/3}\tilde{\rho}^{-4/3}(\omega_k - x2^{-2j/3}\tilde{\rho}^{2/3})^{1/2}\Big(1 + \frac{x2^{-2j/3}\tilde{\rho}^{2/3}}{(\omega_k - x2^{-2j/3}\tilde{\rho}^{2/3})}\Big).\end{aligned} \quad (102)$$

The main term in each of the two brackets in (102) is ~ 1. We distinguish two main regimes :

- If $|t| > 4a$, $x \le a$, then the last two terms in (101) (corresponding to the derivatives of the phase functions the Airy factors $A_\pm$) are small compared to the second term in the right hand side of (95) and therefore the phase is stationary in $\tilde\rho$ for $2^{-j}|y|\sqrt{1-|\Theta'|^2} \sim |t|(2^{-4j/3}\omega_k)^{1/2}$; for such values, the parameter $2^{-j}|y|$ of $\sqrt{1-|\Theta'|^2}$ is large ; moreover, $|\partial^2_{\tilde\rho}\phi^{m=0,\pm_1,\pm_2}_{k,j}| \sim |t|(2^{-4j/3}\omega_k)^{1/2}$ (as in this regime the second order derivatives of the Airy functions remain much smaller than $\partial^2_{\tilde\rho}\phi^{m=0}_{k,j}$) ; we conclude exactly as in the proof of Proposition 20, as the stationary phase applies in both $\tilde\rho$ and Θ'. The estimates are even better than in (99) due to the decay of the Airy factors.
- Let $|t| \le 4a$, then $t\sqrt{2^{-4j/3}\omega_k} \mp_1 a\sqrt{2^{-4j/3}\omega_k - a2^{-2j}} \mp_2 x\sqrt{2^{-4j/3}\omega_k - x2^{-2j}}$ may be close to 0, in which case $\partial^2_{\tilde\rho}\phi^{m=0,\pm_1,\pm_2}_{k,j}$ may be small. Moreover, we may also have $2^{-j}|y|$ small there where the phase is stationary with respect to $\tilde\rho$: when this is the case we cannot apply the stationary phase with respect to Θ'. However, as for $x \le a$ the symbols of $A_\pm(x2^{-2j/3}\tilde\rho^{2/3} - \omega_k)$ and $A_\pm(x2^{-2j/3}\tilde\rho^{2/3} - \omega_k)$ decay like $\omega_k^{-1/4}$, we bound $|G^{m=0}_{j,1-\chi_0}(t,\cdot)|$ as follows

$$
\begin{aligned}
|G^{m=0}_{j,1-\chi_0}(t,\cdot)| &\le 2^{-j(1+(d-2)+2/3)} \sum_{k \lesssim 2^{2j}} \frac{1}{L'(\omega_k)\omega_k^{1/4+1/4}} \le 2^{-j(1+(d-2)+2/3)} 2^{2j/3} \\
&\le 2^{-j(d-1)} \lesssim \frac{2^{-2j(d-1)/3}}{|t|^{(d-1)/2}}, \quad \text{as } t \lesssim 4a \le 16 \times 2^{2j_0/3} < 16 \times 2^{2j/3},
\end{aligned}
$$

where, in order to estimate the sum over k we have used that $L'(\omega_k) \sim \sqrt{2\omega_k}$ and $\omega_k \sim k^{2/3}$. Summing up over $j > j_0$ achieves the proof.

□

In Proposition 21 we have considered only values $j > j_0$ so that the Airy factors could be written as in (7). The next lemma deals with $j \le j_0 < 3(j+2)$.

Proposition 22. *There exists a constant $C = C(d)$ independent of j_0, M, such that the following holds*

$$
\Big| \sum_{j \le j_0 < 3(j+2),\, 2^j \gtrsim t^{(1-\epsilon)/4}} G^{m=0}_{j,1-\chi_0}(t,\cdot) \Big| \le \frac{C}{|t|^{(d-1)/2}}.
$$

Proof of Proposition 22. Notice that if $a \in 2^{2j_0/3}[\frac{1}{2}, 2]$ is chosen such that $2^{j_0} \le t^{(1-\epsilon)/4}$ for some $\epsilon > 0$, then Proposition 20 applies for all $j \le j_0$ and we conclude that $G^{m=0}_{j,1-\chi_0}(t,\cdot)$ is bounded by $C/|t|^{(d-1)/2}$. Let j_0 such that $2^{j_0} \gtrsim t^{(1-\epsilon)/4}$ $\forall \epsilon > 0$.

For values k such that $\omega_k \le \frac{1}{4} 2^{2(j_0-j)/3}$ we obtain trivial contributions in (93) using the decay of the Airy functions. Therefore we split the sum over k in two parts, according to whether $\omega_k \sim 2^{2(j_0-j)/3}$ or $\omega_k \ge 2^{2(j_0-j+2)/3}$. In the second case, we obtain $\frac{4}{5}\omega_k \ge a2^{-2j/3}\tilde\rho^{2/3}$ and using (7) yields again four phase functions as in (100); in this case the proof follows as in Lemma 21 for $|t| > 4a$. When $|t| \le 4a$ we have $|t| \le 4a \le 8 \times 2^{2j_0/3}$. As $j \le j_0 < 3(j+2)$, we write $j = [j_0/3] - 1 + l$, with $l \in [0, 2([j_0]+1)/3] \cap \mathbb{N}$ and we obtain, as in (103),

$$\begin{aligned}\sum_{j\leq j_0<3(j+2)} |G^{m=0}_{j,1-\chi_0}(t,\cdot)| &\lesssim \sum_{j\leq j_0<3(j+2)} 2^{-j(d-1)} \\ &= \sum_{j=[j_0/3]-1+l, l\leq 2([j_0]+1)/3} 2^{-([j_0]/3-1+l)(d-1)} \\ &\leq \frac{4^{d-1}}{|t|^{(d-1)/2}} \sum_{l\geq 0} 2^{-l(d-1)} \leq \frac{4^{d-1}}{|t|^{(d-1)/2}}.\end{aligned} \tag{103}$$

Let $\omega_k \sim 2^{2(j_0-j)/3}$: we must obtain dispersive bounds for $\sum_{j\leq j_0\leq 3(j+2)} G^{m=0,\#}_{j,1-\chi_0}(t,\cdot)$, where

$$G^{m=0,\#}_{j,1-\chi_0}(t,x,a,y) := \sum_{k\geq 1}\int_0^\infty \int_{\Theta'\in\mathbb{R}^{d-2}} e^{\frac{i}{h}\rho|y|\sqrt{1-|\Theta'|^2}} c(\Theta')d\Theta'\rho^{d-2}e^{it\sqrt{\lambda_k(\rho)}}\phi(\rho)$$

$$\phi(\sqrt{\lambda_k(\rho)})\tilde{\psi}(\frac{\omega_k}{2^{2(j_0-j)/3}})(1-\chi_0)\Big(\frac{t\lambda_k(\rho)}{M}\Big)\psi_2(2^j\rho)\frac{\rho^{2/3}}{L'(\omega_k)}Ai(x\rho^{2/3}-\omega_k)Ai(a\rho^{2/3}-\omega_k)d\rho. \tag{104}$$

Here we have introduced a cut-off $\tilde{\psi}$ supported in $[\frac{1}{4}, 4]$ and defined the "tangent" flow as the restriction of $G^{m=0}_{j,1-\chi_0}$ to the sum over k such that $\omega_k \sim 2^{2(j_0-j)/3}$. We need the following lemma, whose proof is similar to that of Proposition 20 :

Lemma 8. *Let $t \geq \frac{2}{3}M$. There exists a constant $C = C(d) > 0$ such that for small $\epsilon > 0$ the following holds*

$$\Big|\sum_{2^{2(j_0-j)}\leq t^{1-\epsilon}} G^{m=0,\#}_{j,1-\chi_0}\Big| \leq \frac{C}{|t|^{(d-1)/2}}. \tag{105}$$

Proof of Lemma 8. We follow the same approach as in Proposition 20 : as t is sufficiently large and $\omega_k \sim 2^{2(j_0-j)/3}$ is small for j close to j_0, we can consider the Airy factors as part of the symbol. As in (98), this is possible as long as

$$t\sqrt{2^{-4j/3}\omega_k} \gg \omega_k^3.$$

Write $\omega_k = 2^{2(j_0-j)/3}\frac{\omega_k}{2^{2(j_0-j)/3}}$: the last inequality reads as $t \gg \frac{2^{2(j_0-j)}}{\sqrt{2^{-4j/3}\omega_k}}\Big(\frac{\omega_k}{2^{2(j_0-j)/3}}\Big)^3$. As $2^{-4j/3}\tilde{\rho}^{4/3}\omega_k \leq 4$, $\tilde{\rho} \in [\frac{3}{2}, 2]$ and $\frac{\omega_k}{2^{2(j_0-j)/3}} \in [\frac{1}{4}, 4]$ on the support of $\tilde{\psi}$ and $\psi(\tilde{\rho})$, in order to apply the stationary phase with the Airy factors in the symbol it will be enough to require $2^{2(j_0-j)} \leq t^{1-\epsilon}$ for some $\epsilon > 0$ (and $t \gtrsim M$). The dispersive bounds follow like in (99) :

$$|G^{m=0,\#}_{j,1-\chi_0}(t,\cdot)| \lesssim \sum_{k\sim 2^{j_0-j}(\lesssim 2^{2j})} 2^{-j}\frac{2^{-j(d-2)}}{(|t|\sqrt{2^{-4j/3}\omega_k})^{(d-2)/2}} \times \frac{2^{-2j/3}}{L'(\omega_k)}\frac{1}{(t\sqrt{2^{-4j/3}\omega_k})^{1/2}}. \tag{106}$$

□

Let now j such that $2^{2(j_0-j)} \geq t^{1-\epsilon}$ for some small $\epsilon > 0$ and k such that $\frac{\omega_k}{2^{2(j_0-j)/3}} \in [\frac{1}{4}, 4]$. Replacing the Airy factors of (104) by their the integral formulas (21) yields a new phase

$$\phi^{m=0}_{k,j} + \frac{s^3}{3} - s(\omega_k - a2^{-2j/3}\tilde{\rho}^{2/3}) + \frac{\sigma^3}{3} - \sigma(\omega_k - x2^{-2j/3}\tilde{\rho}^{2/3}), \tag{107}$$

whose critical points satisfy $s^2+a2^{-2j/3}\tilde\rho^{2/3}=\omega_k$ and $\sigma^2+x2^{-2j/3}\tilde\rho^{2/3}=\omega_k$ and whose critical values equal $\phi_{k,j}^{m=0,\pm_1,\pm_2}$ defined in (100). If $|s|\geq\frac98\sqrt{\omega_k}$ or $|\sigma|\geq\frac98\sqrt{\omega_k}$, repeated integrations by parts provide a contribution $O(\omega_k^{-n})$ for all $n\in\mathbb{N}$: as $\omega_k\sim 2^{2(j_0-j)/3}$ and $2^{2(j_0-j)}\geq t^{1-\epsilon}$, then $O(\omega_k^{-n})=O(|t|^{-n})$. In the following we let $|s|,|\sigma|\leq\frac98\sqrt{\omega_k}$. If $|s|,|\sigma|\geq\frac18\sqrt{\omega_k}$, then the stationary phase applies in both s,σ, the critical value of the phase becomes $\phi_{k,j}^{m=0,\pm_1,\pm_2}$ and we conclude as in Proposition 21. Let $|s|\leq\frac18\sqrt{\omega_k}$ or $|\sigma|\leq\frac18\sqrt{\omega_k}$, then $|s+\sigma|\leq\frac54\sqrt{\omega_k}$ and $|s-\sigma|\leq\frac54\sqrt{\omega_k}$.

Let first $|t|\geq 20a$, then the phase is stationary in $\tilde\rho$ if $2^{-j}|y|\sqrt{1-|\Theta'|^2}\sim|t|(2^{-4j/3}\omega_k)^{1/2}$: as a consequence, the stationary phase can applies in Θ': indeed, the phase is stationary in $\tilde\rho$ when

$$2^{-j}|y|\sqrt{1-|\Theta'|^2}+t\frac{2^{-2j}\tilde\rho+\frac23 2^{-4j/3}\tilde\rho^{1/3}\omega_k}{\sqrt{2^{-2j}\tilde\rho^2+2^{-4j/3}\tilde\rho^{4/3}\omega_k}}-(s+\sigma)\omega_k=0, \tag{108}$$

where $|s+\sigma|\leq\frac54\sqrt{\omega_k}$. Since j_0 is large (recall that $2^{j_0-j}\geq t^{(1-\epsilon)/2}$) and so is j (recall that we are dealing with $j\leq j_0\leq 3(j+2)$), the middle term in (108) equals

$$|t|\frac{2^{-2j}\tilde\rho+\frac23 2^{-4j/3}\tilde\rho^{1/3}\omega_k}{\sqrt{2^{-2j}\tilde\rho^2+2^{-4j/3}\tilde\rho^{4/3}\omega_k}}=\frac23|t|\tilde\rho^{-1/3}\sqrt{2^{-4j/3}\omega_k}\Big(1+\frac{\tilde\rho^{2/3}}{(2^{2j/3}\omega_k)}+O((2^{2j/3}\omega_k)^{-2})\Big).$$

As $a\in 2^{2j_0/3}[\frac12,2]$, $2^{2(j_0-j)/3}\geq\frac14\omega_k$ and $\tilde\rho\in[\frac34,2]$, we have

$$\begin{aligned}|t|\frac{2^{-2j}\tilde\rho+\frac23 2^{-4j/3}\tilde\rho^{1/3}\omega_k}{\sqrt{2^{-2j}\tilde\rho^2+2^{-4j/3}\tilde\rho^{4/3}\omega_k}}&\geq\frac23\times 20a\times 2^{-2j/3}\sqrt{\omega_k}\times\tilde\rho^{-1/3}\\&\geq\frac23\times 20\times\frac12 2^{2j_0/3}\times 2^{-2j/3}\sqrt{\omega_k}\times\tilde\rho^{-1/3}\\&\geq\frac{20}{3\times4\times\tilde\rho^{1/3}}\omega_k^{3/2}\geq(5/3)2^{-1/3}\omega_k^{3/2},\end{aligned}$$

and as $(5/3)\times 2^{-/3}\sim 1,32>5/4$ it follows that (108) can hold only if $2^{-j}|y|\sim|t|(2^{-4j/3}\omega_k)^{1/2}\gtrsim M$, so the stationary phase applies in Θ' as the parameter is sufficiently large. Moreover, for $|t|\geq 20a$, the second order derivative in $\tilde\rho$ is $\sim|t|(2^{-4j/3}\omega_k)^{1/2}$, so the stationary phase applies also in $\tilde\rho$ and the estimates we obtain are exactly as in (106).

Let now $t\leq 20a$ and $|s|,|\sigma|\leq\frac98\sqrt{\omega_k}$ with $|s+\sigma|\leq\frac54\sqrt{\omega_k}$. Notice that in this case the stationary phase in Θ' may not apply at all as the parameter $2^{-j}|y|$ may remain small. After the change of variable $\rho=2^{-j}\tilde\rho$, the sum $\sum_{j\leq j_0\leq 3(j+2)}G^{m=0,\#}_{j,1-\chi_0}(t,\cdot)$ can be bounded as follows

$$\begin{aligned}\sum_{j\leq j_0\leq3(j+2)}G^{m=0,\#}_{j,1-\chi_0}(t,\cdot)\leq&\sum_{j\leq j_0\leq3(j+2)}2^{-(d-1)j-2j/3}\int_0^\infty\tilde\rho^{d-2+2/3}\psi_2(\tilde\rho)\\&\times\Big|\sum_{k\sim 2^{j_0-j}}\frac{1}{L'(\omega_k)}Ai(x(2^{-j}\tilde\rho)^{2/3}-\omega_k)Ai(a(2^{-j}\tilde\rho)^{2/3}-\omega_k)\Big|d\tilde\rho\\\lesssim&\sum_{j\leq j_0\leq3(j+2)}2^{-(d-1)j-2j/3}2^{(j_0-j)/3}\leq\frac{2\times7^{d-1}}{t^{\frac{d-1}{2}}},\end{aligned} \tag{109}$$

where we have used that $k\sim 2^{j_0-j}$ on the support of $\tilde\psi(\frac{\omega_k}{2^{2(j_0-j)/3}})$ and then applied the Cauchy-Schwarz inequality followed by (63) with $L\sim 2^{j_0-j}$ to obtain the first sum in the second line. To deduce the last inequality we have used $|t|\leq 20a\leq$

$40 \times 2^{2j_0/3}$ which gives $2^{-j_0(d-1)/3} < \frac{7^{d-1}}{|t|^{(d-1)/2}}$ together with the fact that the sum over j is taken over $j_0/3 - j \leq 5/3$ so it is convergent. □

6.0.2. *The Klein-Gordon flow.* Let now $m = 1$, then the term in brackets in (96) is

$$f_{k,j}(\tilde{\rho}^{2/3}) := \frac{2}{9}(2^{-4j/3}\omega_k)\tilde{\rho}^{-2/3}\Big(1-(2^{-4j/3}\omega_k)\tilde{\rho}^{4/3}\Big) - \frac{1}{9}(2^{-4j/3}\omega_k)2^{-2j}\tilde{\rho}^{4/3} + 2^{-2j}. \tag{110}$$

The following expansions will be useful (see [14, (2.52), (2.64)])): $\omega_1 = 2.3381074105$, $\omega_4 = 6.7867080901$, and for $j \geq 2$, $\omega_{k=2^{2j}} = (\frac{3\pi}{8})^{2/3} \times 2^{4(j+1)/3}(1 + O(2^{-2(j+1)}))$.

We now separate two different situations: the first one was alluded to in the introduction and uncovers a new effect that leads to a worse decay estimate for Klein-Gordon. The second case is dealt with as we did for the wave equation.

1. Let $j = 0$ and $k = 1$: we claim that $\partial^2_{\tilde{\rho}}\phi^{m=1}_{0,1}$ may vanish for some $\tilde{\rho}$ near 1 for all t. Let $j = 0$ and $k = 1$ and let $z = \tilde{\rho}^{2/3}$: the function $f_{1,0}(z) := 1 + \frac{2}{9}z^{-1}\omega_1 - \frac{1}{9}\omega_1 z^2 - \frac{2}{9}z\omega_1^2$ satisfies $f_{1,0}(1) = 1 + \frac{1}{9}\omega_1 - \frac{2}{9}\omega_1^2 \in (0, 04, 0, 05)$ and $f'_{1,0}(z) < 0$ for all $z > 0$, hence $f_{0,1}$ is strictly decreasing, so the second derivative of $\phi^{m=1}_{1,0}$ does cancel for some $\tilde{\rho}$ very close to 1. In the same way, for each $j \geq 1$, there exists at most one value $k(j) \geq 2$, $k(j) \sim 2^{2j}$, such that $f_{k,j}(z)$ vanishes for some z near 1. We have $2^{-4j/3}\omega_{k(j)} \sim 1$.

   Let $j \geq 0$ and $k = k(j)$ and let $a \sim 2^{2j_0/3}$ for some $j_0 \geq 0$. As $a2^{-2j/3} \lesssim \omega_{k(j)} \sim 2^{4j/3}$, we must have $j_0 \leq 3j + c_0$ for some fixed c_0 depending only on the support of the ψ_2, ϕ. If t is large but such that $t \leq M_1 a$ for some $M_1 > 2$, then $2^{-2j_0/3} \leq M_1/t$ and we can proceed as in (109) : the change of variables $\rho = 2^{-j}\tilde{\rho}$ yields a factor $2^{-(d-1)j-2j/3}$ as in (109). Using (63), the sum over $k \sim 2^{2j}$ of the Airy factors yields in turn a factor $2^{2j/3}$. Writing

$$2^{-(d-1)j} = 2^{-(d-1)(j-j_0/3)} \times 2^{-(d-1)j_0/3} \lesssim 2^{-(d-1)(j-j_0/3)}t^{-(d-1)/2},$$

   and using that the sum over j is taken for $j > j_0/3$, provide bounds like $t^{-(d-1)/2}$ for the sum over $j, k(j)$. We are left with the case $t > M_1 a$. Writing the Airy factors under their integral form yields phase functions as in (107), where $\phi^{m=0}_{k,j}$ is replaced by $\phi^{m=1}_{k,j}$ and where $k = k(j)$. The first order derivative of this new phase with respect to $\tilde{\rho}$ equals $\partial_{\tilde{\rho}}\phi^{m=1}_{k(j),j} + \frac{2}{3}as2^{-2j/3}\tilde{\rho}^{-1/3} + \frac{2}{3}a\sigma 2^{-2j/3}\tilde{\rho}^{-1/3}$ and using $s^2, \sigma^2 \leq 2\omega_{k(j)} \lesssim 2^{4j/3}$ it follows that the absolute value of last two terms in this derivative are bounded by $c_1 a$ and $c_1 x$ for some fixed constant $c_1 > 1$; moreover, we recall that, by symmetry of the Green function (and its spectral localizations), we can assume $x \leq a$. The first order derivative of $\phi^{m=1}_{k,j}$ is given in (95) and for $k = k(j)$, the factor of t is $\sim 2^{-4j/3}\omega_k \sim 1$. Therefore, if M_1 is chosen sufficiently large, then the phase may be stationary in $\tilde{\rho}$ when $2^{-j}|y| \sim t$; in particular, as t is large, the stationary phase with respect to θ applies. Near $\tilde{\rho}$ such that $\partial^2_{\tilde{\rho}}\phi^{m=1}_{k(j),j} = 0$ we have $\partial^3_{\tilde{\rho}}\phi^{m=1}_{k(j),j} \sim t$ and using again that $t > M_1 a \geq M_1 x$ with M_1 large implies that the third order derivative behaves like t. Applying Van der Corput lemma yields a factor $t^{-1/3}$. For each j we have $\frac{1}{L'(\omega_{k(j)})} \sim \frac{1}{\sqrt{\omega_{k(j)}}} \sim 2^{-2j/3}$. Eventually, the sum over j and $k \sim 2^{2j}$ may be bounded by $Ct^{-(d-2)/2-1/3}$ for some uniform constant $C > 0$.

Notice that we cannot do better than that for $m = 1$: using (9) yields a sum over $N \sim t$ and as t is large, the number of waves which provide important contributions is proportional to t, which yields a loss worse than the one obtained using the gallery modes.

2. For $j \geq 0$ and $k \neq k(j)$, then $\partial_\rho^2 \phi_{j,k}^{m=1} \neq 0$ on the support of ψ_2. This situation can be dealt with exactly as in the case of the wave flow and provide the same kind of bounds.

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