Analysis of the geometrical effects on dispersive equations

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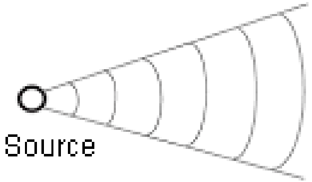
Mathematical description of waves, at least to first approximation, is the same in many different settings.

- **The (scalar) wave equation**

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)u = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^d, \quad \text{with } \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

- A close relative is **the Schrödinger equation** (especially in the ”semi-classical” setting $h \to 0$) :

$$ih\partial_t v + h^2 \Delta v = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^d.$$



Dispersive decay is a **quantitative version** of this picture.

My main contributions

- **Inside convex domains:** Schrödinger & wave equations

  - **optimal** dispersive estimates (for both equations) :

    - ⋆ approximate solutions going over infinitely many caustics;
    - ⋆ sharp pointwise bounds for the Green function;
    - ⋆ long time estimates in the Friedlander domain for wave and Klein-Gordon equations.

  - Strichartz estimates : positive results (**better** than expected from dispersion) and **counterexamples** (worst than without boundary)

- **In exterior domains:** Schrödinger & wave equations

  - **sharp** dispersion in **3D**, **sharp** Strichartz for Schrödinger $\forall D$;

  - **counterexamples** to dispersion in higher dimensions $D \geq 4$:

    - ⋆ highlight strong diffractive effects.

The dispersive estimates measure the uniform decay properties of the evolution flow as a function of time.

- The wave flow : $h \in (0,1)$, $\chi \in C_0^\infty(1/2,2)$

$$\sup \left| \chi(hD_t) e^{\pm it|\sqrt{-\Delta_{\mathbb{R}^d}}}(\delta_{Q_0}) \right| \lesssim \frac{1}{h^d} \min \left\{ 1, \left(\frac{h}{|t|} \right)^{\frac{d-1}{2}} \right\}$$

- The semi-classical Schrödinger flow (for $t \geq h$)

$$\sup \left| \chi(hD_t) e^{\pm i \frac{t}{h} \Delta_{\mathbb{R}^d}}(\delta_{Q_0}) \right| \lesssim \frac{1}{h^d} \min \left\{ 1, \left(\frac{h}{|t|} \right)^{\frac{d}{2}} \right\}.$$

Q_0 **= the source point,** t **= the elapsed time,** $1/h$ **= the frequency.**

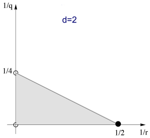
For waves, this holds in $\mathbb{R}^d$ or on manifolds without boundary *as long as time is less than the injectivity radius*.

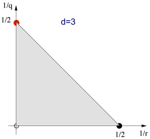
The Strichartz estimates measure average decay (L^2 data).

Admissible indices (q,r) : $q,r \geq 2$, $(q,r,\alpha) \neq (2,\infty,1)$, $\frac{1}{q} \leq \alpha(\frac{1}{2}-\frac{1}{r})$.

- for the wave flow : $(\partial_t^2 - \Delta)u = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$

$$h^{(d-\alpha)(\frac{1}{2}-\frac{1}{r})}\|\chi(hD_t)u\|_{L^q([0,T],L^r_x)} \lesssim \|u_0\|_{L^2} + \|hu_1\|_{L^2}.$$

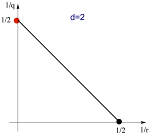


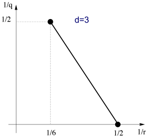


(q,r) is a wave-admissible pair if $\alpha_{W,d} = \frac{d-1}{2}$.

- for the semi-classical Schrödinger : $ih\partial_t v + h^2\Delta v = 0$, $v|_{t=0} = v_0$

$$h^{(d-\alpha)(\frac{1}{2}-\frac{1}{r})}\|\chi(hD_t)v\|_{L^q([0,T],L^r_x)} \lesssim \|v_0\|_{L^2}.$$





(q,r) is a Schrödinger-admissible pair if $\alpha_{S,d} = \frac{d}{2}$.

$\mathbb{R}^d$ with flat metric (wave and Schrödinger): Strichartz, Pecher, Ginibre-Velo, Lindblad-Sogge, Keel-Tao...
$\partial\Omega = \emptyset$ (wave): Kapitanski, Mockenhaupt-Seeger-Sogge, Smith, Bahouri-Chemin, Tataru...

$\partial\Omega = \emptyset$ (Schrödinger): Staffilani-Tataru, Burq-Gérard-Tzvetkov...

Overall view of the mathematical challenges

Goal :

- Study wave dispersion and concentration near a boundary (in **highly non-trivial** geometries)
- Develop **new tools** (efficient wave packets methods, sharp quantitative refinements of propagation of singularities, etc);
- Apply to **nonlinear** problems, **control** theory, etc.

Approximate solutions in domains with boundary

$\simeq$ 1980 : Boundary problems and propagation of singularities

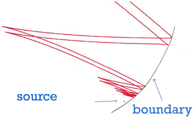
- available approximate solutions Melrose-Taylor, Eskin
- or microlocal energy methods Melrose-Sjöstrand, Ivrii

- do **NOT** provide an accurate description of the amplitude of the wave ... have **NO** use in obtaining dispersion
- do **NOT** capture the separation of optimal wave packets

Geometry of the wavefront

Propagation of a spherical wave: singularities are located on **the sphere of radius t**, centered at the source point (like in the picture).

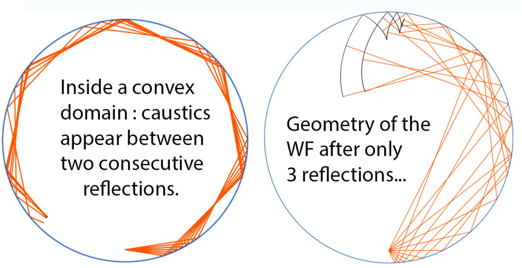
If non-empty boundary: the "sphere" of radius t = **WAVEFRONT** may undergo dramatic changes compared to the flat case !



Part of the **wavefront** near a point of strict convexity after only 5 reflections.

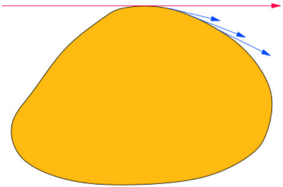
- Possible **dispersive estimates** should reflect **the geometry** of the domain and especially its **boundary**.

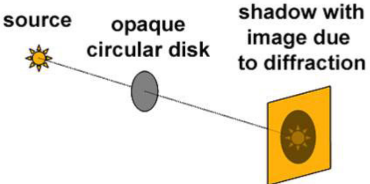
⋆ **The wavefront inside a convex** : the "sphere" of radius t soon **degenerates** and develops singularities in arbitrarily small times.



The wave **shrinks** in size between two consecutive reflections and its **maximum increases**.

⋆ **Near a concave boundary** : rays can stick to the boundary and re-release energy near the "shadow region", producing **diffractive effects** (e.g. the Poisson-Arago spot).





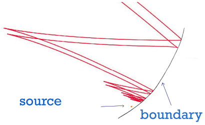
State of the art in 2010 : general domains

$\star$ $\Omega = \mathbb{R}^d \setminus \Theta$, Θ non-trapping, Δ_D : Burq-Gérard-Tzvetkov, Robbiano-Zuily
(in connection with local smoothing or local energy decay)

$\star$ Ω compact, $\partial\Omega \neq \emptyset$: Smith-Sogge, Koch-Tataru, Anton, Blair-Smith-Sogge, etc

$\Rightarrow$ reduction to the boundary**-less** case with **Lipschitz** metric across an interface :

- may handle **any** boundary (higher order tangency points ...)
- **BUT blind by design** to the full effect of dispersion !
  (reduces to wave packets that cross the boundary only once)



Dispersion in the exterior of a ball

Theorem : ([I. & Lebeau], 2020) Let $\Omega_d = \mathbb{R}^d \setminus B_d(0,1)$.

- If $d = 3$, the **dispersive estimates** for the wave and Schrödinger equations inside Ω_3 with Dirichlet condition **hold true**.
- If $d \geq 4$, these estimates fail at the **Poisson-Arago spot** .

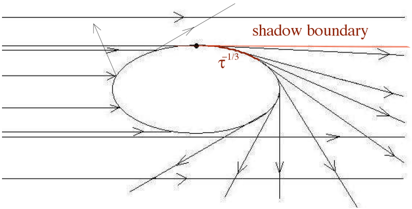
⋆ Recall : Strichartz (without loss) for waves [Smith & Sogge, 1995] and for Schrödinger [I., 2010]. For dispersion, [Li,Smith & Zhang, 2012] outside a ball, only for spherically symmetric data.

⋆ For $d \geq 4 \Rightarrow$ first example of a domain on which global Strichartz estimates do hold like in $\mathbb{R}^d$ while dispersion fails.

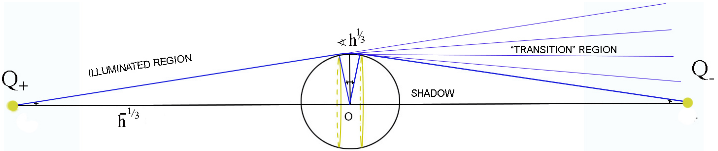
⋆ A loss in dispersion occurs only for obstacles which "look very much" like a sphere (at least viewed from specific locations).

⋆ For **wave and Schrödinger** equations, the mathematical landscape is now well understood.

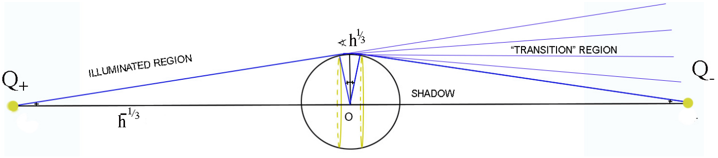
Diffraction ???



Theorem: ([Hargé & Lebeau], 1994 - Keller's theorem for C^∞ boundary) The **decreasing rate** in the shadow region is of the form $e^{-C\tau^{1/3}}$, $C = C(\partial\Omega)$, $\tau \sim$ frequency.



Estimates at the Poisson spot



if $Q_{\pm}(r) =$ **source** / **observation** points at (same) distance r from the ball, symmetric w.r.t. the center of the unit ball $B_d(0,1)$ of $\mathbb{R}^d$, then

- **Wave flow:** take $r \sim h^{-1/3}$, $t \sim 2h^{-1/3}$

$$\left|(\chi(hD_t)e^{i2h^{-1/3}\sqrt{|\Delta|}}(\delta_{Q_-})\right|(Q_+) \sim \frac{1}{h^d}\left(\frac{h}{2h^{-1/3}}\right)^{-\frac{d-1}{2}} h^{-\frac{d-3}{3}},$$

- **(classical) Schrödinger flow:** take $r \sim h^{-1/6}$, $t \sim h^{1/3}$

$$\left|(\chi(hD_t)e^{ih^{1/3}\Delta}(\delta_{Q_-})\right|(Q_+) \sim (h^{1/3})^{-\frac{d}{2}} h^{-\frac{d-3}{6}}.$$

Construction of a parametrix outside a convex in $3D$

⋆ If the source Q_0 is "far" and the observation point Q is "close" to $\partial\Omega$

- use [Melrose-Taylor], [Zworski] : yields a parametrix for Q near the apparent contour of Q_0 (see [Smith-Sogge, 1995], [Zworski,1990]).

Theorem [Melrose & Taylor], [Zworski] : $\exists \theta, \zeta$ phase functions **near a glancing point**, $\exists\, p_0, p_1$ symbols (with p_0 elliptic, $p_1|_{\partial\Omega} = 0$) s.t.

for every solution V to $(\tau^{\mathbf{2}} + \Delta)\mathbf{V} \in \mathbf{O}_{\mathbf{C}^\infty}(\tau^{-\infty})$ $\quad \exists F$ s.t.

$$V(\tau, Q, Q_0) = T_\tau(F), \quad Q \text{ near the glancing point},$$

$$T_\tau(F)(Q, Q_0) := \left(\frac{\tau}{2\pi}\right)^2 \int e^{i\tau\theta(Q,\eta)} \left(p_0 Ai + p_1 \tau^{-1/3} Ai'\right)(\tau^{2/3}\zeta(Q,\eta))\hat{F}(\tau\eta)d\eta.$$

- apply to $V = \widehat{u_{free}}(\tau, Q, Q_0) = \frac{\tau}{|Q-Q_0|} e^{-i\tau|Q-Q_0|}$ and find $\hat{F}$;
- replace $\hat{F}$ then use it to bound the "outgoing wave"

$$\left(\frac{\tau}{2\pi}\right)^2 \int e^{i\tau\theta} \left[p_0 A_+(\tau^{2/3}\zeta) + p_1 \tau^{-1/3} A'_+(\tau^{2/3}\zeta)\right] \frac{Ai}{A_+}(\tau^{2/3}\zeta|_{\partial\Omega})\hat{F}(\tau\eta)d\eta.$$

⋆ If the source Q_0 and the observation point Q are "far" from $\partial\Omega$

- Reduce the problem to obtaining estimates for

$$U(t,Q,Q_0) := \int_{\partial\Omega} \frac{(\partial_n u_{free}|_{\partial\Omega} - \mathbf{N}(u_{free}|_{\partial\Omega}))(t-|Q-P|,P,Q_0)}{4\pi|P-Q|} d\sigma(P).$$

- use **NOW** [Melrose-Taylor] to obtain $(\partial_n u_{free}|_{\partial\Omega} - \mathbf{N}(u_{free}|_{\partial\Omega}))(t,P,Q_0)$ in terms of Airy functions for P near the apparent contour of Q_0.
- higher dimensions : for the Poisson spot, rotational symmetry.
- for a general concave boundary finding F doesn't help...

Construction of a parametrix outside a convex in $3D$

⋆ If Q_0 and Q are "very close" to $\partial\Omega$: [Melrose-Taylor] **not available**

- use spherical harmonics $Y_{m,j}$ (eigenfunctions of $-\Delta_{\mathbb{S}^2}$) to obtain

$$\chi(hD_t)U(t,Q,Q_0) = \sum_{m\geq 0} Z^m_{\frac{Q}{|Q|}}\left(\frac{Q_0}{|Q_0|}\right)\int_0^\infty e^{it\tau}\chi(h\tau)G_{m+1/2}(|Q|,|Q_0|,\tau)d\tau,$$

$$G_{m+1/2}(r,s,\tau) = \frac{\pi}{2i\sqrt{rs}}\Big(J_{m+1/2}(s\tau) - \frac{J_{m+1/2}(\tau)}{H^{(1)}_{m+1/2}(\tau)}H^{(1)}_{m+1/2}(s\tau)\Big)H^{(1)}_{m+1/2}(r\tau),$$

- most delicate situation : $\tau/m = 1 + O(m^{-2/3})$ when

  $H^{(1)}_{m+1/2}$, $J_{m+1/2}$ read in terms of $Ai(\tau^{2/3}\zeta)$ (same as in [Melrose-Taylor] ! $\Rightarrow$ discrete sum instead of integral formula)

⋆ for Schrödinger equation : use the Kanaï transform

⋆ for small frequencies : use the exterior Dirichlet problem for the Helmholtz equation and acoustic surface potentials for general C^2 boundaries.

Convex boundaries

Theorem: ([I., Lebeau & Planchon, '14]; [I., Lascar, Lebeau & Planchon, '20])
Let (Ω, g) be a strictly convex domain; $Q_a \in \Omega$ at distance $a > 0$ from $\partial\Omega$, δ_{Q_a} = Dirac at Q_a. The Dirichlet wave flow satisfies, for $h < |t| \lesssim 1$

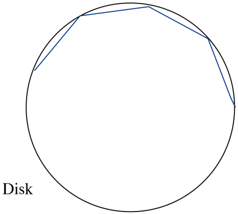
$$\|\chi(hD_t)e^{it\sqrt{-\Delta_g}}(\delta_{Q_a})\|_{L^\infty(\Omega)} \lesssim \frac{1}{h^d}\left(\frac{h}{|t|}\right)^{\frac{(d-1)}{2}}\left[a^{\frac{1}{4}}(h/|t|)^{-\frac{1}{4}} + (h/|t|)^{-\frac{1}{6}}\right].$$

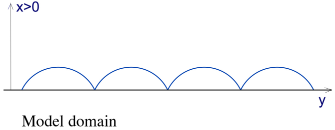
- The result is optimal because of the presence of swallowtail singularities in the wave front set.
- First result describing exactly the amplitude of the wave over infinitely many reflections

Model for convex boundaries

Disk: $r \leq 1$
$\Delta_{disk} = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2$

Model domain:
$\Omega_2 = \{(x, y) \in \mathbb{R}^2 |, x > 0, y \in \mathbb{R}\}$
$\Delta_F = \partial_x^2 + (1 + x)\partial_y^2$





Same to first order under $r = 1 - x/2$, $\theta = y$.

Model for convex boundaries

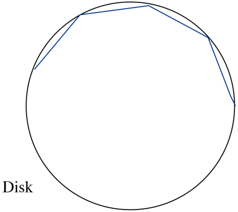
Disk: $r \leq 1$

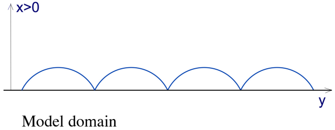
$\Delta_{disk} = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2$

Model domain:

$\Omega_2 = \{(x, y) \in \mathbb{R}^2 |, x > 0, y \in \mathbb{R}\}$

$\Delta_F = \partial_x^2 + (1 + x)\partial_y^2$





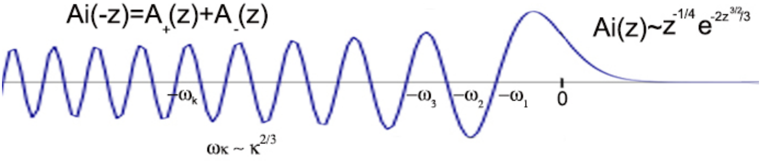
⋆ The operator $-\partial_x^2 + (1 + x)\eta^2$ has eigenfunctions and eigenvalues:

$$e_k(x, \eta) = \frac{\eta^{1/3}}{\sqrt{L'(\omega_k)}} Ai\left(\eta^{2/3}x - \omega_k\right) \text{ associated to } \lambda_k(\eta) = \eta^2 + \omega_k \eta^{4/3}$$

⋆ $(e_k)_{k \geq 1}$ forms an $L^2(0, \infty)$ orthonormal basis, $Ai(-\omega_k) = 0$.

Airy function $Ai''(z) = zAi(z)$

- Integral formula : $Ai(-z) = (2\pi)^{-1} \int e^{i(\sigma^3/3 - z\sigma)} d\sigma$.



- Let $L(\omega) = \pi + i \log \frac{A_-(\omega)}{A_+(\omega)}$ where $A_\pm(\omega) \sim \frac{1}{\omega^{1/4}} e^{\pm\frac{2}{3} i \omega^{\frac{3}{2}}}$. Then

$$L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} + \frac{\pi}{2} - O(\omega^{-\frac{3}{2}}), \quad L'(\omega_k) = \int_0^\infty Ai^2(x - \omega_k)\, dx\,.$$

- "Airy-Poisson" formula :

$$2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \phi(\omega_k) = \sum_{N \in \mathbb{Z}} \int e^{-iNL(\omega)} \phi(\omega)\, d\omega\,.$$

$$e^{-iNL(\omega)} = (-1)^N \Big(\frac{A_-(\omega)}{A_+(\omega)}\Big)^N.$$

Parametrix for $\partial_t^2 u - (\partial_x^2 + (1+x)\partial_y^2)u = 0$ for $x > 0$, $y \in \mathbb{R}$, $u|_{x=0} = 0$

- Seek $u(t,x,y) = \int e^{iy\eta}\chi(h\eta)w(t,x,\eta)d\eta$, $\chi \in C_0^\infty(\frac{1}{2}, 2)$, $h \in (0,1)$,

$$\partial_t^2 w - (\partial_x^2 - (1+x)\eta^2)w = 0, \quad w|_{x=0} = 0.$$

- If $w|_{t=0} \in L^2(0,\infty)$,

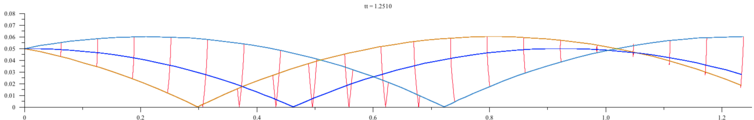
$$w(t,x,\eta) = \sum_{k\geq 1} e^{it\sqrt{\lambda_k(\eta)}} e_k(x,\eta) < e_k(\cdot,\eta), w|_{t=0} >_{L^2(0,\infty)} .$$

- Dirac distribution : $\delta_{x=a} = \sum_{k\geq 1} e_k(x,\eta)e_k(a,\eta)$, $\forall \eta \neq 0$.
- Let $w|_{t=0}(x,\eta) = \chi(h\sqrt{-\Delta_F})\delta_{x=a}$,

$$w(t,x,\eta) = \sum_{k=1}^{C/h} e^{it\sqrt{\lambda_k(\eta)}} e_k(x,\eta)e_k(a,\eta).$$

- Localize w.r.t. $(-\partial_x^2/\eta^2 + x)$: as $(-\partial_x^2/\eta^2 + x)e_k = \omega_k \eta^{-2/3} e_k$

$$\psi\Big((-\partial_x^2/\eta^2 + x)/a\Big)w(t,x,\eta) = \sum_{k=1}^{C/h} e^{it\sqrt{\lambda_k(\eta)}} e_k(x,\eta)e_k(a,\eta)\psi(\omega_k\eta^{-2/3}/a).$$



Worst packet : $x+(\xi/\eta)^2 \sim a$, small angle $|\xi/\eta| \lesssim \sqrt{a}$

Introducing $\psi(\omega_k \eta^{-2/3}/a)$ reduces the sum to

$$w_a(t,x,\eta) := \sum_{k\sim\lambda} e^{it\sqrt{\eta^2+\omega_k\eta^{4/3}}} \frac{\eta^{2/3}}{L'(\omega_k)} Ai(\eta^{\frac{2}{3}}x-\omega_k)Ai(\eta^{\frac{2}{3}}a-\omega_k).$$

As $\omega_k \sim k^{2/3}$, $\eta \sim 1/h \Rightarrow$ the sum reduces to $k \sim \lambda := \frac{a^{3/2}}{h}$.

⋆ If $a \lesssim h^{2/3}$ then $\lambda \lesssim 1$: bounded number of "gallery modes".

Using Airy-Poisson formula for w_a : $w_a(t,x,\eta) = \sum_{N\in\mathbb{Z}} w_{N,a}(t,x,\eta)$,

$$w_{N,a}(t,x,\eta) = \frac{1}{2\pi}\int e^{-iNL(\omega)+it\sqrt{\eta^2+\omega\eta^{4/3}}}\psi(\omega\eta^{-2/3}/a)$$
$$\times \eta^{2/3} Ai(\eta^{\frac{2}{3}}x-\omega)Ai(\eta^{\frac{2}{3}}a-\omega)d\omega.$$

⋆ For a general convex domain : no spectral decomposition to start with...

⋆ $a \sim h^{2/3}$: the geometry becomes irrelevant ; use gallery modes.

- For a general convex domain (Ω, g) : the range $0 < a \lesssim h^{2/3}$ requires to properly construct the "gallery modes" and prove that their decay properties are uniform with respect to their discrete parameter.
- When $0 < a < h^{1-\epsilon}$: even deciding how to chose the initial data in order the Dirichlet condition to be satisfied becomes non trivial as

$$\chi_0(hD_x)\chi(hD_y)\delta_{(a,0)} = \int e^{i((x-a)\xi + y\cdot\eta)}\chi_0(h\xi)\chi(h\eta)d\xi d\eta = \frac{1}{h^d}\widehat{\chi}_0\Big(\frac{x-a}{h}\Big)\widehat{\chi}\Big(\frac{y}{h}\Big).$$

$$\widehat{\chi}_0\Big(\frac{x-a}{h}\Big) = O(h^\infty) \quad \Longleftrightarrow \quad a > h^{1-\epsilon}.$$

Two representations for $u(t,x,y) = \int e^{iy\eta}\chi(h\eta)w(t,x,\eta)d\eta$

$\star$ $a \sim h^{2/3}$: the geometry becomes irrelevant ; use gallery modes. .

$\star$ If $a \gg h^{2/3}$: how many terms in the sum over reflexions ?

Lemma : Let $\mathcal{N}_a(t,x,y)$ denote the set of N with "significant contributions" in $u = \sum_N u_N$ (s.t. $\sum_{N \notin \mathcal{N}_a(t,x,y)} u_N = O(h^\infty)$), then $\forall t$

$$\mathcal{N}_a(t,x,y) \subset \{N \sim t/\sqrt{a}\},$$

$$\#\mathcal{N}_a(t,x,y) \sim O(1) + O(|t|h^2 a^{-7/2}).$$

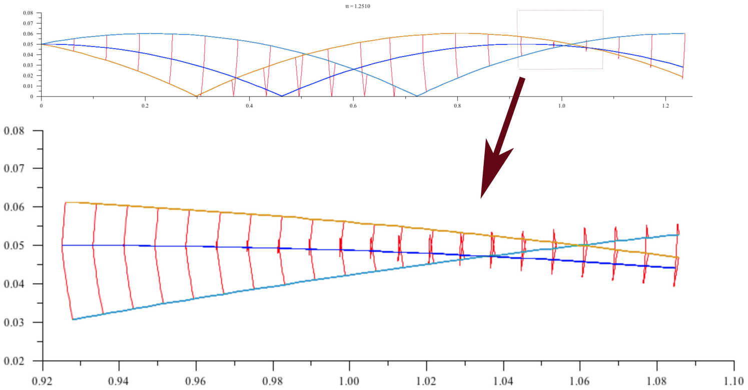
- If $a \gg h^{4/7}$ and $|t| \lesssim 1$: the $(u_N)_N$ do not "overlap much" : at fixed t, only a finite number of u_N and $\|u\|_{L^\infty} = \sup_N \|u_N\|_{L^\infty}$.
- If $h^{2/3} \ll a \lesssim h^{4/7}$: estimate each u_N and sum up all terms.
- If $a \lesssim h^{1/2}$: gain dispersion along the tangential variable.

$w_N = \eta^{\frac{2}{3}} \int e^{-iNL(\omega)+it\sqrt{\eta^2+\omega\eta^{4/3}}} \psi(\omega/(a\eta^{2/3}))A(\eta^{\frac{2}{3}}x-\omega)A(\eta^{\frac{2}{3}}a-\omega)d\omega.$

Therefore, with $\omega = \eta^{2/3}\alpha$ and $\eta = \theta/h$ we have $\theta \sim 1$, $\alpha \sim a$ and

$$u_N(t,x,y) = \frac{1}{(2\pi h)^3} \int e^{\frac{i}{h}\Phi_N} \chi(\theta)\psi(\alpha/a)\,d\sigma\, ds\, d\alpha\, d\theta.$$

$$\Phi_N := \theta\Big(y + t\sqrt{1+\alpha} + \frac{\sigma^3}{3} + \sigma(x-\alpha) + \frac{s^3}{3} + s(a-\alpha) - \frac{4}{3}N\alpha^{3/2}\Big) + l.o.t..$$



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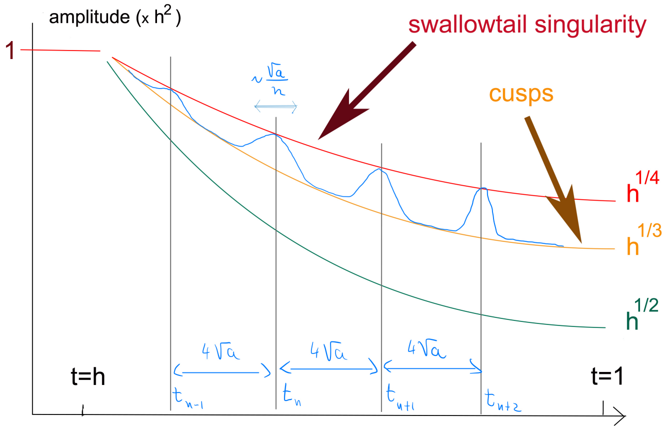
$$\Phi_N|_{\alpha_c} = \theta\Big[y + \frac{s^3}{3} + sa + \frac{\sigma^3}{3} + \sigma x - \frac{1}{24N^2}\Big(\frac{t}{2} - s - \sigma\Big)^3 + l.o.t.\Big].$$



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Therefore, with $\omega = \eta^{2/3}\alpha$ and $\eta = \theta/h$ we have $\theta \sim 1$, $\alpha \sim a$ and

$$u_N(t,x,y) = \frac{1}{(2\pi h)^3} \int e^{\frac{i}{h}\Phi_N} \chi(\theta)\psi(\alpha/a) d\sigma ds d\alpha d\theta.$$

Let $\lambda := a^{3/2}/h \gg 1$ (as $a \gg h^{2/3}$) and $T := t/\sqrt{a} \sim N \in \mathcal{N}_a(t,x,y)$.

Proposition (W1) : For $N < \lambda^{1/3}$ we have

- If $|T-4N| \lesssim 1/N$ then $|u_N(t,x,y)| \lesssim \frac{1}{h^2} \frac{h^{1/3}}{((N/\lambda^{1/3})^{1/4}+|N(T-4N)|^{1/6})}$.
- If $|T-4N| \gtrsim 1/N$ then $|u_N(t,x,y)| \lesssim \frac{1}{h^2} \frac{h^{1/3}}{(1+|N(T-4N)|^{1/2})}$.

Proposition (W2) : For $N \geq \lambda^{1/3}$, we have

- If $N < \lambda$ then $|u_N(t,x,y)| \lesssim \frac{1}{h^2} \frac{h^{1/3}}{((N/\lambda^{1/3})^{1/2}+\lambda^{1/6}|T-4N|^{1/2})}$.
- If $N \geq \lambda$ then $|u_N(t,x,y)| \lesssim \frac{1}{h^2} \frac{h^{1/3}\sqrt{\lambda/N}}{(N/\lambda^{1/3})^{1/2}}$ (gain due to integration in θ) .
- If $a > h^{1/3}$ we always have $N \lesssim \lambda^{1/3}$.

The sharp bounds for u_N yield dispersion and better Strichartz

Theorem : [I. '20]) (long time dispersion for waves, Klein-Gordon)
If $(\partial_t^2 - \Delta_F + m^2)u^m = 0$ in Ω_d, $m \in \{0,1\}$ with data $(u_0, u_1) = (\delta_{(a,0)}, 0)$,

$$|\chi(h\sqrt{-\Delta_F})u^m(t,\cdot)| \lesssim \frac{1}{h^d} \min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-1}{2}-\frac{1}{4}}\right\}. \quad (1)$$

Let $\phi \in C_0^\infty((-2,2))$ equal to 1 on $[0, \frac{3}{2}]$.

$$|\phi(\sqrt{-\Delta_F})u^{m=0}(t,\cdot)| \lesssim \min\left\{1, \frac{1}{|t|^{\frac{d-1}{2}}}\right\}. \quad (2)$$

$$|\phi(\sqrt{-\Delta_F})u^{m=1}(t,\cdot)| \lesssim \min\left\{1, \frac{1}{|t|^{\frac{d-1}{2}-\frac{1}{6}}}\right\}. \quad (3)$$

Theorem : ([I., Lebeau & Planchon, '20]) Strichartz estimates hold true on (Ω_2, g_F) for (q, r) such that

$$\frac{1}{q} \leq \left(\frac{1}{2} - \frac{1}{9}\right)\left(\frac{1}{2} - \frac{1}{r}\right).$$

In particular, $\alpha_{W,d=2} \geq \frac{d-1}{2} - \frac{1}{9}$; for $r = +\infty$, we have $q \geq 5 + 1/7$.

Remark : For $d = 2$, Strichartz estimates with $\alpha_{W,2} \geq \frac{1}{2} - \frac{1}{6}$ had been proved by [Blair, Smith & Sogge, '08]) for arbitrary boundary (no convexity assumption).

Go back to the parametrix and chose a different data $w|_{t=0}(x,\eta)$, $h\eta \sim 1$

Theorem : ([I., Lebeau & Planchon, '20]) Strichartz estimates may hold true on (Ω_2, g_F) only if

$$\frac{1}{q} \le \left(\frac{1}{2} - \frac{1}{10}\right)\left(\frac{1}{2} - \frac{1}{r}\right). \tag{4}$$

In particular, for $r = +\infty$, we have $q \ge 5$. This happen for $a \sim h^{1/3}$.

Idea of proof : let $\lambda = \frac{a^{3/2}}{h} \gg 1$, $a \gtrsim h^{1/2}$ and set, for some large $1 \ll M \ll \lambda$

$$w|_{t=0}(x, \theta/h) = \int e^{i\lambda\theta((x/a-1)\sigma+\sigma^3/3+i\sigma^2/M)} d\sigma, \quad \theta \sim 1.$$

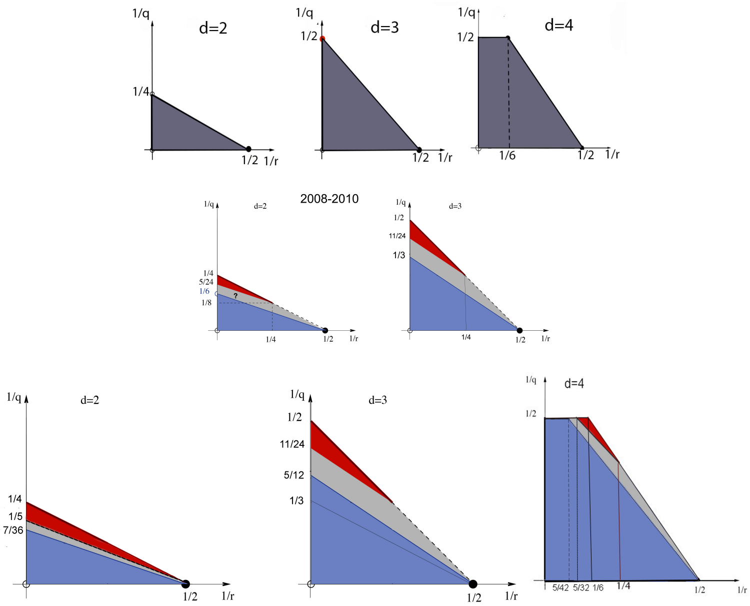
$$w|_{t=0}(0, \theta/h) = \frac{2\pi}{(\lambda\theta)^{1/3}} e^{-\frac{\lambda\theta}{2M}(1-\frac{2}{3}\frac{1}{4M^2})} Ai\Big((\lambda\theta)^{2/3}(-1 + \frac{1}{4M^2})\Big) = O(\lambda^{-\infty}).$$

Explicit L^2 norms : $\|u_0\|_{L^2(\Omega_2)} \lesssim (M/(\lambda a))^{1/4}$.

$\forall |t| \lesssim 1$, $\exists!\ N = [\frac{t}{4\sqrt{a}}]$ such that $u(t,x,y) = u_N(t,x,y) + O(\lambda^{-\infty})$

$|u(4N\sqrt{a} + t, a, y_N)| \sim \frac{2\pi}{\lambda^{1/3}} |Ai(0)|$ on a time interval of size $\frac{\sqrt{a}}{M}$ yields $\|u\|_{L^q((0,M), L^\infty(\Omega_2))} \gtrsim a^{1/(2q)} \lambda^{-1/3}$. Take $M \sim \lambda^{1/3}$.

Picture for Strichartz in convex domains



Semi-classical Schrödinger equation in (Ω_d, g_F)

Theorem : ([I.'20]) The solution to $ih\partial_t v + h^2\Delta_F v = 0$, $v|_{\partial\Omega_d} = 0$ with data $v_0(x,y) = \chi(hD_y)\delta_{x=a,y=0}$ satisfies, for $h < |t| \lesssim 1$

$$\|\chi(hD_t)v(t,\cdot)\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d}\left(\frac{h}{|t|}\right)^{\frac{d}{2}-\frac{1}{4}}. \tag{5}$$

$\forall\ \sqrt{a} < |t| \lesssim 1$ and every $|t|h^{1/3} \ll a \leq 1$, the bound saturates, as

$$\|\chi(hD_t)v(t,\cdot)\|_{L^\infty(\Omega_d)} \sim \frac{a^{\frac{1}{4}}}{h^d}\left(\frac{h}{|t|}\right)^{\frac{d}{2}-\frac{1}{4}}. \tag{6}$$

Corollary : Strichartz estimates hold for $\frac{1}{q} \leq \left(\frac{d}{2}-\frac{1}{4}\right)\left(\frac{1}{2}-\frac{1}{r}\right)$.

Remark : one gallery mode ($a \lesssim h^{2/3}$) yields a loss of $\frac{1}{6}$ in Strichartz for Schrödinger.
[Blair, Smith & Sogge '11], Ω compact :
$\frac{1}{q} \leq \left(1-\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{r}\right)$ for $d = 2$; $\frac{1}{q} \leq \left(\frac{d}{2}-\frac{d-2}{2}\right)\left(\frac{1}{2}-\frac{1}{r}\right)$ for $d \geq 3$.
Worst regime for Strichartz seems to be $a \sim h^{1/2}$.

Same construction as for waves + Kanaï transform

In the same way, for $d = 2$, $v = \sum_N v_N$ where

$$v_N(t,x,y) = \frac{1}{(2\pi h)^3} \int e^{\frac{i}{h}\tilde{\Phi}_N} \chi(\theta) \psi(\alpha/a) d\sigma ds d\alpha d\theta,$$

$$\tilde{\Phi}_N := \theta\Big(y + t\theta(1+\alpha) + \frac{\sigma^3}{3} + \sigma(x-\alpha) + \frac{s^3}{3} + s(a-\alpha) - \frac{4}{3}N\alpha^{3/2}\Big) + l.o.t..$$

⋆ Stationary phase w.r.t. θ yields $(h/|t|)^{1/2}$ and forces $|y| \sim 2|t|$.

⋆ Critical value of the phase after stationary phase w.r.t. θ, α :

$$\tilde{\Phi}_N|_{\theta_c,\alpha_c} = -\frac{y^2}{4t} - \frac{y}{2t}\Big[\frac{s^3}{3} + sa + \frac{\sigma^3}{3} + \sigma x - \frac{1}{12N^2}\Big(\frac{y}{2} + s + \sigma\Big)^3 + l.o.t.\Big].$$

Recall $\Phi_N|_{\alpha_c} = \theta\Big[y + \frac{s^3}{3} + sa + \frac{\sigma^3}{3} + \sigma x - \frac{1}{24N^2}\Big(\frac{t}{2} - s - \sigma\Big)^3 + l.o.t.\Big].$

Proposition (S1) : For $N < \lambda^{1/3}$, $N \sim Y := y/\sqrt{a}$, we have $\forall t$

- If $|Y - 4N| \lesssim 1/N$, then $|v_N(t,x,y)| \lesssim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2} \frac{h^{1/3}}{((N/\lambda^{1/3})^{1/4} + |N(Y-4N)|^{1/6})}$.
- If $|Y - 4N| \gtrsim 1/N$, then $|v_N(t,x,y)| \lesssim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2} \frac{h^{1/3}}{(1+|N(Y-4N)|^{1/2})}$.

$\star$ If $th^{1/3} \lesssim a$ then $\frac{t}{\sqrt{a}} \sim N \lesssim \lambda^{1/3} = \frac{\sqrt{a}}{h^{1/3}} \Rightarrow \forall t$, at $(x,y) = (a, 4N\sqrt{a})$:

$$
\begin{aligned}
|v(t,a,4N\sqrt{a})| &= |v_N(t,a,4N\sqrt{a}) + \sum_{M\neq N} v_M(t,a,4N\sqrt{a})| \\
&= \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2}\left(\frac{h^{1/3}}{(N/\lambda^{1/3})^{1/4}} + O(h^{1/3})\right) \\
&\sim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2}\left(a\frac{h}{|t|}\right)^{1/4}.
\end{aligned}
$$

Proposition (S2) : For $N \geq \lambda^{1/3}$, $N \sim Y := y/\sqrt{a}$, we have

- $|v_N(t,x,y)| \lesssim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2} \frac{h^{1/3}}{((N/\lambda^{1/3})^{1/2} + \lambda^{1/6}|Y - 4N|^{1/2})}$.

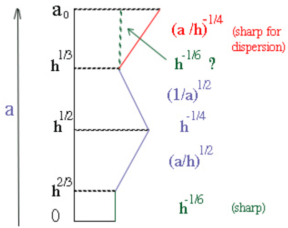
$\star$ If $th^{1/3} \gtrsim a$ then $N \gtrsim \lambda^{1/3}$ and

$$\|\sum_N v_N(t,\cdot)\|_{L^\infty(\Omega_2)} \lesssim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2}\left(\frac{ht}{a}\right)^{1/2}$$

which forces $a \gtrsim (ht)^{1/2}$.

- For $a \lesssim (ht)^{1/2}$ the spectral sum yields $|v(t,\cdot)| \lesssim \frac{1}{h^2}\left(\frac{h}{|t|}\right)^{1/2} a^{1/2}$.

- When $a \sim (ht)^{1/2} \iff N \sim \lambda \Rightarrow$ seems to be the most difficult case.



Open problems

- diffractive effects for higher order tangency points : if the ray has infinite order tangency with the boundary, even deciding what should be the continuation of a ray striking the boundary is difficult... (see [Taylor, 1976])
- general bounded domains: significant difficulties when curvature changes its sign :
  - ⋆ exhibit the separation of wave packets ;
  - ⋆ classify the type and order of caustics that may appear;
  - ⋆ saddle like boundary ; **NO** canonical model to start with.
- exterior domains : clarify if and how non-trapping rays help improving dispersive effects at large time scales;
- concentration of Laplace eigenfunctions (spectral projectors estimates);
- ...

MERCI !