DISPERSIVE ESTIMATES FOR THE WAVE EQUATION OUTSIDE A CYLINDER IN $\mathbb{R}^3$

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ABSTRACT. We consider the wave equation with Dirichlet boundary conditions in the exterior of a cylinder in $\mathbb{R}^3$ and we construct a sharp global in time parametrix to derive sharp dispersive estimates for all frequencies (low and high) and, as a corollary Strichartz estimates matching the $\mathbb{R}^3$ case.

1. General setting

We consider the linear wave equation on an exterior domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let Δ_D be the Laplacian with constant coefficients and Dirichlet boundary conditions:

$$\begin{cases} (\partial_t^2 - \Delta_D)u(x,t) = 0, \quad x \in \Omega, \\ u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x), \\ u(x,t) = 0, \quad x \in \partial\Omega\,. \end{cases} \tag{1.1}$$

Regarding solutions of the linear wave equation (1.1), a basic homogeneous local estimate says that on any smooth Riemannian manifold (Ω, g) *without* boundary, the wave flow satisfies (for $T < \infty$)

$$\|u\|_{L^q(0,T)L^r(\Omega)} \leq C_T\big(\,||u_0||_{\dot{H}^\beta(\Omega)} + ||u_1||_{\dot{H}^{\beta-1}(\Omega)}\big)\,, \tag{1.2}$$

where $\beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ and the pair (q,r) is wave-admissible, i.e such that $\frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}$ and $(q,r,d) \neq (2,\infty,3)$. Here $\dot{H}^\beta$ denotes the homogeneous L^2 Sobolev space over Ω. If (1.2) holds for $T = \infty$, Strichartz estimates are said to be global. Such inequalities were established long ago for Minkowski space (flat metrics) and can be generalized to any smooth Riemannian manifold (Ω, g) because of the local character of the estimate (finite propagation speed). They are known to be sharp on every Riemannian manifold (Ω, g) with $\partial\Omega = \emptyset$.

The aforementioned results for $\mathbb{R}^d$ and manifolds without boundary have been understood for sometime. Euclidean results go back to R.Strichartz's pioneering work [18], where he proved the particular case $q = r$ for the wave and Schrödinger equations. This was later generalized to mixed $L^q_t L^r_x$ norms by J.Ginibre and G.Velo [4] for Schrödinger equations, where (q,r) is sharp admissible and $q > 2$; wave estimates were obtained independently by J.Ginibre and G.Velo [5] and H.Lindblad and C.Sogge [7], following earlier work by L.Kapitanski [8]. Endpoint cases for both equations were finally settled by M.Keel and T.Tao [9]. On manifolds without boundary, by finite speed of propagation it suffices to work in coordinate charts and to establish estimates for variable coefficients operators in $\mathbb{R}^d$. For operators with $C^{1,1}$ coefficients, Strichartz estimates were shown by H.Smith [15] (see also D.Tataru [19] for C^α coefficients of the metric).

The canonical path leading to such Strichartz estimates is to obtain a stronger, fixed time, dispersion estimate, which is then combined with energy conservation, interpolation and a duality argument to obtain (1.2). Let $(\Omega, g) = (\mathbb{R}^d, (\delta_{i,j}))$ and $e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}$ be the half-wave propagators and $\chi \in C_0^\infty(]0,\infty[)$. The following dispersion inequality holds:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\{1,(h/|t|)^{\frac{d-1}{2}}\}. \tag{1.3}$$

The authors were supported by ERC grant ANADEL 757 996.
Key words : Dispersive estimates, wave equation, Dirichlet boundary condition.
AMS subject classification : 35Rxx, 58Jxx.

Our aim in the present paper is to prove dispersion for (1.1) when $\partial\Omega$ is a cylinder in $\mathbb{R}^3$: introducing cylindrical coordinates in $\mathbb{R}^{\nVDash}$, our domain becomes $\Omega = \{(r,\theta,z), r \geq 1, \theta \in [0,2\pi), z \in \mathbb{R}\}$ and $\Delta_D = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial_r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$. With h a small parameter and $\tau = h\partial_t/i$, $\eta = h\partial_y/i$, $\xi = h\partial_x/i$, $\vartheta = h\partial_z/i$, the characteristic set of $\partial_t^2 - \Delta_D$ is $\tau^2 = \xi^2 + \frac{1}{r^2}\eta^2 + \vartheta^2$ and the boundary is $\{r = 1\}$. Recall that in [2], the authors have constructed a global in time parametrix for the wave equation outside a ball in $\mathbb{R}^3$, which allowed them to obtain sharp dispersive bounds. In the particular case of [2] the model domain was $\{(r,\theta,\omega), r \geq 1, \theta \in [0,\pi), \omega \in [0,2\pi)\}$ and the Laplace operator was given by $\Delta_F := \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial_r} + \frac{1}{r^2}(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\omega^2})$. The main difficulty came from rays that hit the boundary without being deviated (corresponding to $\xi = 0$, $\eta = 1$ and r near 1; in fact, due to the rotational symmetry, in the exterior of the ball the characteristic equation is $\xi^2 + \frac{1}{r^2}\eta^2 = \tau^2$) : for this regime, the most efficient tool is the Melrose - Taylor parametrix, as it provides us with the form of the solution to (1.1) near diffractive points $\xi = 0$, $r = 1$ (recall that this parametrix was first used by H.Smith and Ch.Sogge in [17] to obtain sharp Strichartz bounds for waves). In the case of the exterior of a cylinder, the "diffractive regime" would correspond to $(\eta/\tau)^2 + (\vartheta/\tau)^2 = 1$, $\xi = 0$, $r = 1$ (instead of $(\eta/\tau)^2 = 1$, $\xi = 0$, $r = 1$ of [2]) : it turns out that when ϑ/τ is very close to 1 the Melrose and Taylor parametrix fails to apply (essentially because one cannot perform any kind of stationary phase arguments anymore in the oscillatory integrals that allow to obtain such a parametrix). In particular, the situation $\vartheta/\tau = 1$ correspond to rays that (start and) remain close to the boundary for all time and at our knowledge has been encountered only in [10] where the author studies dispersive bounds for (1.1) in the interior of a cylindrical domain $\{(r,\theta,z), r \leq 1, \theta \in [0,2\pi), z \in \mathbb{R}\} \subset \mathbb{R}^3$ with Dirichlet laplacian $\Delta_D = \partial_r^2 + (2-r)\partial_\theta^2 + \partial_z^2$ (and obtains a "sharp loss" of $1/4$ due to swallowtail type singularities in the wave front set) ; notice however that in [10] the time is bounded (as in very long time one would have serious trouble with the case $\vartheta = 0$), so when $\vartheta/\tau \sim 1$ the estimates follow easy by Sobolev embedding (and a parametrix is naturally obtained in terms of a sum gallery modes). In our situation of the exterior of a cylinder, the parametrix is global in time, which makes this situation $\vartheta/\tau \sim 1$ more difficult (and the case $1 - \vartheta/\tau \sim 2^{-j}$ already very delicate when compared to the exterior of a ball).

Theorem 1.1. *Let $\Theta \subset \mathbb{R}^3$ be the infinite cylinder in $\mathbb{R}^3$ and set $\Omega = \mathbb{R}^3 \setminus \Theta$. Let Δ_D denote the Laplace operator in Ω with Dirichlet boundary condition and let $\chi \in C_0^\infty(0,\infty)$.*
Dispersion holds for the wave flow in Ω like in $\mathbb{R}^3$:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_D}}\|_{L^1(\Omega)\to L^\infty(\Omega)} \lesssim h^{-3}\min\{1, \frac{h}{|t|}\}. \tag{1.4}$$

Theorem 1.2. *Under the assumptions of Theorem 1.1, the Strichartz estimates for the wave flow outside a cylinder in $\mathbb{R}^3$ hold as in the flat case.*

We can assume, without loss of generality that $\mathrm{dist}(Q_0, \partial\Omega) \geq \mathrm{dist}(Q, \partial\Omega)$ indeed, when this is not the case we can use the symmetry of the Green function to change Q_0 and Q.

1.0.1. *The fundamental solution in $\mathbb{R}^3$.* Let Δ denote the Laplacian in $\mathbb{R}^3$ and consider

$$\begin{cases} (\partial_t^2 - \Delta)U_{free} = F, \;\; x \in \mathbb{R}^3, \\ U_{free}|_{t=0} = u_0, \; \partial_t U_{free}|_{t=0} = u_1. \end{cases} \tag{1.5}$$

Proposition 1.3. *The solution to (1.5) reads, for all $t \neq 0$, as*

$$U_{free}(t) = \partial_t R(t) * u_0 + R(t) * u_1 + \int_0^t R(t-s) * F(s)ds,$$

where the Fourier transform in space of R is given by $\widehat{R(t,\xi)} = \frac{\sin(t|\xi|)}{|\xi|}$.

Let $Q_0 \in \mathbb{R}^3$ and let $u_{free}(Q, Q_0, t)$ denote the solution to the free wave equation (1.5) with $F = 0$, $u_0 = \delta_{Q_0}$ and $u_1 = 0$, where δ_{Q_0} is the Dirac distribution at Q_0. Using Proposition 1.3, $u_{free}(Q, Q_0, t)$ is given by :

$$u_{free}(Q, Q_0, t) := \frac{1}{(2\pi)^3}\int e^{i(Q-Q_0)\xi}\cos(t|\xi|)d\xi. \tag{1.6}$$

Let $w_{in}(Q,Q_0,\tau) := \widehat{1_{t>0}u_{free}}(Q,Q_0,\tau)$ denote the Fourier transform in time of $u_{free}(Q,Q_0,t)|_{t>0}$, then the following holds :

$$w_{in}(Q,Q_0,\tau) = \frac{i\tau}{4\pi}\frac{e^{-i\tau|Q-Q_0|}}{|Q-Q_0|}. \tag{1.7}$$

1.0.2. *The Neumann operator.* Let $\Theta \subset \mathbb{R}^3$ be the infinite cylinder in $\mathbb{R}^3$ and set $\Omega := \mathbb{R}^3 \setminus \Theta$. We define $K : \mathcal{D}'(\mathbb{R}_+ \times \partial\Omega) \to \mathcal{D}'(\mathbb{R}_+ \times \Omega)$ as follows: for a distribution $f \in \mathcal{D}'(\mathbb{R} \times \partial\Omega)$ with support in $\{t \geq 0\}$, $K(f)$ is defined as the solution to the linear wave equation with data on the boundary equal to f, that is

$$K(f) = g, \quad \text{where} \quad \begin{cases} (\partial_t^2 - \Delta)g = 0, & x \in \Omega, \\ \text{supp}(g) \subset \{t \geq 0\}, & g|_{\partial\Omega} = f. \end{cases} \tag{1.8}$$

We then define the Neumann operator $\mathcal{N} : \mathcal{D}'(\mathbb{R}_+ \times \partial\Omega) \to \mathcal{D}'(\mathbb{R}_+ \times \partial\Omega)$ as

$$\mathcal{N}(f) = \partial_\nu K(f)|_{\partial\Omega}, \tag{1.9}$$

where $\vec{\nu}$ is the outward unit normal to $\partial\Omega$ pointing towards Ω.

1.0.3. *The wave equation outside a cylinder in* $\mathbb{R}^3$. We introduce a cylindrical coordinates system as follows : a point of Q of Ω with coordinates $(x_1,x_2,x_3) \in \mathbb{R}^3$ is defined by (r,θ,z) where $r > 1$, $\theta \in [0,2\pi)$ and $z \in \mathbb{R}$ and where

$$x_1 = r\cos(\theta), \quad x_2 = r\sin(\theta), \quad x_3 = z,$$

Consider the equation (1.1) with initial data $(\delta_{Q_0}, 0)$, where $Q_0 \in \Omega$ is an arbitrary point

$$\begin{cases} (\partial_t^2 - \Delta_D)u = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u|_{t=0} = \delta_{Q_0},\ \partial_t u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0. \end{cases} \tag{1.10}$$

Let $u(Q,Q_0,t) = \cos(t\sqrt{-\Delta_D})(\delta_{Q_0})(Q)$ denote the solution to (1.10). Let u_{free} be the solution to the free wave equation (1.5) with $F = 0$, $u_0 = \delta_{Q_0}$ and $u_1 = 0$ given in (1.6). The Fourier transform in time of $u^+_{free} = 1_{t>0}u_{free}$ equals w_{in} defined in (1.7). As $Q_0 \notin \partial\Omega$, for any sufficiently small time $|t| \ll d(Q_0,\partial\Omega)$, the solution to (1.10) in Ω is just u_{free}. From (1.8), $u^+ := 1_{t>0}u$ reads as a sum of the incoming wave, u^+_{free}, and the reflected wave, $K(-u^+_{free}|_{\partial\Omega})$,

$$u^+ = u^+_{free} - K(u^+_{free}|_{\partial\Omega}). \tag{1.11}$$

Moreover, $\partial_\nu u^+|_{\partial\Omega} = \partial_\nu u^+_{free}|_{\partial\Omega} - \mathcal{N}(u^+_{free}|_{\partial\Omega})$. We also introduce

$$\underline{u}(Q,Q_0,t) := \begin{cases} u(Q,Q_0,t), \text{ if } Q \in \Omega, \\ 0, \text{ if } Q \in \overline{\Theta}. \end{cases} \tag{1.12}$$

Then, using Duhamel formula, $\underline{U}$ reads as follows

$$\underline{u}|_{t>0} = u^+_{free} - \Box_+^{-1}\Big((\partial_\nu u^+)|_{\partial\Omega}\Big), \tag{1.13}$$

where $\Box_+^{-1}F(t) = \int_{-\infty}^t R(t-t') * F(t')dt'$ if $\text{supp}(F) \subset \{t' \geq 0\}$. Using the explicit form of R yields

$$\Box_+^{-1}\Big((\partial_\nu U^+)|_{\partial\Omega}\Big)(Q,Q_0,t) = \frac{1}{4\pi}\int_{\partial\Omega}\frac{\partial_\nu U^+(P,Q_0,t-|Q-P|)}{|Q-P|}d\sigma(P). \tag{1.14}$$

Let $h_0 \in (0,1)$ be sufficiently small, let $h \in (0,h_0)$ and let $\chi \in C_0^\infty([\frac{1}{2},2])$ be a smooth cutoff equal to 1 on $[\frac{3}{4},\frac{3}{2}]$ and such that $0 \leq \chi \leq 1$. As we are interested in evaluating $\chi(hD_t)\underline{u}(Q,Q_0,t)$, let

$$u^\# := \Box_+^{-1}\Big(\partial_\nu u^+|_{\partial\Omega}\Big), \quad u^+_{free,h} := \chi(hD_t)u^+_{free}, \quad u^\#_h := \chi(hD_t)u^\#(Q,Q_0,t). \tag{1.15}$$

As the (frequency localized) free wave flow $u_{free,h}$ does satisfy the usual dispersive estimates, in order to prove Theorems 1.1 for the wave equation and large frequencies we are reduced to evaluating $\chi(hD_t)u^+(Q,Q_0,t) := u^+_h(Q,Q_0,t)$ or $u^\#_h(Q,Q_0,t)$. Using (1.14), we have

Proposition 1.4. *For $Q \in \Omega$, we have*

$$u_h^{\#}(Q, Q_0, t) = \int e^{it\tau}\chi(h\tau)\int_{P\in\partial\Omega}\mathcal{F}(\partial_\nu u^+|_{\partial\Omega})(P, Q_0, \tau)\frac{1}{4\pi|Q-P|}e^{-i\tau|Q-P|}d\sigma(P)d\tau, \quad (1.16)$$

where $\mathcal{F}(\partial_\nu u^+|_{\partial\Omega})(P, Q_0, \tau)$ denotes the Fourier transform in time of $\partial_\nu u^+|_{\partial\Omega}(P, Q_0, t)$.

As the estimates in Theorems 1.1 are global in time, to achieve its proof we need to consider also the case of small frequencies ($h \sim 1$), situation that will be dealt with separately.

2. High-frequency case. Parametrix for (1.1) when $d(Q_0, \partial\Omega) \geq \varepsilon_0$

In this section we consider $d(Q_0, \partial\Omega) \geq \varepsilon_0$ for fixed $\varepsilon_0 = \sqrt{2} - 1$. We consider the source point to be of the form $Q_0 = (s, 0, 0)$, where $s - 1 > \varepsilon_0$ represents the distance from Q_0 to the boundary and where $r_{Q_0} = s$, $\theta_{Q_0} = 0$, $z_{Q_0} = 0$. Let Q be an arbitrary point of Ω, then $Q := (r\cos\theta, r\sin\theta, z)$. We introduce the distance between Q and Q_0 as follows

$$\tilde{\phi}(r, \theta, z, s) := |Q - Q_0| = \sqrt{r^2 - 2sr\cos\theta + s^2 + z^2}. \quad (2.1)$$

For a source point Q_0 as above, we define its apparent contour $\mathcal{C}_{Q_0}$ as the set of points $P \in \partial\Omega$ such that the ray Q_0P is tangent to $\partial\Omega$: in other words,

$$\mathcal{C}_{Q_0} := \{P \in \partial\Omega \text{ with coordinates } (1, \theta, z) \text{ such that } \partial_r\tilde{\phi}(1, \theta, z, s) = 0\}.$$

As $\partial_r\tilde{\phi} = (r - s\cos\theta)/\tilde{\phi}$ cancels on the boundary $r = 1$ when $\cos\theta = \frac{1}{s}$, we find $\mathcal{C}_{Q_0} := \{P = (1, \arccos(1/s), z), z \in \mathbb{R}\}$. In the following we set $\theta_* := \arccos(1/s) = \frac{\pi}{2} - \arcsin(1/s)$.

Choice of coordinates. In a general setting, let $(x, Y) \in \mathbb{R}_+ \times \partial\Omega$ be normal coordinates such that $x \to (x, Y)$ is the ray orthogonal to $\partial\Omega$ at $Y \in \partial\Omega$. Any point in $Q \in \Omega$ can be written under the form $Q = Y + x\vec{\nu}_Y$, where Y is the orthogonal projection of Q on $\partial\Omega$ and $\vec{\nu}_Y$ the outward unit normal to $\partial\Omega$ pointing towards Ω. The dual variable to (x, Y) is denoted (ξ, Ξ). In this coordinate system, the principal symbol of $-\Delta$ on $T^*(\mathbb{R}^3)$ is written $<(\xi, \Xi), (\xi, \Xi)>_g := \xi^2 + |\Xi|_g^2$, where $|\Xi|_g^2 = \sum_{i,j} g^{i,j}(x, Y)\Xi_i\Xi_j =: r(x, Y, \Xi)$; here the coefficients $g^{i,j}$ belong to a bounded set of C^∞ and the principal part is uniformly elliptic. The principal symbol of $\partial_t^2 - \Delta$ is given by $p(x, Y, \xi, \Xi, \tau) := -\tau^2 + \xi^2 + r(x, Y, \Xi)$. The time variable and its dual are t and τ. We let $\mathcal{Q} = \{(x, Y, t, \xi, \Xi, \tau), x = 0\}$, $\mathcal{P} = \{(x, Y, t, \xi, \Xi, \tau), p(x, Y, \xi, \Xi, \tau) = 0\}$. The cotangent bundle of $\partial\Omega \times \mathbb{R}$ is the quotient of $\mathcal{Q}$ by the action of translation in ξ, and we take as coordinates (Y, t, Ξ, τ). A point $(Y, t, \Xi, \tau) \in T^*(\partial\Omega \times \mathbb{R})$ is classified as one of three distinct types: it is said to be *hyperbolic* if $\tau^2 > r(0, Y, \Xi)$, so that there are two distinct nonzero real solutions ξ to $p(0, Y, \xi, \Xi, \tau) = 0$. These two solutions yield two distinct bicharacteristics, one of which enters Ω as t increases (the *incoming ray*) and one which exits Ω as t increases (the *outgoing ray*). The point is *elliptic* if $\tau^2 < r(0, Y, \Xi)$, so there are no real solutions ξ to $p(0, Y, \xi, \Xi, \tau) = 0$. In the remaining case $\tau^2 = r(0, Y, \Xi)$, there is a unique solution to $p(0, Y, \xi, \Xi, \tau) = 0$ which yields a glancing ray, and the point is said to be a *glancing point.*

Definition 2.1. *A point $(0, Y_0, t_0, \xi_0, \Xi_0, \tau_0) \in \mathcal{Q} \cap \mathcal{P} \subset T^*(\partial\Omega \times \mathbb{R})$ is called a glancing point if the bicharacteristic passing through it is a glancing ray. This is equivalent to the condition $\partial_\xi(\xi^2 + r(x, Y, \Xi))|_{(0,Y_0,\xi_0,\Xi_0,\tau_0)} = 0$, that is $\xi_0 = 0$.*

Remark 2.2. *If the glancing ray has exactly second order with the boundary we have moreover $H_px|_{(0,Y_0,\xi_0,\Xi_0,\tau_0)} = \partial_x(\xi^2 + r(x, Y, \Xi))|_{(0,Y_0,\xi_0,\Xi_0,\tau_0)} = \partial_x r(0, Y_0, \Xi_0) \neq 0$. Otherwise we can also define the glancing set of order at least $k \geq 2$ by the equations*

$$p = 0, \quad H_p^j x = 0, \text{ for } 0 \leq j \leq k,$$

where H_p is the Hamiltonian associated to p. Notice that the convexity or concavity of the boundary with respect to tangential bicharacteristics leads to drastic differences in the behaviour of the broken bicharacteristic flow (see [6, Chapter 24.3] *for details).*

For our domain we introduce the following normal coordinate system

$$r = 1 + x, \ x \geq 0; \ y := \pi/2 - \theta, \theta \in [0, 2\pi), z \in \mathbb{R}. \quad (2.2)$$

In these (normal) cylindrical coordinates, our Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{1}{(1+x)}\frac{\partial}{\partial_x} + \frac{1}{(1+x)^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2.3}$$

In the general notations we have $Y = (y, z)$, $\Xi = (\eta, \vartheta)$ and the symbol associated to (2.3) is $p(x, \xi, \eta, \vartheta, \tau) = -\tau^2 + \xi^2 + (1 + x)^{-2}\eta^2 + \vartheta^2$, $r(x, \eta, \vartheta) = (1 + x)^{-2}\eta^2 + \vartheta^2$, and the glancing condition becomes

$$\vartheta^2 + \frac{\eta^2}{(1+x)^2} + \xi^2 = \tau^2, \quad x = 0, \quad \xi = 0. \tag{2.4}$$

If a glancing ray has exactly second order contact with the boundary we have in addition $\eta^2 \frac{d}{dx}(1+x)^{-2}|_{x=0} = -\eta^2/2 < 0$. hence $\eta \neq 0$. As $h\tau \sim 1$ on the support of $\chi(hD_t)$, we will set $\alpha = \eta/\tau$, $\gamma = \vartheta/\tau$ and the glancing condition (2.4) reads as $\alpha^2 + \gamma^2 = 1$, while the hyperbolic and elliptic cases read as $1 - \alpha^2 - \gamma^2 > 0$ and $1 - \alpha^2 - \gamma^2 < 0$, respectively. A point in $T^*(\partial\Omega \times \mathbb{R})$ such that $1 \geq \alpha^2 > 0$ may be a glancing point of order exactly two. When $\alpha = 0$, then it is a glancing point of order ∞ (as, in this case, $H_p^j x = 0$ for all $j \geq 1$).

Remark 2.3. *Notice also that, when $1-\gamma^2-\alpha^2 \geq 1/16$, then on the boundary $\xi^2 = \tau^2(1-\gamma^2-\alpha^2) \geq \tau^2/16$ in which case the corresponding point in the cotangent bundle is transverse. The proof of Theorem 1.1 for such points follows exactly as in the case of the half-space, so we will focus on the situation $1-\gamma^2-\alpha^2 \leq 1/16$, when $\xi/\tau \lesssim 1/4$.*

In the coordinates (2.2) we have $x_{Q_0} = s - 1 > \varepsilon_0$, $y_{Q_0} = \frac{\pi}{2}$, $z_{Q_0} = 0$. Also, the apparent contour reads as $\mathcal{C}_{Q_0} := \{P = (0, \arcsin(1/s), z), z \in \mathbb{R}\}$. For a point $Q \in \Omega$ we write $Q = ((1 + x)\sin y, (1 + x)\cos y, z)$ and the distance $|Q - Q_0|$ in the new coordinates becomes $\phi(x, y, z, s) := \tilde{\phi}(1 + x, y, \pi/2 - y, z, s) = \sqrt{(1 + x)^2 - 2s(1 + x)\sin y + s^2 + z^2}$. If Q is our observation point, we will assume (without loss of generality) that $y_Q \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

2.0.1. *Construction of the free and the reflected waves $u^+_{h,free}$ and u^+_h, $h \in (0, h_0)$.* We aim at obtaining dispersive bounds either for $u^\#_h(Q, Q_0, t)$ or $u^+_h(Q, Q_0, t)$ introduced in (1.15): when $dist(Q, \partial\Omega)$ is large enough, it will be convenient to bound $u^\#_h(Q, Q_0, t)$ and hence $\underline{u}_h(Q, Q_0, t)$, while when Q is close to boundary we will bound directly $u^+_h(Q, Q_0, t)$. In both cases, it is indispensable to obtain the precise form of $u^+_h = u^+_{free,h} - \chi(hD_t)K(\partial_x u^+_{free}|_{\partial\Omega})$.

Recall that $u^+_{free}(Q, Q_0, t) = \int e^{i\tau t} w_{in}(Q, Q_0, \tau)d\tau$, $u^+_{free,h} = \chi(hD_t)u^+_{free}$, with w_{in} defined in (1.7). Let $\chi_1(\beta)$ be a smooth function supported near 1, equal to 1 for $1 \geq \beta \geq 1/81$, equal to 0 for $\beta \leq 1/100$ and such that $0 \leq \chi_1 \leq 1$. We define

$$w_1(Q, Q_0, \tau) = \frac{\tau^2}{(2\pi)^2}\frac{i\tau}{4\pi}\int \frac{\chi_1(1-\gamma^2)}{\phi(x, \tilde{y}, \tilde{z}, s)} e^{i\tau(y-\tilde{y})\alpha+(z-\tilde{z})\gamma)} e^{-i\tau\phi(x,\tilde{y},\tilde{z},s)} d\alpha d\gamma d\tilde{y} d\tilde{z}, \tag{2.5}$$

whose symbol is supported for $1 - \gamma^2 \geq 1/100$. Let also $\chi \in C_0^\infty(1/4, 4)$ be equal to 1 near 1 such that $1 - \chi_1(\beta) = \sum_{j\geq 4} \chi(2^{2j}\beta)$. Write $w_{in} = w_1 + \sum_{j\geq 4} w_{2^{-2j}}$, where

$$w_{2^{-2j}}(Q, Q_0, \tau) = \frac{\tau^2}{(2\pi)^2}\frac{i\tau}{4\pi}\int \frac{\chi(2^{2j}(1-\gamma^2))}{\phi(x, \tilde{y}, \tilde{z}, s)} e^{i\tau(y-\tilde{y})\alpha+(z-\tilde{z})\gamma)} e^{-i\tau\phi(x,\tilde{y},\tilde{z},s)} d\alpha d\gamma d\tilde{y} d\tilde{z}. \tag{2.6}$$

Write $u^+_{free} = u^+_{free,1} + \sum_{j\geq 4} u^+_{free,2^{-2j}}$. We separate the cases $\sqrt{1-\gamma^2} \geq \frac{1}{10}$, $\sqrt{1-\gamma^2} \sim 2^{-j}$.

2.1. **The case $\sqrt{1-\gamma^2} \geq 1/10$.** On the support of $\chi_1(1 - \gamma^2)$ we have $\sqrt{1-\gamma^2} \geq 1/10$. We may have either $\alpha^2 \sim 1 - \gamma^2$, when the possible glancing points have exactly second order contact with the boundary or $|\alpha| \leq \frac{1}{4}\sqrt{1-\gamma^2}$. We separate these two cases :

Let $\chi_0 \in C_0^\infty([-2, 2])$ and equal to 1 on $[-\frac{3}{2}, \frac{3}{2}]$, let $\varepsilon_1 > 0$ small enough to be chosen later and set $\chi_{\varepsilon_1}(\cdot) := \chi_0((\cdot - 1)/\varepsilon_1)$. We let $w_{1,gl}$ be defined by (2.5) with additional cutoff $\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}})$ supported for

$|\frac{\alpha}{\sqrt{1-\gamma^2}}-1|\leq\varepsilon_1$. Define $w_{1,he}$ as in (2.5) with additional cutoff $1-\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}})$. We let

$$u^+_{free,1,gl}:=\int e^{it\tau}w_{1,gl}d\tau,\quad u^+_{free,1,he}:=\int e^{it\tau}w_{1,he}d\gamma,$$

then $u^+_{free,1}=u^+_{free,1,gl}+u^+_{free,1,he}$ and $u^+_{free,h,1}=\chi(hD_t)u^+_{free,1}$.

2.1.1. *The glancing part (with second order contact) of u^+_1.* We deal with $w_{1,gl}$. The goal of this section is to explicitly describe the form of $u^+_{1,gl}:=u^+_{free,1,gl}-K(u^+_{free,1,gl}|_{\partial\Omega})$.

Proposition 2.4. *Microlocally near a glancing point of exactly second order contact with the boundary there exist smooth phase functions $\iota(x,y,z,\alpha,\gamma)$ and $\zeta(x,y,z,\alpha,\gamma)$ such that $\phi_\pm=\iota\pm(-\zeta)^{3/2}$ satisfy the eikonal equation and there exist symbols a, b satisfying the transport equation such that, for any parameters α,γ in a conic neighborhood of a glancing direction,*

$$G_\tau(x,y,z,\alpha,\gamma):=e^{\mathrm{i}\tau\iota(x,y,z,\alpha,\gamma)}\Big(aA_+(\tau^{2/3}\zeta)+b\tau^{-1/3}A'_+(\tau^{2/3}\zeta)\Big)A_+^{-1}(\tau^{2/3}\zeta_0)\tag{2.7}$$

satisfies for $\tau>1$ sufficiently large

$$(\tau^2+\Delta)G_\tau=e^{\mathrm{i}\tau\iota(x,y,z,\alpha,\gamma)}\Big(a_\infty A_+(\tau^{2/3}\zeta)+b_\infty\tau^{-1/3}A'_+(\tau^{2/3}\zeta)\Big)A_+^{-1}(\tau^{2/3}\zeta_0),$$

where the symbols verify a_∞, $b_\infty\in O(\tau^{-\infty})$ and where we set $\zeta_0=\zeta|_{\partial\Omega}=\zeta(0,y,z,\alpha,\gamma)$. Moreover the phases and the symbols satisfy the following properties:

- *θ and ζ are homogeneous of degree 0 and $-1/3$ and satisfy $|\nabla(\theta\pm\frac{2}{3}(-\zeta)^{3/2})|^2_g=1$. The eikonal equation takes the form*
$$\{\ \langle d\iota,d\iota\rangle-\zeta\langle d\zeta,d\zeta\rangle=1,\quad\langle d\iota,d\zeta\rangle=0,$$
where $\langle\cdot,\cdot\rangle$ is the polarization of p;
- *the phase ζ_0 is independent of y,z so that $\zeta_0(\alpha,\gamma)$ vanishes at a glancing direction;*
- *the diffractive condition means that $\partial_x\zeta|_{\partial\Omega}<0$ in a neighborhood of the glancing point, i.e. $\zeta\leq\zeta_0$ near $\partial\Omega$ in $\Omega\times\mathbb{R}^d$;*
- *the symbols $a(x,y,z,\alpha,\gamma)$ and $b(x,y,z,\alpha,\gamma)$ belongs to the class $\mathcal{S}^0_{(1,0)}$ (defined below) and satisfy the appropriate transport equations. Moreover $a|_{x=0}$ is elliptic at the glancing point with essential support included in a small, conic neighborhood of it, while $b|_{x=0}$ is $O(\tau^{-\infty})$.*

The functions ι and ζ of the Melrose-Taylor parametrix solve the system of equations

$$\begin{cases}(\partial_x\iota)^2+\dfrac{(\partial_y\iota)^2}{(1+x)^2}+(\partial_z\iota)^2-\zeta\Big((\partial_x\zeta)^2+\dfrac{(\partial_y\zeta)^2}{(1+x)^2}+(\partial_z\zeta)^2\Big)=1,\\ \partial_x\theta\partial_x\zeta+\dfrac{\partial_y\theta\partial_y\zeta}{(1+x)^2}+\partial_z\theta\partial_z\zeta=0.\end{cases}\tag{2.8}$$

The system (2.8) admits a pair of solutions of the form

$$\iota(y,z,\alpha,\gamma)=y\alpha+z\gamma,\quad\zeta(x,\alpha,\gamma)=\alpha^{2/3}\tilde\zeta((1+x)\sqrt{1-\gamma^2}/\alpha),\tag{2.9}$$

where for $\rho:=(1+x)\frac{\sqrt{1-\gamma^2}}{\alpha}$, $\tilde\zeta$ is the (unique) solution to $\frac{1}{\rho^2}-\tilde\zeta(\rho)[\tilde\zeta'(\rho)]^2=1$, $\tilde\zeta(1)=0$.

Lemma 2.5. *The equation $-\tilde\zeta(\partial_\rho\tilde\zeta)^2+1/\rho^2=1$, $\tilde\zeta(1)=0$ has a unique solution of the form*

$$\frac{2}{3}[-\tilde\zeta(\rho)]^{3/2}=\int_1^\rho\frac{\sqrt{w^2-1}}{w}dw=\sqrt{\rho^2-1}-\arccos\Big(\frac{1}{\rho}\Big),$$

if $\rho>1$, while for $\rho<1$ we have

$$\frac{2}{3}[\tilde\zeta(\rho)]^{3/2}=\int_\rho^1\frac{\sqrt{1-w^2}}{w}dw=\log[(1+\sqrt{1-\rho^2})/\rho]-\sqrt{1-\rho^2}.$$

We note that at $\rho=1$ we have $\tilde\zeta=0$ and $\lim_{\rho\to1}\frac{(-\tilde\zeta)(\rho)}{\rho-1}=2^{1/3}$.

Let $\tilde{\chi}_1(1-\gamma^2) \in C_0^\infty$ be supported for $\sqrt{1-\gamma^2} > 1/16$, and equal to 1 on the support of $\chi_1(1-\gamma^2)$ introduced in Section 2.0.1 to defined w_1. For α, γ on the support of $\tilde{\chi}_1$ such that $|\frac{\alpha}{\sqrt{1-\gamma^2}} - 1| \leq \varepsilon_1$ and for (x, y, z) near $(0, \arcsin(1/s), z)$, Proposition 2.4 applies as the glancing rays passing through these points have second order contact with the boundary.

Corollary 2.6. *Consider the following operator* $M_\tau : \mathcal{E}'(\mathbb{R}^2) \to \mathcal{D}(\mathbb{R}^3)$

$$M_\tau(f)(x, y, z) := \Big(\frac{\tau}{2\pi}\Big)^2 \int G_\tau(x, y, z, \alpha, \gamma)\tilde{\chi}_1(1-\gamma^2)\widehat{f}(\tau\alpha, \tau\gamma)d\alpha d\gamma.$$

Near the glancing region $(\tau^2 + \Delta)M_\tau(f) \in O(\tau^{-\infty})$ *(up to the boundary) for all* $f \in \mathcal{E}'(\mathbb{R}^2)$*. Moreover, the restriction to the boundary* $M_\tau(f)|_{\partial\Omega} =: J_\tau(f)$ *defined by*

$$J_\tau(f)(y, z) = \Big(\frac{\tau}{2\pi}\Big)^2 \int e^{\mathrm{i}\tau(\iota(y,z,\alpha,\gamma)-\tilde{y}\alpha-\tilde{z}\gamma)}a(0, y, z, \alpha, \gamma, \tau)\tilde{\chi}_1(1-\gamma^2)f(\tilde{y}, \tilde{z})d\alpha d\gamma d\tilde{y}d\tilde{z},$$

has a microlocal inverse J_τ^{-1} *as* $a(x, y, z, \alpha, \gamma, \tau)$ *is the elliptic symbol of Proposition 2.4.*

As observed in [21] (in a general setting), the free solutions of $(\tau^2 + \Delta)w(., \tau) \in O_{C^\infty}(\tau^{-\infty})$, essentially supported in a conic neighborhood of a glancing point are parameterized by

$$\Big(\frac{\tau}{2\pi}\Big)^2 \int e^{i\tau\iota(y,z,\alpha,\gamma)}\Big(aA(\tau^{2/3}\zeta(x, \alpha, \gamma)) + b\tau^{-1/3}A'(\tau^{2/3}\zeta(x, \alpha, \gamma))\Big)\tilde{\chi}_1(1-\gamma^2)\widehat{F}(\tau\alpha, \tau\gamma)d\alpha d\gamma. \tag{2.10}$$

We therefore define the following operator $T_\tau : \mathcal{E}'(\mathbb{R}^2) \to \mathcal{D}(\mathbb{R}^3)$ for $F \in \mathcal{E}'(\mathbb{R}^2)$

$$T_\tau(F)(x, y, z) = \Big(\frac{\tau}{2\pi}\Big)^2 \tau^{1/3} \int e^{\mathrm{i}\tau(y\alpha+z\gamma+\frac{\sigma^3}{3}+\sigma\zeta(x,\alpha,\gamma))}(a + b\frac{\sigma}{\mathrm{i}})\tilde{\chi}_1(1-\gamma^2)\widehat{F}(\tau\alpha, \tau\gamma)d\alpha d\gamma. \tag{2.11}$$

According to [16, Lemma A.2], the operator T_τ is an elliptic FIO in a conic neighborhood of a glancing point and $(\tau^2 + \Delta)T_\tau(F) \in O_{C^\infty}(\tau^{-\infty})$. In particular, one allows $x < 0$.

Lemma 2.7. *For every solution* $w(Q, \tau)$ *to* $(\tau^2 + \Delta)w(\cdot, \tau) \in O_{C^\infty}(\tau^{-\infty})$ *near the glancing region there exists a unique* F_τ *such that* $w(\cdot, \tau) = T_\tau(F_\tau)$.

We apply Lemma 2.7 to $w_{1,gl}$. In the coordinates (2.2), recall that $(x, y, z)_{Q_0} = (s - 1, 0, 0)$ and that the apparent contour of Q_0 is defined by $\theta = \theta_* := \arccos(1/s) = \pi/2 - \arcsin(1/s)$. With $y = \pi/2 - \theta$, this becomes $\mathcal{C}_{Q_0} = \{P(x, y, z) \text{ with } x = 0, y = y_* := \arcsin(1/s), z \in \mathbb{R}\}$.

Lemma 2.8. *Let* $Q_0(s - 1, 0, 0)$ *as before with* $s \geq \sqrt{2}$, $y_* = \arcsin(1/s)$ *and assume* $\tau > 1$ *is large enough. Then there exists an unique function* $F_{1,\tau}$ *satisfying* $w_{1,gl}(x, y, z, \tau) = T_\tau(F_{1,\tau})(x, y, z)$ *for* (x, y) *in a neigborhood of* $(0, y_*)$*. It has the following form*

$$\widehat{F}(\tau\alpha, \tau\gamma) = \tau^{\frac{1}{6}}e^{-\mathrm{i}\tau\sqrt{1-\gamma^2}\Gamma_0(\frac{\alpha}{\sqrt{1-\gamma^2}},s)}f(\alpha, \gamma, \tau)\frac{\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}})\tilde{\chi}_1(1-\gamma^2)}{(1-\gamma^2)^{5/12}(s^2-1)^{1/4}}, \tag{2.12}$$

where $f(\alpha, \gamma, \tau)$ *is an elliptic symbol of order* 0*,* $\tilde{\chi}_1(1-\tau^2)$ *is the same as in Corollary 2.6 and* χ_{ε_1} *is the smooth cut-off supported in a* ε_1*-neighborhood of* 1 *introduced in the beginning of this section. The phase function* Γ_0 *reads as* $\Gamma_0(\tilde{\alpha}, s) = y_*\tilde{\alpha} + \sqrt{s^2-1} + \frac{(1-\tilde{\alpha})^2}{2\sqrt{s^2-1}}(1 + O(1-\tilde{\alpha}))$.

The proof of Lemma 2.8 is postponed to Section 2.1.2. Our goal is to describe, microlocally near a bicharacteristic tangent to the boundary, $u^+_{1,gl,h} := \chi(hD_t)u^+_{1,gl}$, where $u^+_{1,gl}(\cdot, t) := u^+_{free,1,gl}(\cdot, t) - K(t)(u^+_{free,1,gl}|_{\partial\Omega})$, where the outgoing part of the parametrix to the Dirichlet problem reads as

$$K(t)(u^+_{free,1,gl}|_{\partial\Omega}) = \int e^{\mathrm{i}t\tau}M_\tau \circ J_\tau^{-1}(w_{1,gl}|_{\partial\Omega})d\tau.$$

With this setting we can state the following proposition.

Proposition 2.9. *For* $Q = (x, y, z)$ *near the glancing region we have*

$$u^+_{1,gl}(Q, Q_0, t) = \frac{1}{(2\pi)^2}\int e^{\mathrm{i}t\tau}\big(w_{1,gl}(x, y, z, \tau) - M_\tau(J_\tau^{-1}(w_{1,gl}|_{\partial\Omega})(x, y, z)\big)d\tau,$$

where, near the glancing regime, and for $F_{1,\tau}$ provided by Lemma 2.8, $w_{1,gl}(\cdot,\tau) = T_\tau(F_{1,\tau})$ and $M_\tau(J_\tau^{-1}(w_{1,gl}|_{\partial\Omega}))$ reads as

$$\Big(\frac{\tau}{2\pi}\Big)^2 \int e^{i\tau(y\alpha+z\gamma)} \Big(aA_+(\tau^{2/3}\zeta) + b\tau^{-1/3}A'_+(\tau^{2/3}\zeta)\Big) \frac{A(\tau^{2/3}\zeta_0)}{A_+(\tau^{2/3}\zeta_0)}\tilde{\chi}_1(1-\gamma^2)\widehat{F}_{1,\tau}(\tau\alpha,\tau\gamma)d\alpha d\gamma.$$

Corollary 2.10. *For $P=(0,y,z)\in\partial\Omega$ near $\mathcal{C}_{Q_0}$ we have*

$$\begin{aligned}\mathcal{F}(\partial_x u^+_{1,gl})(P,Q_0,\tau) = \Big(\frac{\tau}{2\pi}\Big)^2 \tau^{5/6} \int e^{i\tau(y\alpha+z\gamma-\sqrt{1-\gamma^2}\Gamma_0(\frac{\alpha}{\sqrt{1-\gamma^2}},s))} \\ \times\, \tilde{b}_\partial f \frac{\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}})\tilde{\chi}_1(1-\gamma^2)}{(s^2-1)^{1/4}} \frac{(1-\gamma^2)^{-5/12+1/3}}{A_+(\tau^{2/3}\zeta_0(\alpha,\gamma))} d\alpha d\gamma,\end{aligned} \tag{2.13}$$

where $\zeta_0(\alpha,\gamma) = \alpha^{2/3}\tilde{\zeta}(\sqrt{1-\gamma^2}/\alpha)$ with $\tilde{\zeta}$ defined in Lemma 2.5. Here, as $\tilde{\alpha} = \frac{\alpha}{\sqrt{1-\gamma^2}} \sim 1$, $\tilde{b}_\partial(y,z,\tilde{\alpha},\tau)$ is an elliptic symbol of order 0 in τ with main contribution $\frac{a_0}{\tilde{\alpha}^{1/3}}(\partial_\rho\tilde{\zeta})(\rho)|_{\rho=\frac{1}{\tilde{\alpha}}}$, that reads as an asymptotic expansion with small parameter $(\tau\sqrt{1-\gamma^2})^{-1}$.

Proof. As $\partial_x u_1^+|_{\partial\Omega} = \partial_x u^+_{free,1}|_{\partial\Omega} - \partial_x(K(t)(u_{free,1}|_{\partial\Omega}))|_{\partial\Omega} = \partial_x u^+_{free,1}|_{\partial\Omega} - \mathcal{N}(u^+_{free,1}|_{\partial\Omega})$, we compute the normal derivatives of each term using Proposition 2.9 and then take the difference. As such, for $P=(0,y,z)$ near the glancing region, we obtain the following

$$\mathcal{F}(\partial_x u^+_{1,gl})(P,Q_0,\tau) = \Big(\frac{\tau}{2\pi}\Big)^2 \int e^{i\tau(y\alpha+z\gamma)}\tau^{2/3} b_\partial \Big(A' - A'_+\frac{A}{A_+}\Big)(\tau^{2/3}\zeta_0)\widehat{F}_{1,\tau}(\tau\alpha,\tau\gamma)d\alpha d\gamma, \tag{2.14}$$

where $b_\partial = a(0,y,z,\alpha,\gamma)(\partial_x\zeta)(0,\alpha,\gamma) + \tau^{-1}\partial_x b(0,y,z,\alpha,\gamma)$ and where $a|_{x=0}$ is elliptic and for $\tilde{\alpha} = \frac{\alpha}{\sqrt{1-\gamma^2}} \sim 1$ on the support of χ_{ε_1}, $\partial_x\zeta|_{\partial\Omega} = \sqrt{1-\gamma^2}^{2/3}\frac{1}{\tilde{\alpha}^{1/3}}(\partial_\rho\tilde{\zeta})(\rho)|_{\rho=\frac{1}{\tilde{\alpha}}}$. Let $\tilde{b}_\partial(y,z,\tilde{\alpha},\tau) := \frac{a_0}{\tilde{\alpha}^{1/3}}(\partial_\rho\tilde{\zeta})(\rho)|_{\rho=\frac{1}{\tilde{\alpha}}}$, then $\tilde{b}_\partial$ is elliptic, close to 1 on the support of the symbol and $b_\partial = \sqrt{1-\gamma^2}^{2/3}\tilde{b}_\partial + O(\tau^{-1})$. Replacing $\widehat{F}_{1,\tau}$ by (2.12) and using the Wronskian relation $A'(z)A_+(z) - A'_+(z)A(z) = ie^{-i\pi/3}$ allows to conclude. ☐

2.1.2. *Explicit computation of $F_{1,\tau}$. Proof of Lemma 2.7.* In this part we obtain the unique function $F_{1,\tau}$ given in Lemma 2.7. Notice that we may follow closely the approach in [2, Section 3.1.1] (as the glancing order contact is exactly 2) to provide a detailed proof. Instead, we use the explicit form of $w_{1,gl}$ and write it under the form (2.11). After the changes of variables $\alpha = \sqrt{1-\gamma^2}\tilde{\alpha}$ and $\tilde{z} = \phi(x,\tilde{y},0,s)z_1$, we write $w_{1,gl}(Q,Q_0,\tau)$ as follows

$$\frac{\tau^2}{(2\pi)^2}\frac{i\tau}{4\pi}\int \frac{\chi_1(1-\gamma^2)}{\sqrt{1+z_1^2}}\chi_{\varepsilon_1}(\tilde{\alpha})\sqrt{1-\gamma^2}e^{i\tau(z\gamma+\sqrt{1-\gamma^2}(y-\tilde{y})\tilde{\alpha}-\phi(x,\tilde{y},0,s)(z_1\gamma+\sqrt{1+z_1^2}))}d\tilde{\alpha}d\gamma d\tilde{y}d\tilde{z}_1.$$

The critical point w.r.t. z_1 satisfies $\gamma + \frac{z_1}{\sqrt{1+z_1^2}} = 0$ and there the second order derivative equals $\frac{\phi(x,\tilde{y},0,s)}{\sqrt{1+z_1^2}^3} = \phi(x,\tilde{y},0,s)\sqrt{1-\gamma^2}^3$. The stationary phase applies (as long as $\tau\sqrt{1-\gamma^2}^3 \geq \tau^\epsilon$ for some $\epsilon>0$; in our case $\epsilon=1$) and yields, for some constant $C\neq 0$ and modulo $O(\tau^{-\infty})$,

$$w_{1,gl}(Q,Q_0,\tau) = C\tau^{2+1/2}\int \frac{\chi_1(1-\gamma^2)\chi_{\varepsilon_1}(\tilde{\alpha})}{\phi^{1/2}(x,\tilde{y},0,s)}(1-\gamma^2)^{\frac12+\frac12-\frac34}e^{i\tau(z\gamma+\sqrt{1-\gamma^2}((y-\tilde{y})\tilde{\alpha}-\phi(x,\tilde{y},0,s)))}d\tilde{\alpha}d\gamma d\tilde{y}.$$

Next we show that, for $\tilde{\alpha}$ near 1, we can perform a suitable change of variable w.r.t. $\tilde{y}$ such that the phase $\tilde{y}\tilde{\alpha} + \phi(x,\tilde{y},0,s)$ transforms into an Airy type phase function of the form $\sigma^3/3 + \sigma\tilde{\zeta}(\frac{1+x}{\tilde{\alpha}}) + \Gamma_0(\tilde{\alpha},s)$, where $\tilde{\zeta}$ is the function defined in (2.9). Let $\varphi(x,\tilde{y},\tilde{\alpha},s) := \tilde{y}\tilde{\alpha} + \phi(x,\tilde{y},0,s)$. As $\partial_{\tilde{y}}\phi(x,\tilde{y},0,s) = -\frac{s(1+x)\cos\tilde{y}}{\phi}$, $\partial^2_{\tilde{y}}\phi(x,\tilde{y},0,s) = \frac{s(1+x)\sin\tilde{y}-(\partial_{\tilde{y}}\phi)^2}{\phi}$, then $\partial^2_{\tilde{y}}\varphi(x,\tilde{y},\tilde{\alpha},s) = 0$ when $\tilde{y} = y_*(x) := \arcsin\big(\frac{1+x}{s}\big)$ and there $\partial_x\phi(x,\tilde{y},0,s)_{|y_*(x)} = 0$ and $\partial_{\tilde{y}}\phi(x,\tilde{y},0,s)_{|y_*(x)} = -(1+x)$. For $\tilde{y}$ near $y_*(x)$ there are two critical points, denoted $y_\pm(x)$ and satisfying

$$s(1+x)\sin(y_\pm(x)) = \tilde{\alpha}^2 \pm \sqrt{s^2-\tilde{\alpha}^2}\sqrt{(1+x)^2-\tilde{\alpha}^2} \tag{2.15}$$

and there we can explicitly compute $\phi(x,y_\pm(x),0,s) = \sqrt{s^2-\tilde{\alpha}^2} \mp \sqrt{(1+x)^2-\tilde{\alpha}^2}$.

Lemma 2.11. *Let $\tilde y = y_*(x) + Y$. There exists a unique change of variables $\mathtt{Y} \mapsto \sigma$ which is smooth and satisfying $\frac{d\mathtt{Y}}{d\sigma} \notin \{0, \infty\}$ such that, for $\tilde\zeta$ given by Lemma 2.5, we have*

$$\varphi(x, y_*(x) + \mathtt{Y}, \tilde\alpha, s) = \frac{\sigma^3}{3} + \sigma\tilde\alpha^{2/3}\tilde\zeta(\frac{1+x}{\tilde\alpha}) + \Gamma_0(\tilde\alpha, s), \tag{2.16}$$

and where $\Gamma_0(\tilde\alpha, s) := \sqrt{s^2-1} + \arcsin(\frac{1}{s})\tilde\alpha + \frac{(1-\tilde\alpha)^2}{2\sqrt{s^2-1}}(1 + O(1-\tilde\alpha))$.

Proof. As the phase φ has degenerate critical points of order exactly two, it follows from [3] that there exists a unique change of variables $\mathtt{Y} \mapsto \sigma$ which is smooth and satisfying $\frac{d\mathtt{Y}}{d\sigma} \notin \{0, \infty\}$ and that there exist smooth functions $\zeta^\#(x, \tilde\alpha, s)$ and $\Gamma(x, \tilde\alpha, s)$ such that

$$\varphi(x, y_*(x) + \mathtt{Y}, \tilde\alpha, s) = \frac{\sigma^3}{3} + \sigma\zeta^\#(x, \tilde\alpha, s) + \Gamma(x, \tilde\alpha, s). \tag{2.17}$$

As the change of coordinates is regular the critical points $\mathtt{Y}_\pm := y_\pm(x) - y_*(x)$ of φ must correspond to the critical points of the new phase function, i.e. $\sigma_\pm = \pm\sqrt{-\zeta^\#(x, \tilde\alpha, s)}$. Write $\zeta^\#(x, \tilde\alpha, s) := \tilde\alpha^{\frac{2}{3}}\tilde\zeta^\#(\frac{1+x}{\tilde\alpha}, \tilde\alpha, s)$. We will show that $\tilde\zeta^\#$ satisfies the same equation as $\tilde\zeta$ in (2.5). As the critical values of the two functions in (2.17) must coincide we have

$$\varphi(x, y_*(x) + \mathtt{Y}_\pm, \tilde\alpha, s) = \mp\frac{2}{3}(-\zeta^\#)^{\frac{3}{2}}(x, \tilde\alpha, s) + \Gamma(x, \tilde\alpha, s), \tag{2.18}$$

from which we deduce the following

$$\frac{4}{3}\tilde\alpha(-\tilde\zeta^\#)^{\frac{3}{2}}(\frac{1+x}{\tilde\alpha}, x, s) = \varphi(x, y_-, \alpha, \gamma, s) - \varphi(x, y_+, \tilde\alpha, s). \tag{2.19}$$

Taking the derivative with respect to x in the equation (2.19) yields (with $y_\pm = y_*(x) + \mathtt{Y}_\pm$)

$$\begin{aligned} 2(-\partial_x\tilde\zeta^\#)(-\tilde\zeta^\#)^{\frac{1}{2}} =& \partial_x\phi(x, y_*(x) + \mathtt{Y}_-, 0, s) - \partial_x\phi(x, y_*(x) + \mathtt{Y}_+, 0, s) \\ & - \partial_x y_+\partial_y\varphi(x, y_+(x), \tilde\alpha, s) + \partial_x y_-\partial_y\varphi(x, y_-(x), \tilde\alpha, s). \end{aligned} \tag{2.20}$$

The last two terms in the second line of (2.20) vanish as $y_\pm(x)$ are the critical points of the function φ with respect to y ; for the same reason we have that $\partial_y\phi(x, y_\pm(x), 0, s) = -\tilde\alpha$. As $\phi(x, y, 0, s)$ satisfies the eikonal equation $(\partial_x\phi)^2(x, y, 0, s) + \frac{1}{(1+x)^2}(\partial_y\phi)^2(x, y, 0, s) = 1$, then $(\partial_x\phi(x, y_\pm(x), 0, s))^2 = 1 - \frac{\tilde\alpha^2}{(1+x)^2}$. Moreover, $\partial_x\phi_{|y_\pm} = \frac{s}{\phi(x, y_\pm, 0, s)}(\tilde\rho - \sin(y_\pm))$ (with $\tilde\rho = \frac{1+x}{s}$) which is non positive in the "y_+ case" and positive in the "y_- case". Eventually we obtain, using (2.20) and the right signs of $\partial_x\phi$, $-\tilde\zeta^\#[-\partial_x\tilde\zeta^\#]^2 = 1 - \frac{\tilde\alpha^2}{(1+x)^2}$, which is the same equation as in Lemma 2.5 with $\rho = \frac{1+x}{\tilde\alpha} = (1+x)\frac{\sqrt{1-\gamma^2}}{\alpha}$. As the degenerate critical point occurs at $\sigma = 0$, hence at $\zeta^\# = 0$, we deduce by uniqueness of the solution that $\tilde\zeta^\# = \tilde\zeta = \tilde\zeta(\frac{1+x}{\tilde\alpha})$.

Next, we compute the explicit form of the function $\Gamma(x, \tilde\alpha, s)$. Taking the sum in (2.18) gives $\Gamma(x, \tilde\alpha, s) = \frac{1}{2}(\varphi(x, y_+(x), \tilde\alpha, s) + \varphi(x, y_-(x), \alpha, \gamma, s))$; taking the derivative w.r.t. x yields $\partial_x\Gamma(x, \tilde\alpha, s) = 0$. As such, Γ is independent of x and we define $\Gamma_0(\tilde\alpha, s) := \Gamma(0, \tilde\alpha, s)$, then

$$\Gamma_0(\tilde\alpha, s) = \frac{1}{2}\big((y_+ + y_-)\tilde\alpha + \phi(0, y_+, 0, s) + \phi(0, y_-, 0, s)\big),$$

where $y_\pm = y_\pm(0)$. For small $x \geq 0$ and for y in a neighborhood of $y_* = y_*(0)$, y remains sufficiently close to $y_*(x)$: shrinking the support if necessary, we may assume $|y - y_*(x)| < 1/2$. For $|y_\pm - y_*| < 1/2$ we may compute, using (2.15) with $x = 0$, the first approximation of $y_\pm$: we have

$$y_\pm = \arcsin\Big(\frac{\tilde\alpha^2}{s} \pm \sqrt{1-\tilde\alpha^2}\sqrt{1 - \frac{\tilde\alpha^2}{s^2}}\Big), \quad y_* = \arcsin(\frac{1}{s}). \tag{2.21}$$

As $\Gamma_0(\tilde\alpha, s) = \frac{1}{2}(\varphi(0, y_+, \tilde\alpha, s) + \varphi(0, y_-, \tilde\alpha, s))$ and $\partial_y\varphi|_{y_\pm} = 0$ we get $\partial_{\tilde\alpha}\Gamma_0 = \frac{1}{2}(y_+ + y_-) + \frac{1}{2}\sum_\pm \partial_{\tilde\alpha}y_\pm\partial_y\varphi|_{y_\pm} = \frac{1}{2}(y_+ + y_-)$. This yields $\Gamma_0(1, s) = \sqrt{s^2-1} + \arcsin\frac{1}{s}$ and $\partial_{\tilde\alpha}\Gamma_0(1, s) = \arcsin(1/s)$. We need the higher order derivatives : using (2.21), it follows that $(y_+ + y_-)$ reads as an asymptotic expansion of even powers of $\sqrt{1-\tilde\alpha^2}$ and with main term $\arcsin(\frac{\tilde\alpha^2}{s})$. From (2.21), we find, with $Z_\pm = \frac{\tilde\alpha^2}{s} \pm \sqrt{1-\tilde\alpha^2}\sqrt{1 - \frac{\tilde\alpha^2}{s^2}}$,

$Z_\pm|_{\tilde\alpha=1}=\frac{1}{s}$

$$\frac12\partial_{\tilde\alpha}(y_++y_-)=\frac{\tilde\alpha}{s}\Big(\frac{1}{\sqrt{1-Z_+^2}}+\frac{1}{\sqrt{1-Z_-^2}}\Big)-\frac{\tilde\alpha(s^2+1-2\tilde\alpha^2)}{2s^2\sqrt{1-\tilde\alpha^2}\sqrt{1-\frac{\tilde\alpha^2}{s^2}}}\Big(\frac{1}{\sqrt{1-Z_+^2}}-\frac{1}{\sqrt{1-Z_-^2}}\Big).$$

As $\Big(\frac{1}{\sqrt{1-Z_+^2}}-\frac{1}{\sqrt{1-Z_-^2}}\Big)=\frac{Z_+^2-Z_-^2}{\sqrt{1-Z_+^2}\sqrt{1-Z_-^2}(\sqrt{1-Z_+^2}+\sqrt{1-Z_-^2})}$ and $Z_+^2-Z_-^2=4\frac{\tilde\alpha^2}{s}\sqrt{1-\tilde\alpha^2}\sqrt{1-\frac{\tilde\alpha^2}{s^2}}$,

$$\frac12\partial_{\tilde\alpha}(y_++y_-)=\frac{\tilde\alpha}{s}\Big(\frac{1}{\sqrt{1-Z_+^2}}+\frac{1}{\sqrt{1-Z_-^2}}\Big)-\frac{2\tilde\alpha^3(s^2+1-2\tilde\alpha^2)}{s^3\sqrt{1-Z_+^2}\sqrt{1-Z_-^2}(\sqrt{1-Z_+^2}+\sqrt{1-Z_-^2})}.$$

At $\tilde\alpha=1$ we find $\partial^2_{\tilde\alpha}\Gamma_0(1,s)=\frac12\partial_{\tilde\alpha}(y_++y_-)|_{\tilde\alpha=1}=\frac{1}{\sqrt{s^2-1}}$. In the same way we notice that all the higher order derivatives of Γ_0 come with a factor $\frac{1}{\sqrt{s^2-1}}$. The proof is achieved. □

After the changes of coordinates $\tilde y=y_*(x)+Y$, $Y\to\sigma$, $\sigma=(\tau\sqrt{1-\gamma^2})^{-1/3}\tilde\sigma$ we obtain $w_{1,gl}(Q,Q_0,\tau)$ as follows (with $Y=Y(\sigma)=Y((\tau\sqrt{1-\gamma^2})^{-1/3}\tilde\sigma)$)

$$\tau^{2+\frac12-\frac13}\int\frac{\chi_1(1-\gamma^2)(1-\gamma^2)^{\frac14-\frac16}\chi_{\varepsilon_1}(\tilde\alpha)}{\phi^{1/2}(x,y_*(x)+Y,0,s)}\frac{dY}{d\sigma}e^{i\tau(z\gamma+\sqrt{1-\gamma^2}(y\tilde\alpha-\Gamma_0(\tilde\alpha,s)))}e^{-i(\frac{\tilde\sigma^3}{3}+\tilde\sigma(\tau\sqrt{1-\gamma^2})^{2/3}\tilde\zeta(\frac{1+x}{\tilde\alpha}))}d\tilde\sigma d\tilde\alpha d\gamma.$$

At this point we let again $\alpha=\sqrt{1-\gamma^2}\tilde\alpha$. Following [3], we integrate by parts w.r.t. $\tilde\sigma$ and apply the Malgrange theorem to obtain $w_{1,gl}$ under the form (2.11), for a function $\widehat F_{1,\tau}$ with phase $-\tau\sqrt{1-\gamma^2}\Gamma_0(\frac{\sqrt{1-\gamma^2}}{\alpha},s)$ and symbol $\frac{\tau^{\frac12-\frac13}}{(s^2-1)^{1/4}}(1-\gamma^2)^{\frac1{12}-\frac12}\chi_1(1-\gamma^2)f$, where f is an elliptic symbol. For details see [2, Section 3.1.1].

2.1.3. *The "non-glancing" parts of u_1^+.* In this part we deal with $w_{1,he}$ and coordinates (x,φ,z) with $x\geq0$, $\theta=\pi/2-y\in[0,2\pi)$, $z\in\mathbb{R}$, as they seem more convenient. The goal of this section is to explicitly describe the form of $u^+_{h,1,he}:=u^+_{free,h,1,he}-K(u^+_{free,h,1,he}|_{\partial\Omega})$, where we recall that $u^+_{free,h,1,he}:=\int e^{it\tau}\chi(h\tau)w_{1,he}d\tau$ and where $w_{1,he}(Q,Q_0,\tau)$ is given by

$$\frac{\tau^2}{(2\pi)^2}\frac{i\tau}{4\pi}\int\frac{\chi_1(1-\gamma^2)}{\tilde\phi(1+x,\tilde\theta,\tilde z,s)}(1-\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}}))e^{i\tau((\theta-\tilde\theta)\alpha+(z-\tilde z)\gamma)}e^{-i\tau\tilde\phi(1+x,\tilde\theta,\tilde z,s)}d\alpha d\gamma d\tilde\theta d\tilde z, \qquad (2.22)$$

where $\tilde\phi(1+x,\tilde\theta,\tilde z,s)=\sqrt{(1+x)^2-2(1+x)s\cos\tilde\theta+s^2+\tilde z^2}$. We apply the stationary phase with respect to $(\tilde\theta,\alpha)$ and $(\tilde z,\gamma)$ at the critical points $(\alpha,\gamma)=-\nabla_{\theta,z}\tilde\phi(1+x,\theta,z,s)$, $\tilde\theta=\theta$, $\tilde z=z$ and obtain

$$w_{1,he}(Q,Q_0,\tau)=\frac{i\tau}{4\pi}\frac{\sigma_{free,1,he}(x,\theta,z,s,\tau)}{\tilde\phi(1+x,\theta,z,s)}e^{-i\tau\tilde\phi(1+x,\theta,z,s)}\chi_1(1-(\partial_z\tilde\phi)^2)\Big(1-\chi_{\varepsilon_1}(\frac{-\partial_\theta\tilde\phi}{\sqrt{1-(\partial_z\tilde\phi)^2}})\Big), \qquad (2.23)$$

where $\sigma_{free,1,he}$ is a classical symbol, elliptic, that reads as an asymptotic expansion with main contribution 1 and small parameter τ^{-1}. If we denote Σ the factor of $e^{-i\tau\tilde\phi(1+x,\theta,z,s)}$ in (2.23), then $u^+_{free,h,1,he}=\int e^{i\tau(t-\tilde\phi(1+x,\theta,z,s))}\chi(h\tau)\Sigma(x,\theta,z,s,\tau)d\tau$ and the reflected wave $K(u^+_{free,h1,he}|_{\partial\Omega})$ reads as $K(u^+_{free,1,he}|_{\partial\Omega})=\int e^{it\tau}\chi(h\tau)e^{-i\tau\tilde\phi_R(1+x,\theta,z,s)}\Sigma_R d\tau$, where $\tilde\phi_R$ satisfies the eikonal equation (2.8) and the boundary condition $\tilde\phi_R|_{x=0}=\tilde\phi|_{x=0}$ and $\partial_x\tilde\phi_R|_{x=0}=-\partial_x\tilde\phi|_{x=0}$. The symbol Σ_R is an asymptotic expansion with small parameter τ^{-1} that reads as $\Sigma_R(\cdot,\tau)=\sum_k\tau^{-k}\Sigma_{R,k}$, where $\Sigma_{R,k}$ solve a system of the transport equations and it is determined such that the Dirichlet boundary condition to be satisfied, that is

$$\Sigma_R|_{x=0}=\Sigma|_{x=0}=\frac{i\tau}{4\pi}\frac{\sigma(0,\theta,z,s,\tau)}{\tilde\phi(1,\theta,z,s)}\chi_1(1-(\partial_z\tilde\phi)^2)\Big(1-\chi_{\varepsilon_1}(\frac{-\partial_\theta\tilde\phi}{\sqrt{1-(\partial_z\tilde\phi)^2}})\Big)|_{x=0}. \qquad (2.24)$$

As such, we obtain $\partial_x u^+_{h,1,he}|_{x=0}=\int e^{it\tau}\chi(h\tau)(-i)\tau e^{-i\tau\tilde\phi(1,\theta,z,s)}\tilde\Sigma(\theta,z,s,\tau)d\tau$, where $\tilde\Sigma$ is a classical symbol that reads as an asymptotic expansion with small parameter τ^{-1} and whose main contribution equals $2i\partial_x\tilde\phi|_{x=0}\Sigma|_{x=0}$, with $\Sigma|_{x=0}$ given in (2.24).

Remark 2.12. *On the support of* $\chi_1(1-(\partial_z\tilde{\phi})^2|_{x=0})$ *we have* $\sqrt{1-(\partial_z\tilde{\phi})^2|_{x=0}} \geq 1/10$ *while on the support of* $1-\chi_{\varepsilon_1}$ *we have* $1-\frac{(-\partial_\theta\tilde{\phi}|_{x=0})}{\sqrt{1-(\partial_z\tilde{\phi})^2|_{x=0}}} \geq \varepsilon_1$ *hence, using the eikonal equation at* $x=0$*, we obtain a lower bound for* $\partial_x\tilde{\phi}|_{x=0}$ *as follows :*

$$(\partial_x\tilde{\phi})^2|_{x=0} = 1-\frac{1}{(1+x)^2}(\partial_\theta\tilde{\phi})^2-(\partial_z\tilde{\phi})^2\Big|_{x=0} \geq c(\varepsilon_1)>0,$$

where $c(\varepsilon_1)$ *depends only on* ε_1 *(and on the support of* χ_1*). As* $\partial_x\tilde{\phi}|_{x=0}=\frac{1-s\cos\theta}{\tilde{\phi}(1,\theta,z,s)}$*, this lower bound implies that* $s|\cos\theta_*-\cos\varphi|\geq c(\varepsilon_1)\tilde{\phi}(1,\theta,z,s)$*, where* $\theta_*=\arccos(1/s)$*.*

Using Remark 2.12, we write

$$\partial_x u^+_{h,1,he}|_{x=0} = \int e^{it\tau}\chi(h\tau)e^{-i\tau\tilde{\phi}(1,\theta,z,s)}\frac{\tau^2}{\tilde{\phi}(1,\theta,z,s)}\sigma_{1,he}(\theta,z,s,\tau)d\tau, \tag{2.25}$$

where $\sigma_{1,he}$ is a classical symbol, supported for $s(\cos\theta_*-\cos\theta)\geq c(\varepsilon_1)\tilde{\phi}(1,\theta,z,s)$ and with main contribution $\partial_x\tilde{\phi}(1,\theta,z,s)\chi_1(1-(\partial_z\phi)^2)\Big(1-\chi_{\varepsilon_1}(\frac{(-\partial_\theta\tilde{\phi})}{\sqrt{1-(\partial_z\tilde{\phi})^2}})\Big)|_{x=0}$.

2.2. **The case $\sqrt{1-\gamma^2}\sim 2^{-j}$ such that $\tau 2^{-3j}s\geq M$ for some large $M\gg 1$.** In the previous case $\sqrt{1-\gamma^2}\geq\frac{1}{10}$ we have kept track of all factors $(1-\gamma^2)$: we may easily notice that, as long as $\tau\sqrt{1-\gamma^2}^3 s\geq\tau^\epsilon$ for some $\epsilon>0$, we may proceed exactly in the same way as before as all stationary phases apply. For each $4\leq j\leq\frac{1-\epsilon}{3}\log_2(\tau)$, we obtain an unique $F_{2^{-2j},\tau}$ such that $w_{2^{-2j},gl}=T_\tau(F_{2^{-2j},\tau})$, which has exactly the same form as in (2.12). For the "non-glancing" part we construct the reflected wave exactly as in the previous section.

Let $\tau\sqrt{1-\gamma^2}^3 s\in[M,\tau^\epsilon]$ for some large $M\gg 1$ and any small $\epsilon>0$ and let $1-\gamma^2=2^{-2j}\varphi$, $\varphi\sim 1$. With $\Phi(z_1,\cdot):=z\gamma+(y-\tilde{y})\alpha-\phi(x,\tilde{y},0,s)(z_1\gamma+\sqrt{1+z_1^2})$, we have

$$w_{2^{-2j},gl}(Q,Q_0,\tau)=\frac{\tau^2}{(2\pi)^2}\frac{i\tau}{4\pi}\int\frac{\chi_1(2^{2j}(1-\gamma^2))}{\sqrt{1+z_1^2}}\chi_{\varepsilon_1}(\frac{\alpha}{\sqrt{1-\gamma^2}})e^{i\tau\Phi}d\alpha d\gamma d\tilde{y}d\tilde{z}_1. \tag{2.26}$$

The critical point is $z_{1,c}=-\frac{\gamma}{\sqrt{1-\gamma^2}}$ and $\partial^2_{z_1}\Phi|_{z_{1,c}}=\sqrt{1-\gamma^2}^3$, $|\partial^k_{z_1}\Phi_{z_{1,c}}|\leq 1-\gamma^2\sim 2^{-4j}$. As $\tau 2^{-3j}s\leq\tau^\epsilon$ for any small $\epsilon>0$ and $s\geq\sqrt{2}$, then $2^{-j}\leq\tau^{-(1-\epsilon)/3}s^{-1/3}$ and $\tau 2^{-4j}s\leq\tau^\epsilon\times\tau^{-(1-\epsilon)/3}s^{-1/3}\leq\tau^{(4\epsilon-1)/3}$. For $\epsilon<1/8$, it follows that the phase $\Phi(z_1,\cdot)-\Phi(z_{1,c},\cdot)-\frac{1}{2}(z_1-z_{1,c})^2\partial^2\Phi(z_{1,c},\cdot)$ is not oscillating and the corresponding exponential factor can be brought into the symbol. Therefore we may replace the phase by

$$\begin{aligned}\tilde{\Phi} &:= \Phi(z_{1,c},\cdot)+\frac{1}{2}(z_1+\frac{\gamma}{\sqrt{1-\gamma^2}})^2\phi(x,\tilde{y},0,s)\sqrt{1-\gamma^2}^3\\ &= z\gamma+\sqrt{1-\gamma^2}\Big((y-\tilde{y})\tilde{\alpha}-\phi(x,\tilde{y},0,s)\Big)(1-\frac{1}{2}(z_1\sqrt{1-\gamma^2}+\gamma)^2),\end{aligned} \tag{2.27}$$

where we have set $\alpha=\sqrt{1-\gamma^2}(1-\frac{1}{2}(z_1\sqrt{1-\gamma^2}+\gamma)^2)$. We can now use Lemma 2.11 and make the same change of variable $\tilde{y}\to\sigma$ that transforms $\Big((y-\tilde{y})\tilde{\alpha}-\phi(x,\tilde{y},0,s)\Big)$ into an Airy phase function. We then proceed exactly in the same way as for $\sqrt{1-\gamma^2}\geq\frac{1}{10}$ to obtain $F_\tau(\tilde{y},\tilde{z})$ and then $\mathcal{F}(\partial_x u^+)$ as in (2.13) (the integration by parts in σ, which allows to obtain Airy type terms, is done in a similar way as its parameter $\tau\sqrt{1-\gamma}$ is large).

When dealing with the "non-glancing" part, one may notice that for $\tau 2^{-3j}s\geq M$ we can also proceed as in section 2.1.3 : for $\tau 2^{-3j}s\geq\tau^\epsilon$ with $\epsilon>0$ this is obvious, while for $\tau^\epsilon>\tau 2^{-3j}s\geq M$ for large $M\gg 1$ we have $2^j\geq\tau^{(1-\epsilon)/3}s^{1/3}$ and we may apply the stationary phase with respect to $\tilde{z},\gamma$ as, at the critical point, the determinant of the Hessian is bounded from below by $\tau 2^{-2j}s\geq M2^j\geq\tau^{(1-\epsilon)/3}$. For small $0<h\leq h_0<1$, we define

$$j(s,h):=\sup\{j,2^{-3j}s\geq Mh\}. \tag{2.28}$$

For all $j \leq j(s,h)$ we have $2^{-3j}s \geq Mh$, find

$$\partial_x u^+_{h,2^{-2j},he}|_{x=0} = \int e^{it\tau}\chi(h\tau)e^{-i\tau\tilde{\phi}(1,\theta,z,s)}\frac{\tau^2}{\tilde{\phi}(1,\theta,z,s)}2^{-j}\sigma_{2^{-2j},he}(\theta,z,s,\tau)d\tau, \tag{2.29}$$

where $\sigma_{2^{-2j},he}$ is an asymptotic expansion with small parameter $(\tau 2^{-3j}s)^{-1}$.

3. High-frequency case. Dispersive estimates when $d(Q_0,\partial\Omega) \geq \varepsilon_0$

3.1. Dispersive bounds for the glancing part when $d(Q,\partial\Omega) \geq \varepsilon_0 := \sqrt{2}-1$.

3.1.1. *Case* $\sqrt{1-\gamma^2} \geq 1/10$, $s \geq 1+\varepsilon_0$ *for fixed* $\varepsilon_0 > 0$. We let $u^\#_{1,gl} := \Box_+^{-1}(\partial_x u^+_{1,gl}|_{\partial\Omega})$, $u^\#_{h,1,gl} = \chi(hD_t)u^\#_{1,gl}$. Let $Q = ((1+x_Q)\sin y_Q, (1+x_Q)\cos y_Q, z_Q)$ in Ω, and assume $d(Q,\partial\Omega) \geq \varepsilon_0 := \sqrt{2}-1$, that is $r := 1+x_Q \geq 1+\varepsilon_0 = \sqrt{2}$. We prove the following

Proposition 3.1. *Let $s \geq r \geq 1+\varepsilon_0$ for some $\varepsilon_0 > 0$. There exists $C = C(\varepsilon) > 0$ such that for all $t > h$, the following holds uniformly with respect to Q, Q_0*

$$|u^\#_{h,1,gl}(Q,Q_0,t)| \leq \frac{C}{h^2}\frac{1}{t}.$$

Proof. Replacing (2.13) in (1.14) we find, modulo $O(\tau^{-\infty})$,

$$u^\#_{h,1,gl}(Q,Q_0,t) = \int \chi(h\tau)\tfrac{e^{it\tau}}{4\pi}\tau \times I_{1,gl}(Q,Q_0,\tau)d\tau, \tag{3.1}$$

where, for a point $P = (\sin y, \cos y, z)$ on $\partial\Omega$ we write $|P-Q| = \phi(x_Q, y-y_Q, z-z_Q, 1)$ and where, after the change of coordinates $\alpha = \sqrt{1-\gamma^2}\tilde{\alpha}$, we obtain

$$\begin{aligned} I_{1,gl}(Q,Q_0,\tau) := \int & \tau^{-1+2+\frac{5}{6}} e^{i\tau(z\gamma+\sqrt{1-\gamma^2}(y\tilde{\alpha}-\Gamma_0(\tilde{\alpha},s)) - \phi(x_Q,y-y_Q,z-z_Q,1))} \\ & \times \frac{f(\alpha,\gamma,\tau)\tilde{b}_\partial(y,z,\tilde{\alpha},\tau)}{\phi(x_Q,y-y_Q,z-z_Q,1)}\frac{(1-\gamma^2)^{-\frac{5}{12}+\frac{1}{3}+\frac{1}{2}}\tilde{\chi}_1(1-\gamma^2)\chi_{\varepsilon_1}(\tilde{\alpha})}{(s^2-1)^{1/4}A_+(\tau^{\frac{2}{3}}\zeta_0(\alpha,\gamma))}d\tilde{\alpha}d\gamma dydz. \end{aligned} \tag{3.2}$$

Lemma 3.2. *There exists a constant $C > 0$ such that following estimate holds true, uniformly with respect to Q, Q_0 and t such that $\sqrt{s^2-1+z_Q^2} \sim t$*

$$|I_{1,gl}(Q,Q_0,\tau)| \leq \tfrac{C}{t}.$$

Moreover, for $\frac{t}{\sqrt{s^2-1+z_Q^2}} \notin [1/4,4]$, we have $|I_{1,gl}(Q,Q_0,\tau)| \leq \frac{C}{\sqrt{s^2-1}}$.

If $\frac{t}{\sqrt{s^2-1+z_Q^2}} \in [1/4,4]$, the estimate of Proposition 3.1 follows using (3.1) and Lemma 3.2. If not, the phase of (3.1) is not stationary w.r.t. τ and we proceed by integrations by parts. □

In the remaining of this section we prove Lemma 3.2.

Proof. We apply the stationary phase with respect to z in the integral (3.2): let $r = 1+x_Q$ and set $z = z_Q + \tilde{z}\sqrt{1+r^2-2r\cos(y_Q-y)}$. As $r-1 \geq \varepsilon_0$, this is well defined and $dz/d\tilde{z} = \phi(x_Q, y-y_Q, 0, 1)$. As $\phi(x_Q, y-y_Q, z-z_Q, 1) = \phi(x_Q, y-y_Q, 0, 1)\sqrt{1+\tilde{z}^2}$, we write

$$\begin{aligned} I_{1,gl}(Q,Q_0,\tau) :=& \tau^{\frac{4}{3}+\frac{1}{2}} \int e^{i\tau(z_Q\gamma - \sqrt{1-\gamma^2}(-y\tilde{\alpha}+\Gamma_0(\tilde{\alpha},s)) + \phi(x_Q,y-y_Q,0,1)(\tilde{z}\gamma-\sqrt{1+\tilde{z}^2}))} \\ & \frac{(1-\gamma^2)^{5/12}\tilde{b}_\partial(y,z,\tilde{\alpha},\tau)}{\sqrt{1+\tilde{z}^2}}\frac{f(\alpha,\gamma,\tau)\tilde{\chi}_1(1-\gamma^2)\chi_{\varepsilon_1}(\tilde{\alpha})}{(s^2-1)^{1/4}A_+(\tau^{\frac{2}{3}}\zeta_0(\alpha,\gamma))}d\tilde{\alpha}d\gamma dyd\tilde{z}. \end{aligned}$$

At this point we apply the stationary phase w.r.t. the variable $\tilde{z}$. The critical point is $\tilde{z} = \gamma(1-\gamma^2)^{-1/2}$ and there the second derivative equals $\sqrt{1-\gamma^2}^3\phi(x_Q, y-y_Q, 0, 1)$, which is uniformly bounded from below on the support of $\tilde{\chi}_1$ and for $x_Q = r-1 \geq \varepsilon_0$. At the critical point $\tilde{z}\gamma - \sqrt{1+\tilde{z}^2} = -\sqrt{1-\gamma^2}$ and the critical value of the phase of (3.2) becomes $(y-y_*)\alpha + z_Q\gamma - \sqrt{1-\gamma^2}(\sqrt{s^2-1} + \phi(x_Q, y-y_Q, 0, 1))$. The stationary phase yields a factor $\tau^{-1/2} \times (1-\gamma^2)^{-3/4}$ and the symbol $\frac{\tilde{b}_\partial(1-\gamma^2)^{5/12}}{\sqrt{1+\tilde{z}^2}}$ becomes

$(1-\gamma^2)^{11/12}\bar{b}_\partial(y,z_Q,\alpha,\gamma,\tau)\phi^{-1/2}(x_Q,y-y_Q,0,1)$, where $\bar{b}_\partial$ is of order 0 with main contribution $\tilde{b}_\partial$. We obtain

$$\begin{aligned}I_{1,gl}(Q,Q_0,\tau) :=&\tau^{\frac{4}{3}}\int e^{i\tau(z_Q\gamma-\sqrt{1-\gamma^2}(-y\tilde{\alpha}+\Gamma_0(\tilde{\alpha},s)+\phi(x_Q,y-y_Q,0,1)))}\\ &(1-\gamma^2)^{1/6}\frac{\bar{b}_\partial(y,Q,\tilde{\alpha},\gamma,\tau)}{\phi^{1/2}(x_Q,y-y_Q,0,1)}\frac{f(\alpha,\gamma,\tau)\tilde{\chi}_1(1-\gamma^2)\chi_{\varepsilon_1}(\tilde{\alpha})}{(s^2-1)^{1/4}A_+(\tau^{\frac{2}{3}}\zeta_0(\alpha,\gamma))}d\tilde{\alpha}d\gamma dy.\end{aligned} \tag{3.3}$$

The phase $\phi(x_Q,y-y_Q,0,1)$ has two degenerate critical points of order exactly two at $y=y_Q\pm\arccos(1/r)$, where $r=1+x_Q$. Near $y_Q-\arccos(1/r)$, its first order derivative equals -1, hence for y near this point the phase of $I_{1,gl}$ is non-stationary w.r.t. y and repeated integrations by parts yield a $O(\tau^{-\infty})$ contribution there. Let $y_c := y_Q+\arccos(1/r)$. Notice that, if $y\in[0,2\pi)$ is sufficiently close to y_* on the support of $I_{1,gl}$ (say $|y-y_*|\leq\frac{\pi}{16}$) and is such that $|y-y_c|\geq\frac{\pi}{8}$, then $1-\tilde{\alpha}$ has to be bounded from below by a fixed constant there where the phase of $I_{1,gl}$ is stationary w.r.t. y. Taking ε_1 smaller if necessary, it follows that for such value of y outside a small, fixed neighborhood of y_c, $\tilde{\alpha}$ cannot belong to the support of $\chi_{\varepsilon_1}(\tilde{\alpha})$. We are reduced to study the integral (3.3) for $|y-y_c|\leq\frac{\pi}{8}<1$. Let $\epsilon_1>0$ be small enough. We study separately the cases $|y-y_c|\leq\tau^{-1/3+\epsilon_1}$ and $\tau^{-1/3+\epsilon_1}\lesssim|y-y_c|\leq\frac{\pi}{8}$; to do that, we use againg the smooth cut-off χ_0 supported in $[-2,2]$ and equal to 1 on $[-3/2,3/2]$ and split $I_{1,gl}=I^{\chi_0}_{1,gl}+I^{1-\chi_0}_{1,gl}$, where $I^{\chi}_{1,gl}$ has the form (3.3) with additional cut-off $\chi((y-y_c)\tau^{1/3-\epsilon_1})$.

Case $\tau^{-1/3+\epsilon_1}\leq|y-y_c|\leq\frac{\pi}{8}$ *: study of* $I^{1-\chi_0}_{1,gl}$. We set $\tilde{\alpha}=\tilde{\alpha}(\beta,\tau):=1-\tau^{-2/3}\beta$: as on the support of $\chi_{\varepsilon_1}(\tilde{\alpha})$ we have $1-\tilde{\alpha}\lesssim\varepsilon_1$, it follows that $\tau^{-2/3}\beta\lesssim\varepsilon_1$. This choice of coordinates is motivated by the behaviour of the Airy factor $A_+(\tau^{\frac{2}{3}}\zeta_0(\alpha,\gamma))$: as $\tau^{2/3}\zeta_0(\alpha,\gamma)=\tau^{2/3}\alpha^{2/3}\tilde{\zeta}(\frac{\sqrt{1-\gamma^2}}{\alpha})=\tau^{2/3}\sqrt{1-\gamma^2}^{2/3}\tilde{\alpha}^{2/3}\tilde{\zeta}(\frac{1}{\tilde{\alpha}})$, then

$$\tau\alpha(-\tilde{\zeta})^{3/2}(\frac{1}{\tilde{\alpha}})=\sqrt{2}\sqrt{1-\gamma^2}\beta^{3/2}(1+O(\tau^{-2/3}\beta)), \tag{3.4}$$

where we used Lemma 2.5. As such, for $(\sqrt{2}\sqrt{1-\gamma^2}\beta^{3/2})^{2/3}$ large enough, $A_+(\tau^{\frac{2}{3}}\zeta_0(\alpha,\gamma))$ does oscillate, while for $(\sqrt{2}\sqrt{1-\gamma^2}\beta^{3/2})^{2/3}$ bounded it may be brought into the symbol. Write $1=\chi_0(\beta)+(1-\chi_0)(\beta)$. On the support of $1-\chi_0(\beta)$ the Airy factor may oscillate and the phase function of $I^{1-\chi_0}_{1,gl}$ equals $z_Q\gamma-\sqrt{1-\gamma^2}\varphi$, where we have set

$$\varphi(y,\tilde{\alpha},r):=-y\tilde{\alpha}+\Gamma_0(\tilde{\alpha},s)+\phi(x_Q,y-y_Q,0,1)-\frac{2}{3}(-\tilde{\zeta})^{3/2}(\frac{1}{\tilde{\alpha}}). \tag{3.5}$$

Moreover, we have, with φ defined in (3.5) and $\Sigma=\bar{b}_\partial f$,

$$\begin{aligned}I^{1-\chi_0}_{1,gl}(Q,Q_0,\tau)=\tau^{\frac{4}{3}-\frac{2}{3}}&\int e^{-i\tau(z\gamma-\sqrt{1-\gamma^2}\varphi)}\tilde{\chi}_{\varepsilon_1}(1-\tau^{-2/3}\tilde{\beta})(1-\chi_0)((y-y_c)\tau^{1/3-\epsilon_1})\\ &\beta^{1/4}(1-\gamma^2)^{\frac{1}{12}+\frac{1}{6}}\Sigma(y,\beta,\gamma,\tau)\times\frac{\tilde{\chi}_1(1-\gamma^2)}{(s^2-1)^{1/4}\phi^{1/2}(x_Q,y-y_Q,0,1)}dyd\gamma d\beta,\end{aligned} \tag{3.6}$$

where the new factors $\beta^{1/4}(1-\gamma^2)^{1/12}$ come from the Airy term A_+^{-1} (using (3.4)).

Lemma 3.3. *Let* $y=y_c+\mathtt{Y}$, *where* $y_c=y_Q+\arccos(1/r)$. *There exists a unique change of variables* $\mathtt{Y}\mapsto\sigma$ *which is smooth and satisfying* $\frac{d\mathtt{Y}}{d\sigma}\notin\{0,\infty\}$ *such that, for* $\tilde{\zeta}$ *given by Lemma 2.5, we have*

$$-(y_c+\mathtt{Y})\tilde{\alpha}+\phi(x_Q,y-y_Q+\mathtt{Y},0,1)=\frac{\sigma^3}{3}+\sigma\tilde{\alpha}^{2/3}\tilde{\zeta}(\frac{1}{\tilde{\alpha}})+\tilde{\Gamma}(\tilde{\alpha},r), \tag{3.7}$$

and where $\tilde{\Gamma}(\tilde{\alpha},r):=\sqrt{r^2-1}-y_c\tilde{\alpha}+\frac{(1-\tilde{\alpha})^2}{2\sqrt{r^2-1}}(1+O(1-\tilde{\alpha}))$.

Proof. We proceed exactly as in the proof of Lemma 2.11 (where now $x=0$ and s is replaced by r). As y_c is the degenerate critical point of order 2 of ϕ, there exist a smooth change of variable $\mathtt{Y}\to\sigma$ and smooth phase functions $\zeta^{\#}$ and $\tilde{\Gamma}$ such that the LHS term in (3.7) reads as $\frac{\sigma^3}{3}+\sigma\zeta^{\#}(\tilde{\alpha},r)+\tilde{\Gamma}(\tilde{\alpha},r)$. Exactly as in Lemma 2.11 we obtain that $\zeta^{\#}=\tilde{\alpha}^{2/3}\tilde{\zeta}(\frac{1}{\tilde{\alpha}})$. It remains to determine $\tilde{\Gamma}$. The two critical points satisfy

$$r\cos(\arccos(1/r)+\mathtt{Y}_\pm)=\tilde{\alpha}^2\mp\sqrt{r^2-\tilde{\alpha}^2}\sqrt{1-\tilde{\alpha}^2}.$$

We have as before $\phi(x_Q, y_c+\mathtt{Y}_\pm, 0, 1) = \sqrt{r^2-\tilde\alpha^2}\pm\sqrt{1-\tilde\alpha^2}$. As $\cos(\arccos(1/r)+\mathtt{Y}) = \sin(\arcsin(1/r)-\mathtt{Y})$ we use the computations from Lemma 2.11 to determine $\arcsin(1/r)-\frac{1}{2}(\mathtt{Y}_++\mathtt{Y}_-)$. As $-y_c = -y_Q-\arccos(1/r) = -y_Q-\frac{\pi}{2}+\arcsin(1/r)$ we obtain $\tilde\Gamma(\tilde\alpha, r) = -(y_Q+\frac{\pi}{2})\tilde\alpha+\Gamma_0(\tilde\alpha, r)$ where $\Gamma_0(\tilde\alpha, r)$ is the same as in Lemma 2.11 and compute the derivatives of this new function at $\tilde\alpha = 1$ using those of Γ_0 as follows

$$\tilde\Gamma(1,r) = \sqrt{r^2-1}-y_c, \tilde\Gamma'(1,r) = -y_c, \tilde\Gamma''(1,r) = \frac{1}{\sqrt{r^2-1}}, \tilde\Gamma^{(k)} = \frac{c_k}{\sqrt{r^2-1}}(1+O(\frac{1}{\sqrt{r^2-1}})).$$

□

Using the changes of variable $y\to y_c+\mathtt{Y}$, $\mathtt{Y}\to\sigma$ from the previous Lemma 2.7 yields

$$\varphi(y,\tilde\alpha,r) = \frac{\sigma^3}{3}+\sigma\tilde\alpha^{2/3}\tilde\zeta(\frac{1}{\tilde\alpha})+\tilde\Gamma(\tilde\alpha,r)+\Gamma_0(\tilde\alpha,s)-\frac{2}{3}(-\tilde\zeta)^{3/2}(\frac{1}{\tilde\alpha}).$$

Let $y = y_c+Y$, $Y\to\sigma$ as in the Lemma 3.3 and set moreover $\sigma = \tau^{-1/3}w$: then $\tau^{\epsilon_1}\lesssim |w|$ on the support of the symbol (and if $|\tau^{-1/3}w|\geq\frac{\pi}{4}$ the integral defining $I^{1-\chi_0}_{1,gl}$ is $O(\tau^{-\infty})$). We apply the stationary phase in w near the critical points : let χ be a smooth cut-off supported in a fixed neighborhood of 1 and equal to 1 near 1 and set $\chi_\pm := \chi(\pm\frac{\sqrt{2\beta}}{w})$, $\tilde\alpha = 1-\tau^{-2/3}\beta$; let also $\overline\chi := 1-\chi_+-\chi_-$. Write

$$I^{1-\chi_0}_{1,gl}(Q,Q_0,\tau) = \sum_{\chi\in\{\chi_\pm,\overline\chi\}} I^{1-\chi_0,\chi}_{1,gl},$$

where $I^{1-\chi_0,\chi}_{1,gl}$ are given by (3.6) with additional cutoffs $\chi(\frac{\sqrt{2\beta}}{w})$.

Lemma 3.4. *For w in a small, fixed neighborhood of $\pm\sqrt{2\beta}$, we have*

$$I^{1-\chi_0,\chi_\pm}_{1,gl}(Q,Q_0,\tau) = \tau^{4/3-2/3-1/3}\int e^{-i\tau(z\gamma-\sqrt{1-\gamma^2}\varphi_\pm)}\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta)\Sigma_\pm(\beta,\gamma,\tau)$$
$$\times\frac{(1-\chi_0)(Y(\tau^{-1/3}w)\tau^{1/3-\varepsilon_1})\tilde\chi_1(1-\gamma^2)}{(s^2-1)^{1/4}\phi^{1/2}(x_Q,\arccos(1/r)+Y(\tau^{-1/3}w_\pm),0,1)}d\gamma d\beta, \quad (3.8)$$

where $\varphi_\pm := \mp\frac{2}{3}(-\tilde\zeta)^{3/2}(\frac{1}{\tilde\alpha})+\tilde\Gamma(\tilde\alpha,r)+\Gamma_0(\tilde\alpha,s)-\frac{2}{3}(-\tilde\zeta)^{3/2}(\frac{1}{\tilde\alpha})|_{\tilde\alpha=1-\tau^{-2/3}\beta}$. Here $\Sigma_\pm$ is a classical symbol that reads as an asymptotic expansion with small parameter $\leq\tau^{-\epsilon_1}$ and main contribution $\frac{dY}{d\sigma}\frac{\beta^{1/4}}{\sqrt{|w_\pm|}}\chi_\pm(\frac{\sqrt{2\beta}}{w_\pm})\Sigma(y_c+\tau^{-1/3}w_\pm,\beta,\tau)$.

Proof. The critical points are $w_\pm := \pm\tau^{1/3}\sqrt{-\tilde\zeta(\frac{1}{1-\tau^{-2/3}\beta})}$ which gives $w_\pm = \pm\sqrt{2\beta}(1+O(\sqrt{2\tau^{-2/3}\beta}))$. As $|w|\geq\tau^{\epsilon_1}$ on the support of $(1-\chi_0)(Y(\tau^{-1/3}w)\tau^{1/3-\varepsilon_1})$ and $\frac{w}{\sqrt{2\beta}}\sim\pm1$ on the support of the symbol, we also have $\sqrt{2\beta}\gtrsim\tau^{\epsilon_1}$. At $w_\pm$, the second order derivative of the phase equals $\partial^2_w(\tau\sqrt{1-\gamma^2}\varphi)|_{w_\pm} = \sqrt{1-\gamma^2}w_\pm(1+O(\tau^{-1/3}w_\pm))$. As $\sqrt{1-\gamma^2}\geq 1/16$ on the support of $\tilde\chi_1$ and $|w|\geq\tau^{\epsilon_1}$ on the support of $(1-\chi_0)(w\tau^{-\epsilon_1})$, it follows that $\sqrt{1-\gamma^2}\times w_\pm\gtrsim\tau^{\epsilon_1}$ and the stationary phase applies at $w_\pm$ with a parameter larger than τ^{ϵ_1}. The exponent of the factor $(1-\gamma^2)$ is $1/12+1/6-1/4 = 0$. The factor $\tau^{-1/3}$ before the integral (3.8) comes from the change of variables $\sigma\to w$.

□

We now consider the integral $I^{1-\chi_0,\overline\chi}_{1,gl}(Q,Q_0,\tau)$ whose symbol is supported for $\frac{w^2}{2\beta}\notin[1/2,3/2]$ and show that its contribution is $O(\tau^{-\infty}/t)$.

Lemma 3.5. *The stationary phase applies in γ with large parameter τ and yields*

$$I^{1-\chi_0,\overline\chi}_{1,gl}(Q,Q_0,\tau) = \tau^{\frac{4}{3}-1-\frac{1}{2}}\int e^{-i\tau\sqrt{\varphi^2+z_Q^2}}\tilde\chi_\varepsilon(1-\tau^{-2/3}\beta)(1-\chi_0)(|w|\tau^{-\epsilon_1})$$
$$\beta^{1/4}\overline\chi(\frac{\sqrt{2\beta}}{w})\Sigma(y,\beta,\tau)\Big(\frac{\varphi^2}{\varphi^2+z_Q^2}\Big)\frac{1}{\varphi^{1/2}}\times\frac{\tilde\chi_1(\frac{\varphi^2}{\varphi^2+z_Q^2})}{(s^2-1)^{1/4}\phi^{1/2}(x_Q,y-y_Q,0,1)}d\beta dy, \quad (3.9)$$

where Σ is a classical symbol of order 0 in τ that reads as an asymptotic expansion with small parameter τ^{-1} and main contribution $\tilde b_\partial f$.

Proof. The critical point satisfies : $\gamma_c = -z_Q/\sqrt{\varphi^2+z_Q^2}$ and at γ_c, the second order derivative of the phase equals $\frac{\varphi}{\sqrt{1-\gamma_c^2}^3} = \varphi \times \frac{\sqrt{\varphi^2+z_Q^2}^3}{\varphi^3} \geq \varphi$ and its critical value equals $-\sqrt{z_Q^2+\varphi^2}$. In order to show that the stationary phase applies we will show that $\varphi \geq \sqrt{s^2-1}$. From Lemmas 3.3 and 2.11 we have $\tilde\Gamma(\tilde\alpha, r)+\Gamma_0(\tilde\alpha, s) = \sqrt{r^2-1}+\sqrt{s^2-1}+(y_*-y_c)\tilde\alpha + O(1-\tilde\alpha)$. As y is close to y_* on the support of the symbol $I_{1,gl}$ ($|y-y_*| \leq \frac{\pi}{16}$) and $|y-y_c| \leq \frac{\pi}{8}$ (these constants may be shrunk if necessary), then $|y_c-y_*| \leq \frac{3\pi}{16} < \frac{5}{8}$ while $\sqrt{r^2-1} \geq \sqrt{2}\sqrt{\varepsilon_0} \geq \frac{4}{5}$ for $\varepsilon_0 = \sqrt{2}-1$. Moreover, on the support of $\chi_{\varepsilon_1}(\tilde\alpha)$ we have $|1-\tilde\alpha| \lesssim \varepsilon_1$ so we conclude taking ε_1 small enough compared to ε_0. The stationary phase yields (3.9). The exponent of $\left(\frac{\varphi^2}{\varphi^2+z_Q^2}\right)$ equals $\frac{1}{12}+\frac{1}{3}+\frac{3}{4}$, where the last term comes from the second order derivative. □

Corollary 3.6. *We have $I_{1,gl}^{1-\chi_0,\overline{\chi}}(Q,Q_0,\tau) = O(\tau^{-\infty}/t)$. Moreover, modulo $O(\tau^{-\infty}/t)$,*

$$I_{1,gl}^{1-\chi_0,\chi_\pm}(Q,Q_0,\tau) = \tau^{4/3-1-1/2}\int e^{-i\tau\sqrt{\varphi_\pm^2+z_Q^2}}\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta) \frac{\tilde\Sigma_\pm(\beta,\tau)}{\varphi_\pm^{1/2}} \times \frac{\check\chi_1\left(\frac{\varphi_\pm}{\sqrt{\varphi_\pm^2+z_Q^2}}\right)}{(s^2-1)^{1/4}\phi^{1/2}(x_Q, y_c-y_Q+Y(\tau^{-1/3}w_\pm),0,1)}d\gamma d\beta, \quad (3.10)$$

where $\check\chi_1(\cdot) = (\cdot)^{3/4}\tilde\chi_1$ and $\tilde\Sigma_-$ is a classical symbol with main contribution $\Sigma(\beta,\gamma_c,\tau)$.

Proof. Using Lemma 3.5, $I_{1,gl}^{1-\chi_0,\overline{\chi}}(Q,Q_0,\tau)$ of the form (3.9) : as its symbol is supported for w outside a fixed neighborhood of $w_\pm$, repeated integrations by parts yield $O(\tau^{-\infty}/t)$ where the factor $1/t$ is obtained from the symbol $(\varphi\sqrt{s^2-1})^{-1/2} \lesssim 1/\sqrt{s^2-1} \lesssim 1/t$ as $2\sqrt{s^2-1} \geq \varphi \geq \sqrt{s^2-1}$, as $t \sim \sqrt{\varphi^2+z_Q^2} \leq 16\varphi$ on the support of $\tilde\chi_1$ and $r-1 \geq \varepsilon_0$. To obtain (3.10) we apply the same proof as the one of Lemma 3.5 to (3.8) for the $\pm$ sign. □

Using the Corollary, we obtain $I_{1,gl}^{1-\chi_0}(Q,Q_0,\tau) = \sum_\pm I_{1,gl}^{1-\chi_0,\chi_\pm} + O(\tau^{-\infty}/t)$, where $I_{1,gl}^{1-\chi_0,\chi_\pm}$ are given in (3.10). We are left with the integration with respect to β in the integrals (3.10) whose symbols $(1-\chi_0)(\sqrt{\beta}\tau^{-\epsilon_1})\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta)$ are supported for $\beta \gtrsim \tau^{2\epsilon_1}$ and $\tau^{-2/3}\beta \lesssim \varepsilon_1$. As β takes values in a large interval, we consider separately dyadic intervals where $\beta \sim 2^{2k}$ and then sum all the contributions. Let $\tilde\chi$ supported near 1 and equal to 1 on $[\frac{3}{4},\frac{5}{4}]$ such that

$$(1-\chi_0)(\sqrt{\beta}\tau^{-\epsilon_1})\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta)\sum_k \tilde\chi(\beta 2^{-2k}) = (1-\chi_0)(\sqrt{\beta}\tau^{-\epsilon_1})\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta). \quad (3.11)$$

On the support of $(1-\chi_0)(\sqrt{\beta}\tau^{-\epsilon_1})\chi_{\varepsilon_1}(1-\tau^{-2/3}\beta)$ we have $\tau^{2\epsilon_1} < \beta \lesssim \varepsilon_1\tau^{2/3}$ and for each k in the previous sum, $\tilde\chi(\beta 2^{-2k})$ localize at $\beta \sim 2^{2k}$. The sum is thus taken for $\epsilon_1\log_2(\tau) \leq k < \frac{1}{3}\log_2(\tau)$. Recall that $\varphi|_\pm = \varphi|_{w_\pm}$ where $\varphi_- = \tilde\Gamma(\tilde\alpha,r)+\Gamma_0(\tilde\alpha,s)$ and $\varphi_+ = \varphi_- - \frac{4}{3}(-\tilde\zeta)^{3/2}(\frac{1}{\tilde\alpha})$. We deal separately with the $\pm$ signs. Let $I_{1,gl}^{1-\chi_0,\chi_\pm,k}$ denote the integrals in (3.10) with additional cutoff $\tilde\chi(\beta 2^{-2k})$. Using (3.11) we have

$$I_{1,gl}^{1-\chi_0,\chi_\pm} = \sum_{k=\epsilon_1\log_2\tau}^{(\log_2\tau)/3} I_{1,gl}^{1-\chi_0,\chi_\pm,k}.$$

Lemma 3.7. *There exists a uniform constant $C = C_+(\varepsilon_0,\varepsilon_1)$ depending only on ε_j such that*

$$|I_{1,gl}^{1-\chi_0,\chi_+}| \leq \sum_{k=\epsilon_1\log_2\tau}^{(\log_2\tau)/3} |I_{1,gl}^{1-\chi_0,\chi_+,k}| \leq C_+(\varepsilon_0,\varepsilon_1)/t.$$

Proof. At $w_+ = \sqrt{2\beta}(1+O(\sqrt{2\tau^{-2/3}\beta}))$, the phase φ_+ is stationary when $\tau^{1/3}(y_c-y_*) = 2\sqrt{2\beta}(1+O(\sqrt{\tau^{-2/3}\beta}))$. Let $\beta = 2^{2k}\Xi$ on the support of $\tilde\chi(\beta 2^{-2k})$, with $\Xi\in[1/2,3/2]$. As

$$\begin{aligned}\partial_\Xi(\tau\sqrt{\varphi_+^2+z_Q^2}) &= \frac{\varphi_+}{\sqrt{\varphi_+^2+z_Q^2}}\frac{\partial\beta}{\partial\Xi}(\tau\partial_\beta\varphi_+)\\ &= \frac{\varphi_+}{\sqrt{\varphi_+^2+z_Q^2}}2^{2k+k}\Big(\frac{\tau^{1/3}(y_c-y_*)}{2^k}-2\sqrt{2\Xi}(1+O(\sqrt{\tau^{-2/3}\beta}))\Big),\end{aligned}\tag{3.12}$$

the phase is stationary for $\Xi\sim 1$ only when $\frac{\tau^{1/3}(y_c-y_*)}{2^k}\sim 2\sqrt{2}$; as $\frac{\varphi_+}{\sqrt{\varphi_+^2+z_Q^2}}\geq 1/16$ on the support of $\tilde\chi_1$ and as $2^{3k}\geq \tau^{3\epsilon_1/2}$, it follows that, for $|\frac{\tau^{1/3}(y_c-y_*)}{2^k}-2\sqrt{2}|\geq 4$ and $\Xi\in[1/2,3/2]$, repeated integrations by parts yield a contribution $O(\tau^{-\infty}/t)$. We deduce that there are at most a finite number of values of k for which the phase may be stationary ; for such k the stationary phase applies at the critical point $2\sqrt{2\Xi}\sim\frac{\tau^{1/3}(y_c-y_*)}{2^k}$ as, there, the second order derivative equals $-\frac{\varphi_+}{\sqrt{\varphi_+^2+z_Q^2}}2^{3k}\times\frac{\sqrt2}{\sqrt\Xi}$ and $\Xi\sim 1$. For all such k, the stationary phase yields a factor $2^{2k}\times 2^{-3k/2}\times\sqrt{\varphi_+^2+z_Q^2}^{\,1/2}/\varphi_+^{1/2}$, where the exponent $2k$ comes from the change of variables and the exponent $2^{-3k/2}$ from the second order derivative at $\Xi\sim1$. As $2^{2k}\leq\tau^{2/3}$, the sum over all such k yields at most $2^{k/2}\leq\tau^{1/6}$ and the exponent $1/6$ is canceled by the exponent of $\tau^{4/3-2/3-1/2-1/3}$ from $I^{1-\chi_0,\chi_+}_{1,gl}$. We conclude using that on the support of the symbol $(\varphi_+\sqrt{s^2-1})^{-1/2}\leq C_+(\varepsilon_0,\varepsilon_1)/t$, where $C_+(\varepsilon_0,\varepsilon_1)$ depends only on ε_j. □

Lemma 3.8. *There exists a uniform constant $C=C_-(\varepsilon_0,\varepsilon_1)$ depending only on ε_j such that*

$$\sum_{k=\epsilon_1\log_2(\tau)}^{(\log_2\tau)/3}|I^{1-\chi_0,\chi_-,k}_{1,gl}(Q,Q_0,t)|\lesssim\frac{C_-(\varepsilon_0,\varepsilon_1)}{t},$$

Proof. We have $\varphi_-=\tilde\Gamma(\tilde\alpha,r)+\Gamma_0(\tilde\alpha,s)$ hence, for $\tilde\alpha=1-\tau^{-2/3}\beta$, we have

$$\begin{aligned}\tau\varphi_- &= \tau\Big(\sqrt{r^2-1}+\sqrt{s^2-1}-(y_c-y_*)\tilde\alpha\\ &\quad+\frac{(1-\tilde\alpha)^2}{2}\Big(\frac{1}{\sqrt{r^2-1}}(1+O(1-\tilde\alpha))+\frac{1}{\sqrt{s^2-1}}(1+O(1-\tilde\alpha))\Big)\Big)\\ \tau\partial_\beta\varphi_- &= \tau^{1/3}(y_c-y_*)+\tau^{-1/3}\beta\Big(\frac{1}{\sqrt{r^2-1}}(1+O(\tau^{-2/3}\beta))+\frac{1}{\sqrt{s^2-1}}(1+O(\tau^{-2/3}\beta))\Big).\end{aligned}\tag{3.13}$$

At $\beta=2^{2k}\Xi$ we have $\partial_\Xi(\tau\sqrt{\varphi_-^2+z_Q^2})=\frac{\varphi_-}{\sqrt{\varphi_-^2+z_Q^2}}2^{2k}\partial_\beta(\tau\varphi_-)|_{\beta=2^{2k}\Xi}$ hence

$$\partial_\Xi(\tau\sqrt{\varphi_-^2+z_Q^2})=\frac{2^{4k}\tau^{-1/3}\varphi_-}{\sqrt{\varphi_-^2+z_Q^2}}\Big(\frac{\tau^{2/3}}{2^{2k}}(y_c-y_*)+\Xi\Big(\frac{1}{\sqrt{r^2-1}}(1+O(\tau^{-2/3}2^{2k}))+\frac{1}{\sqrt{s^2-1}}(1+O(\tau^{-2/3}2^{2k}))\Big)\Big),$$

$$\partial^2_\Xi(\tau\sqrt{\varphi_-^2+z_Q^2})|_{\partial_\beta\varphi_-=0}=\frac{2^{4k}\tau^{-1/3}\varphi_-}{\sqrt{\varphi_-^2+z_Q^2}}\Big(\frac{1}{\sqrt{r^2-1}}(1+O(\tau^{-2/3}2^{2k}))+\frac{1}{\sqrt{s^2-1}}(1+O(\tau^{-2/3}2^{2k}))\Big).$$

As $s\geq r$ and for $\frac{\varphi_-}{\sqrt{\varphi_-^2+z_Q^2}}$ on the support of $\tilde\chi_1$ we obtain a lower bound for the second order derivative of $\frac{2^{4k}\tau^{-1/3}}{\sqrt{r^2-1}}$. From now on we can proceed as in the case of φ_+ : if $\frac{2^{4k}\tau^{-1/3}}{\sqrt{r^2-1}}\geq\tau^{\epsilon'}$ for some $\epsilon'>0$, we apply the stationary phase if moreover $\frac{\tau^{2/3}\sqrt{r^2-1}}{2^{2k}}(y_*-y_c)\sim 1$: the last condition reduces the number of such k to at most three values for which we find

$$|I^{1-\chi_0,\chi_-,k}_{1,gl}(Q,Q_0,t)|\lesssim\frac{\tau^{-1/6}}{t}\times\frac{2^{2k}}{\phi^{1/2}}\Big(\frac{\varphi_-}{\sqrt{\varphi_-^2+z_Q^2}}\Big)^{3/4-1/2}\times\Big(\frac{2^{4k}\tau^{-1/3}}{\sqrt{r^2-1}}\Big)^{-1/2}\sim 1/t,\tag{3.14}$$

where we used that $(\varphi\sqrt{s^2-1})^{-1/2} \lesssim 1/t$ and $\phi \geq r-1$ to obtain $\frac{(r^2-1)^{1/4}}{\phi^{1/2}} \lesssim 1$. If $\frac{\tau^{2/3}\sqrt{r^2-1}}{2^{2k}}(y_*-y_c) \notin [1/4,4]$, repeated integrations by parts yield a $O(\tau^{-\infty}/t)$ contribution (and we conclude using that $2^k \lesssim \tau^{1/3}$).

Fix $M>4$ large enough and consider $2^{4k}\tau^{-1/3}\frac{1}{\sqrt{r^2-1}} \in [M^2,\tau^{\epsilon'}]$ for some $\epsilon'>0$. As this parameter is large, we still may apply the stationary phase but we need to verify that the remainders are sufficiently small and that we can bound their sum. There is still a finite number of k for which the phases may be stationary. At the critical points Ξ_c, the stationary phase applies and we obtain, for all $N\geq 1$,

$$I^{1-\chi_0,\chi_-,k}_{1,gl}(Q,Q_0,t) = \tau^{-1/6}e^{-i\tau\sqrt{\varphi_-^2+z_Q^2}}\frac{\tilde{\Sigma}_-(2^{2k}\Xi_c,\tau)}{\varphi_-^{1/2}(s^2-1)^{1/4}}\frac{2^{2k}\times|\partial^2_\Xi(\tau\sqrt{\varphi_-^2+z_Q^2})|^{-1/2}}{\phi^{1/2}(x_Q,\arccos(1/r)+\tau^{-1/3}w_-,0,1)} + O\Big((\frac{2^{4k}\tau^{-1/3}}{\sqrt{r^2-1}})^{-N}\tau^{-1/6}\times\frac{2^{2k}}{\sqrt{s^2-1}(r-1)^{1/4}}\Big), \quad (3.15)$$

where the main contribution of $I^{1-\chi_0,\chi_-,k}_{1,gl}(Q,Q_0,t)$ in the first line still satisfies (3.14) and where the remainder in the second line is $O\Big((\frac{2^{4k}\tau^{-1/3}}{4\sqrt{r^2-1}})^{-N}/t\Big)$. In the second line we used $\phi \geq (r-1)^{1/2}$. The bounds for the remainders follow using $\sup|\partial^2_\Xi(\tau\sqrt{\varphi_-^2+z_Q^2})| \geq \frac{2^{4k}\tau^{-1/3}}{\sqrt{r^2-1}}$. Notice that, taking one derivative of the cutoff $\chi_{\varepsilon_1}(\tau^{-2/3}2^{2k}\Xi) = \chi(\tau^{-2/3}2^{2k}\Xi/\varepsilon_1)$ yields a factor $\tau^{-2/3}2^{2k}/\varepsilon_1$ but, as $\Xi\sim 1$ on the support of $\chi(\beta 2^{-2k}) = \chi(\Xi)$, on the support of $\chi(\tau^{-2/3}2^{2k}\Xi/\varepsilon_1)\chi(\Xi)$ we have $\tau^{-2/3}2^{2k}/\varepsilon_1 \lesssim 1$ hence for $M>4$ sufficiently large this factor doesn't change the contribution of the remainder. For all k s.t the phase is not stationary, integration by parts yields a contribution of at most

$$\tau^{-1/6}\frac{2^{2k}}{t(r-1)^{1/2}}\times(2^{-4k}\tau^{1/3}\sqrt{r^2-1})^{N+1} = \frac{1}{t}\times(2^{-2k}\tau^{1/6}\sqrt{r^2-1}^{1/2})\times(2^{-4k}\tau^{1/3}\sqrt{r^2-1})^N$$

for all $N\geq 0$. Let $N=0$ and sum over k with $2^{4k}\tau^{-1/3}\frac{1}{\sqrt{r^2-1}} \in [M^2,\tau^{\epsilon'}]$, then

$$\frac{1}{t}\Big(\sum_{M^2\leq 2^{4k}\tau^{-1/3}\sqrt{r^2-1}^{-1}\leq\tau^{\epsilon'}} 2^{-2k}\Big)\times\tau^{1/6}\sqrt{r^2-1}^{1/2} \lesssim 1/(Mt).$$

Let now k such that $\tau^{\epsilon_1} \lesssim 2^k$ and $2^{4k}\tau^{-1/3}\frac{1}{\sqrt{r^2-1}} \leq M^2$ for some large, fixed $M>1$. We bound each $I^{1-\chi_0,\chi_-,k}_{1,gl}$ by $\tau^{-1/6}\frac{2^{2k}}{t(r-1)^{1/4}} \lesssim M/t$ using $2^{2k}\leq M\tau^{1/6}\sqrt{r^2-1}^{1/2}$ and conclude. □

Remark 3.9. *In the two previous Lemmas, the bounds for $I^{1-\chi_0,\chi_\pm}_{1,gl}$ come with additional factors $(\frac{\varphi_\pm}{\sqrt{\varphi_\pm^2+z_Q^2}})^{1/4}$. This is useful to keep in mind for the case when $1-\gamma^2$ will be small.*

Let now β on the support of $\chi_0(\beta)$, when, using (3.4), the Airy factor can be brought in the symbol. The phase of $I^{1-\chi_0}_{1,gl}$ equals now $\tau(z_Q\gamma-\sqrt{1-\gamma^2}\varphi_0)$, where $\varphi_0 := (-(y_c-y_*+\tau^{-1/3}w)(1-\tau^{-2/3}\beta)+\sqrt{s^2-1}+\phi(x_Q,\arccos(1/r)+\tau^{-1/3}w,0,1)) \geq \sqrt{s^2-1}$. As $\sqrt{1-\gamma^2}\times w \geq \tau^{\epsilon_1}$ on the support of $\tilde{\chi}_1(1-\gamma^2)(1-\chi_0)(w\tau^{-\epsilon_1})$, it follows that the phase is non-stationary in w as $\beta\leq 2 \ll \tau^{2\epsilon_1} \lesssim w^2/2$ and we integrate by parts to obtain $O(\tau^{-\infty}/t)$.

Case $|y-y_c| \leq 2\tau^{-1/3+\epsilon_1}$: study of $I^{\chi_0}_{1,gl}$. Let again $y = y_c+\tau^{-1/3}w$, with $|w|\leq\tau^{\epsilon_1}$ this time. As $\partial_w\phi(x_Q,\arccos(1/r)+\tau^{-1/3}w,0,1) = \tau^{-1/3}\Big(1-\tau^{-2/3}w^2/2(1+O(\tau^{-1/3}w))\Big)$, the derivative w.r.t. w of phase of $I^{\chi_0}_{1,gl}$ equals

$$\tau^{1-1/3}\sqrt{1-\gamma^2}\Big(1-\tau^{-2/3}\beta-1+\tau^{-2/3}w^2/2(1+O(\tau^{-1/3}w))\Big) = \sqrt{1-\gamma^2}(-\beta+w^2/2(1+O(\tau^{-1/3}w)),$$

hence, for $\beta\geq\tau^{2\epsilon_1}$ we may perform repeated integrations by parts and obtain a $O(\tau^{-\infty}/t)$ contribution (using that the support in w,β is bounded). We may therefore introduce $\chi_0(\beta\tau^{-2\epsilon_1})$ into the symbol of $I^{\chi_0}_{1,gl}$ without changing its contribution modulo $O(\tau^{-\infty}/t)$ terms. If we introduce moreover a cutoff $\chi_0(\beta)$

supported for $\beta \leq 2$, the Airy factor doesn't oscillate and may be brought into the symbol : in this case the phase of $I^{\chi_0}_{1,gl}$ is given by

$$\tau(z_Q\gamma - \sqrt{1-\gamma^2}(-(y_c - y_* + \tau^{-1/3}w)(1-\tau^{-2/3}\beta) + \sqrt{s^2-1} + \phi(x_Q, \arccos(1/r) + \tau^{-1/3}w, 0, 1))).$$

Let $\varphi_0 := -(y_c - y_* + \tau^{-1/3}w)(1-\tau^{-2/3}\beta) + \sqrt{s^2-1} + \phi(x_Q, \arccos(1/r) + \tau^{-1/3}w, 0, 1)$, then $\varphi_0 \geq \sqrt{s^2-1}$ and the stationary phase w.r.t. γ applies exactly as before. The critical point γ satisfies $z_Q = \frac{\gamma}{\sqrt{1-\gamma^2}}\varphi_0$. The corresponding contribution of $I^{\chi_0}_{1,gl}(Q, Q_0, \tau)$ is of the form (3.9) where moreover $\beta \leq 2$ and $|w| \leq \tau^{\epsilon_1}$. We then obtain

$$|I^{\chi_0}_{1,gl}(Q, Q_0, \tau)| \lesssim \frac{\tau^{1/6-1/3}}{\varphi_0^{1/2}(s^2-1)^{1/4}} \times \tau^{\epsilon_1}, \tag{3.16}$$

where the exponents $1/6 - 1/3$ come from (3.9) and the change of variable $y = y_c + \tau^{-1/3}w$, and the factor τ^{ϵ_1} from the size of the support in w. Let now $\beta \in [3/2, \tau^{2\epsilon_1}]$ on the support of $(1-\chi_0(\beta))\chi_0(\beta\tau^{2\epsilon_1})$, when the Airy factor does oscillate. We also have $\varphi \geq \sqrt{s^2-1}$ and the stationary phase w.r.t γ applies. The corresponding contribution of $I^{\chi_0}_{1,gl}(Q, Q_0, \tau)$ may be bounded as in (3.16) but with an additional factor $\tau^{2\epsilon_1}$ arising from the support wr.t. $\beta \leq \tau^{2\epsilon_1}$. Taking $\varepsilon_1 < 1/18$ allows to conclude. □

3.1.2. *Case $\sqrt{1-\gamma^2} \sim 2^{-j}$, $s \geq 1 + \varepsilon_0 = \sqrt{2}$ and $\tau 2^{-3j}s \geq M$ for some sufficiently large $M \gg 1$.* We proceed exactly like before for each such j and then sum up.

3.2. **Dispersive bounds when $d(Q, \partial\Omega) \leq \varepsilon_0 \leq d(Q_0, \partial\Omega)$.** Let $1 \leq r \leq 1+\varepsilon_0 \leq s$ and let $\sqrt{1-\gamma^2} \geq 1/10$ or $\sqrt{1-\gamma^2} \sim 2^{-j}$ such that $\tau 2^{-3j}s \geq M$ for large $M \gg 1$. We proceed as in [2, Section 3.3] : in this case we obtain directly the form of the reflected wave, which holds as the observation point Q is close to the boundary ; formula (1.14) becomes useless since $d(Q, \partial\Omega)$ is arbitrarily small. By Proposition 2.9 we are reduced to prove $|\int \chi(h\tau)\tau e^{it\tau} I(Q_0, Q, \tau)d\tau| \leq \frac{C}{h^2 t}$ for a constant independent of Q_0 and Q, where

$$I(\tau, Q_0, Q) := \tau \int e^{\mathrm{i}\tau(y\alpha + z\gamma)}\Big(aA_+(\tau^{2/3}\zeta) + b\tau^{-1/3}A'_+(\tau^{2/3}\zeta)\Big)\frac{A(\tau^{2/3}\zeta_0)}{A_+(\tau^{2/3}\zeta_0)}\tilde{\chi}_1(1-\gamma^2)\widehat{F}_{1,\tau}(\alpha,\gamma)d\alpha d\gamma.$$

Lemma 3.10. *There exists a constant $C > 0$, such that, for all Q in a small neighborhood of $\mathcal{C}_{Q_0}$, $|y - y_0| \leq \frac{\pi}{16}$ and $t \sim dist(Q_0, \partial\Omega) + dist(Q, \partial\Omega)$ the following holds*

$$\big|I(Q_0, Q, \tau)\big| \leq \frac{C}{t}.$$

The Lemma follows exactly as in [2, Lemma 3.25] because the observation point Q is located near a glancing point of the boundary (notice that, in the case Q far from $\partial\Omega$, the geometry of the obstacle was important and the approach to obtain dispersive bounds in the case of the exterior of the cylinder was different from the one in the exterior of a ball; when Q is near $\partial\Omega$ the same arguments hold in both cases so we do not reproduce the proof here).

3.3. **Dispersive bounds for the "non-glancing" part,** $d(Q_0, \partial\Omega) \geq \varepsilon_0$**.** Let $s \geq 1 + \varepsilon_0 = \sqrt{2}$. We let $u^{\#}_{1,he} := \Box_+^{-1}(\partial_x u^+_{1,he}|_{\partial\Omega})$, $u^{\#}_{h,1,he} = \chi(hD_t)u^{\#}_{1,he}$, where $\partial_x u^+_{1,he}|_{\partial\Omega}$ is defined in (2.25). We also let $u^{\#}_{2^{-2j},he} := \Box_+^{-1}(\partial_x u^+_{2^{-2j},he}|_{\partial\Omega})$, $u^{\#}_{h,2^{-2j},he} = \chi(hD_t)u^{\#}_{2^{-2j},he}$ for $j \geq j(s,r,h)$ where $\partial_x u^+_{2^{-2j},he}|_{\partial\Omega}$ is defined in (2.29) for $j(s,r,h)$ is such that $2^{-3j(s,r,h)} \geq Mh$ for some large $M \gg 1$. We prove the following :

Proposition 3.11. *There exists $C = C(\varepsilon, \varepsilon_0) > 0$ such that for all $t > h$, the following holds uniformly with respect to Q, Q_0 such that $s \geq r \geq 1 + \varepsilon_0$ (where $s = 1 + x_{Q_0}$, $r = 1 + x_Q$) :*

$$|u^{\#}_{h,1,he}(Q, Q_0, t)| \leq \frac{C}{h^2|t|}, \qquad \sum_{j \geq j(s,r,h)} |u^{\#}_{h,2^{-2j},he}(Q, Q_0, t)| \leq \frac{C}{h^2|t|}.$$

Proof. From section 2.1.3, the phase function of $\mathcal{F}(\partial_\nu u^+)(P, Q_0, \tau)$ is $-i\tau|P - Q_0|$ and the symbol is $\frac{\tau^2}{|P-Q_0|}\sigma_{1,he}(P, Q_0, \tau)$ with $\sigma_{1,he}$ a classical symbol of order 0 with respect to τ supported for P with coordinates $(1, \theta, z)$ such that $s(\cos\theta_* - \cos\theta) \geq c(\varepsilon_1)|P - Q_0|$. Moreover, on the support of the symbol $\sigma_{1,he}$ we have

$$1-(\partial_z\tilde\phi(1,\theta,s,z))^2\geq 1/16,\quad 1-\frac{\partial_\theta\tilde\phi(1,\theta,s,z)}{\sqrt{1-(\partial_z\tilde\phi)^2(1,\theta,s,z)}}\geq C(\varepsilon_1),\tag{3.17}$$

for some $C(\varepsilon_1)>0$ depending only on ε_1. The phase of $u^{\#}_{h,1,he}(Q,Q_0,t)$ is $\tau(t-\Phi)$ where $\Phi:=|Q-P|+|P-Q_0|$. In the following we explicitly compute the derivative of the phase Φ of $u^{\#}_{h,1,he}$, where

$$\Phi:=|Q-P|+|Q_0-P|=\tilde\phi(1,\theta-\theta_Q,r_Q,z-z_Q)+\tilde\phi(1,\theta,s,z),$$

where $P=(\cos\theta,\sin\theta,z)$, $Q_0=(s,0,0)$ and $Q=(r_Q\cos\theta_Q,r_Q\sin\theta_Q,z_Q)$. Let $r=r_Q$. The critical points satisfy $\partial_\theta\Phi=\partial_z\Phi=0$, which is equivalent to

$$\begin{cases}\dfrac{z}{\tilde\phi(1,\theta,s,z)}+\dfrac{z-z_Q}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}=0,\\[2mm] \dfrac{s\sin\theta}{\tilde\phi(1,\theta,s,z)}+\dfrac{r\sin(\theta-\theta_Q)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}=0.\end{cases}\tag{3.18}$$

We aim at applying the stationary phase with respect to both θ and z. We evaluate the second order derivatives of Φ at $\nabla_{\theta,z}\Phi=0$. The second order derivative of Φ satisfies

$$\partial^2_{z,z}\Phi|_{\partial_z\Phi=0}=\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\Big(1-\frac{z^2}{\tilde\phi^2(1,\theta,s,z)}\Big).\tag{3.19}$$

Next, as $\partial^2_{\theta,z}\Phi=-\Big(\frac{zs\sin\theta}{\tilde\phi^3(1,\theta,s,z)}+\frac{(z-z_Q)r\sin(\theta-\theta_Q)}{\tilde\phi^3(1,\theta-\theta_Q,r,z-z_Q)}\Big)$, we obtain, using the system (3.18),

$$\partial^2_{\theta,z}\Phi|_{\nabla_{\theta,z}\Phi=0}=-\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\frac{zs\sin\theta}{\tilde\phi^2(1,\theta,s,z)}.\tag{3.20}$$

Finally, we compute

$$\partial^2_{\theta,\theta}\Phi=\frac{s\cos\theta}{\tilde\phi(1,\theta,s,z)}-\frac{s^2\sin^2\theta}{\tilde\phi^3(1,\theta,s,z)}+\frac{r\cos(\theta-\theta_Q)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}-\frac{r^2\sin^2(\theta-\theta_Q)}{\tilde\phi^3(1,\theta-\theta_Q,r,z-z_Q)}.\tag{3.21}$$

To evaluate $\partial^2_{\theta,\theta}\Phi|_{\nabla_{z,\theta}\Phi=0}$ we need a refined analysis of the critical points. Using both equations in (3.18) gives $\frac{s\sin\theta}{\psi(s,\theta)}=-\frac{r\sin(\theta-\theta_Q)}{\psi(r,\theta-\theta_Q)}$, where $\psi(s,\theta)=\sqrt{1-2s\cos\theta+s^2}$, and hence $\theta\in[\theta_Q-\pi,\theta_Q]$. Taking the square in the last equality in (3.18), subtracting 1 and then using the first in (3.18) yields $\frac{s\cos\theta-1}{\tilde\phi(1,\theta,s,z)}=\pm\frac{(r\cos(\theta-\theta_Q)-1)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}$. Depending on the sign, we separate three situations :

Different signs. Consider first the case $\frac{s\cos\theta-1}{\tilde\phi(1,\theta,s,z)}=-\frac{(r\cos(\theta-\theta_Q)-1)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}$ when

$$\frac{s\cos\theta}{\tilde\phi(1,\theta,s,z)}+\frac{r\cos(\theta-\theta_Q)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}=\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}.$$

We find

$$\partial^2_{\theta,\theta}\Phi|_{\nabla_{\theta,z}\Phi=0}=\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\Big(1-\frac{s^2\sin^2\theta}{\tilde\phi^2(1,\theta,s,z)}\Big).\tag{3.22}$$

Using (3.19), (3.22), (3.20), the determinant of the Hessian matrix equals

$$\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)^2\times\frac{(s\cos\theta-1)^2}{\tilde\phi^2(1,\theta,s,z)}|_{\nabla_{\theta,z}\Phi=0}\tag{3.23}$$

and on the support of the symbol $\sigma_{1,he}$ the second factor in (3.23) takes values in $[c^2(\varepsilon_1),1]$. The unique critical point w.r.t. z reads as $z_c=z_Q\times\frac{\psi(s,\theta)}{\psi(s,\theta)+\psi(r,\theta-\theta_Q)}$.

Lemma 3.12. *When $\tau\times\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\geq M$ for some $M>1$ large enough, the usual stationary phase applies for θ,z near the critical points and yields*

$$|\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)|\lesssim\frac{\tau}{t}$$

for $t \sim \tilde\phi(1,\theta_c-\theta_Q,r,z_c-z_Q)+\tilde\phi(1,\theta_c,s,z_c)$. *For* z,θ *outside a fixed neighborhood of the critical points the previous estimate still holds.*

Proof. When $\tau\times\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\geq\tau^{\epsilon'}$ for some $\epsilon'>0$, the stationary phase obviously applies with large parameter $\gtrsim\tau^{\epsilon'}$: then $\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)$ takes the form

$$\begin{aligned}\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)=&\frac{\tau^2\tilde\sigma_{1,he}(\theta,z,s,\tau)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)\tilde\phi(1,\theta,s,z)}e^{i\tau(t-\Phi)_{z_c,\theta_c}}\\&\times\tau^{-1}\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)^{-1}\Big|_{z_c,\theta_c}\\&+O\Big(\frac{\tau^{-\infty}}{\tilde\phi(1,\theta_c-\theta_Q,r,z_c-z_Q)+\tilde\phi(1,\theta_c,s_c,z)}\Big)\end{aligned}\quad(3.24)$$

for some new symbol $\tilde\sigma_{1,he}$ which reads as an asymptotic expansion with main contribution $\sigma_{1,he}$ and small parameter $\lesssim\tau^{-\epsilon'}$. As the main contribution of $\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)$ can be bounded by $\frac{\tau\tilde\sigma_{1,he}(\theta,z,s,\tau)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)+\tilde\phi(1,\theta,s,z)}$, for $t\sim\tilde\phi(1,\theta_c-\theta_Q,r,z_c-z_Q)+\tilde\phi(1,\theta_c,s,z_c)$ this allows to conclude using the integration w.r.t. τ. For t that doesn't satisfy this condition we conclude by integrations by parts, finite speed of propagation and support properties of the symbol. If we replace $\tau^{\epsilon'}$ by some large constant M, the main contribution of $\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)$ can be bounded in the same way, but we need to check the remainder terms which we may bound by

$$\frac{\tau^2}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)\tilde\phi(1,\theta,s,z)}\times\tau^{-1}\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)M^{-N}$$

for all $N\geq1$, which is enough to conclude.

Let now z,θ outside a fixed neighborhood of the critical points. First of all, if $|z|\geq2|t|$ then the phase $\tau(t-\Phi)$ cannot be stationary w.r.t. τ ; let $|z|\leq2|t|$ such that $|\frac{z}{z_c}-1|\geq c$ for some small, fixed constant $c>0$. If $\tau\frac{|z_Q|}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\geq M_1$ for some large $M_1>1$, then, using $\tau\partial_z\Phi=\frac{z_Q}{\tilde\phi_2(1,\theta-\theta_Q,r,z-z_Q)}(\frac{z}{z_c}-1)$, repeated integrations by parts allows to conclude. Let $\tau\frac{|z_Q|}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}<M_1$. As $\tau\partial_z\Phi=\tau\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)z-\tau\frac{z_Q}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}$, then if $\tau\partial_z\Phi\geq M_2$ for some large constant $M_2>1$, repeated integrations by parts allow again to conclude (after checking the size of the remainders); if, instead, $\tau\partial_z\Phi\leq M_2$ then

$$|z|\leq\Big(\frac{M_2}{\tau}+\frac{|z_Q|}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\frac{1}{(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)})}$$

and we directly obtain, using the size of the support of the integrant (z,θ) (with θ bounded)

$$|\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)|\lesssim\frac{\tau^2}{\tilde\phi_1\tilde\phi_2}\frac{M_1+M_2}{\tau}\times\frac{\tilde\phi_1\tilde\phi_2}{\tilde\phi_1+\tilde\phi_2}\lesssim\frac{1}{t},$$

with $\tilde\phi_1=\tilde\phi(1,\theta,s,z)$ and $\tilde\phi_2=\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)$. □

Lemma 3.13. *When* $\tau\times\Big(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\Big)\leq M$ *estimate* (3.24) *still holds for* $t\sim\tilde\phi(1,\theta_c-\theta_Q,r,z_c-z_Q)+\tilde\phi(1,\theta_c,s,z_c)$.

Proof. For $|z/z_c-1|\geq c$ we may proceed as in the second part of the proof of the previous lemma. Let therefore $z/z_c\in[1/4,4]$ and make the change of variables $z=z_c\Xi$. Then

$$\tau\partial_\Xi\Phi=\tau z_c\partial_z\Phi|_{z=z_c\Xi}=\tau z_c\times\frac{z_Q}{\tilde\phi_2}(\Xi-1)=\tau z_Q^2\frac{\tilde\phi_1}{\tilde\phi_2}\frac{1}{\tilde\phi_1+\tilde\phi_2}(\Xi-1).$$

Using (3.19), we obtain the second order derivative at $\Xi=1$ as follows

$$\tau\partial^2_\Xi\Phi|_{\Xi=1}=\tau z_c^2\partial_z^2\Phi|_{z=z_c\Xi}=\tau(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2})z_c^2\frac{\psi_1^2}{\tilde\phi_1^2},$$

where, from the support properties of the symbol, the last factor is bounded from below by a fixed constant. If $\tau(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2})z_c^2 \geq \tilde M$, we apply the stationary phase near $\Xi = 1$ only with respect to Ξ (and not with θ) as in the previous lemma and, using that θ belongs to a compact set, we find the following uniform bound

$$|\mathcal{F}(u^{\#}_{1-\chi_0})(Q,Q_0,\tau)| \lesssim \frac{\tau^2}{\tilde\phi_1\tilde\phi_2}\times z_c \times \tau^{-1/2} z_c^{-1}(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2})^{-1/2} \lesssim \frac{\tau}{\tilde\phi_1+\tilde\phi_1}\times\Big(\tau(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2})\Big)^{1/2} \tag{3.25}$$

and we conclude using the hypothesis $\tau\times\left(\frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}\right)\leq M$. □

Same sign, P in the illuminated regime of Q_0, Q. Consider now the situation $\frac{s\cos\theta-1}{\psi(s,\theta)} = \frac{r\cos(\theta-\theta_Q)-1}{\psi(r,\theta-\theta_Q)}$. The formula (3.19) remains unchanged and $\partial^2_{z,z}\Phi$ is strictly positive. Moreover, from the support condition of $\sigma_{1,he}$ we have $s\cos\theta > 1$ then $r\cos(\theta-\theta_Q) > 1$ and in (3.21) we obtain a lower bound for the sum of the first and third terms at the critical points as follows :

$$\frac{s\cos\theta}{\tilde\phi(1,\theta,s,z)}+\frac{r\cos(\theta-\theta_Q)}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)} \geq \frac{1}{\tilde\phi(1,\theta,s,z)}+\frac{1}{\tilde\phi(1,\theta-\theta_Q,r,z-z_Q)}$$

and we can proceed exactly as in the previous case.

Remark 3.14. *Notice that the positivity condition $s\cos\theta > 1$ is equivalent to $\cos\theta > \cos\theta_*$, where $\theta_* = \arccos(1/s)$, which in turn assures that the point P belongs to the illuminated region of Q (as $\theta < \theta_*$). When both conditions hold ($\cos\theta > 1/s$ and $\cos(\theta-\theta_Q) > 1/r$), the point P belongs to the illuminated regions from Q_0 and Q. In fact, the line Q_0Q is tangent to the boundary when $\arccos(1/s)+\arccos(1/r) = \theta_Q$: if $P\in\partial\Omega$ is such that the cosinus of the angle between QO and OP is larger than $1/r$, then the point Q belongs to the illuminated regime of Q_0. As such, the previous case when $\pm(s\cos\theta-1) > 0$ and $\pm(1-r\cos(\theta-\theta_Q)) > 0$ corresponds to points P which belong to the illuminated regime of only one of the two poins Q_0 and Q. In the last case $s\cos\theta - 1 < 0$ and $1-r\cos(\theta-\theta_Q) < 0$ that will be dealt with in the remaining of this section, P does not belong to the illuminated regions of Q_0, Q.*

Same sign, P in the shadow regime of Q_0, Q. In this case we do not have a lower bound for the determinant of the Hessian matrix as before. Replacing $\frac{r\cos(\theta-\theta_Q)}{\tilde\phi_2} = -\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2}+\frac{s\sin\theta}{\tilde\phi_1}$ in the expression (3.21) yields the following form for the determinant of the Hessian matrix at this critical point :

$$\Big(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2}\Big)\times\frac{(1-s\cos\theta)}{\tilde\phi_1}\times\Big|\Big(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2}\Big)\frac{(1-s\cos\theta)}{\tilde\phi_1}-2\frac{\psi_1^2}{\tilde\phi_1^2}\Big|_{|\nabla_{\theta,z}\Phi=0}. \tag{3.26}$$

Lemma 3.15. *At $\nabla_{\theta,z}\Phi = 0$ the following holds*

$$\frac{\psi_2^2}{\tilde\phi_2^2}-\frac{1}{\tilde\phi_2}\frac{(1-r\cos(\theta-\theta_Q))}{\tilde\phi_2} = \frac{r(r-\cos(\theta-\theta_Q))}{\tilde\phi_2^2} \geq \frac{r(r-1)}{\tilde\phi_2^2}.$$

$$\frac{\psi_1^2}{\tilde\phi_1^2}-\frac{1}{\tilde\phi_1}\frac{(1-s\cos\theta)}{\tilde\phi_1} = \frac{s(s-\cos\theta)}{\tilde\phi_2^2} \geq \frac{s(s-1)}{\tilde\phi_1^2}.$$

The lemma is a direct computation. Taking the sum of the terms in the left hand side and using that $\tilde\phi_j = \psi_j\sqrt{1+\frac{z_Q^2}{(\psi_1+\psi_2)^2}}$ for $j\in\{1,2\}$ and $\frac{(1-s\cos\theta)}{\tilde\phi_1} = \frac{(1-r\cos(\theta-\theta_Q))}{\tilde\phi_2}$ yields at $\nabla_{\theta,z}\Phi = 0$

$$\Big(2\frac{\psi_1^2}{\tilde\phi_1^2}-\Big(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2}\Big)\frac{(1-s\cos\theta)}{\tilde\phi_1}\Big) \geq \frac{s(s-1)}{\tilde\phi_1^2}+\frac{r(r-1)}{\tilde\phi_2^2},$$

which further induces the following lower bound for the determinant of the Hessian matrix

$$\Big(\frac{1}{\tilde\phi_1}+\frac{1}{\tilde\phi_2}\Big)\times\frac{(1-s\cos\theta)}{\tilde\phi_1}\times\Big(\frac{s(s-1)}{\tilde\phi_1^2}+\frac{r(r-1)}{\tilde\phi_2^2}\Big).$$

From now on we may proceed as in the proof of the first case, applying the stationary phase when this determinant is sufficiently large and obtaining bounds using the size of the intervals of integration when the stationary phase fails to apply. Thus, we obtain the equivalent of Lemma 3.12

Lemma 3.16. *When* $\tau \times \Big(\frac{1}{\tilde{\phi}(1,\theta,s,z)} + \frac{1}{\tilde{\phi}(1,\theta-\theta_Q,r,z-z_Q)}\Big)^{1/2}\Big(\frac{s(s-1)}{\tilde{\phi}_1^2} + \frac{r(r-1)}{\tilde{\phi}_2^2}\Big)^{1/2} \geq M$ *for some large* $M > 1$, *the usual stationary phase applies for* θ, z *near the critical points and yields*

$$|\mathcal{F}(u^{\#}_{1-\chi_0})(Q, Q_0, \tau)| \lesssim \frac{\tau}{t}$$

for $t \sim \tilde{\phi}(1, \theta_c - \theta_Q, r, z_c - z_Q) + \tilde{\phi}(1, \theta_c, s, z_c)$. *For* z, θ *outside a fixed neighborhood of the critical points the previous estimate still holds.*

Proof. The main contribution of $\mathcal{F}(u^{\#}_{1-\chi_0})(Q, Q_0, \tau)$ after applying the stationary phase may be bounded as follows

$$\frac{\tau^2}{\tilde{\phi}_1\tilde{\phi}_2} \times \tau^{-1}\Big(\frac{1}{\tilde{\phi}_1} + \frac{1}{\tilde{\phi}_2}\Big)^{-1/2}\Big(\frac{s(s-1)}{\tilde{\phi}_1^2} + \frac{r(r-1)}{\tilde{\phi}_2^2}\Big)^{-1/2} \lesssim \frac{\tau}{\tilde{\phi}_1 + \tilde{\phi}_2} \times \sqrt{\frac{1}{\tilde{\phi}_1} + \frac{1}{\tilde{\phi}_2}}\Big(\frac{\tilde{\phi}_1}{4s} + \frac{\tilde{\phi}_2}{4r}\Big).$$

Using (3.17) on the support of $\sigma_{1,he}$, we obtain the desired estimates. We let the other situations to the reader. ☐

The proof is written for $\sqrt{1-\gamma^2} \geq 1/10$. When $\sqrt{1-\gamma^2} \sim 2^{-j}$ and $j \geq j(s, r, h)$ the phase is the same, only the symbol comes with non positive powers of 2^j, that one may sum up and obtain the second statement of the lemma. ☐

4. High-frequency case. Parametrix and dispersive estimates for $d(Q, \partial\Omega) < \varepsilon_0$ and $d(Q_0, \partial\Omega) < \varepsilon_0$, $\varepsilon_0 = \sqrt{2} - 1$, or for $d(Q, \partial\Omega) \geq \varepsilon_0$ and $d(Q_0, \partial\Omega) \geq \varepsilon_0$ and $\sqrt{1-\gamma^2} \sim 2^{-j}$ with $\tau 2^{-3j} d(Q_0, \partial\Omega) \leq M$

In this section both Q and Q_0 are close to the boundary. For convenience, we will assume this time that $s \leq r \leq \sqrt{2}$. Denote $\mathcal{R}(Q, Q_0, \tau)$ the outgoing solutions of the Helmholtz equation $(\tau^2 + \Delta)w = \delta_{Q_0}$, $w_{|\partial\Omega} = 0$ with $Q_0 = (s, 0, 0)$ where we recall that, in cylindrical coordinates, $\Delta = \partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2} + \partial_z^2$. Then the solution of the wave equation with initial condition $(u_0, u_1) = (\delta_{Q_0}, 0)$ is given by

$$u(Q, Q_0, t) = \int_0^\infty e^{\mathrm{i}t\tau}\mathcal{R}(Q, Q_0, \tau)\frac{d\tau}{\pi}. \tag{4.1}$$

For a given $w_0(r, \theta, z)$, the solution to the *inhomogeneous equation* $(\tau^2 + \Delta)w = w_0$ reads as

$$w(\tau, r, \theta, z) = \int_{\mathbb{R}} e^{\mathrm{i}z\vartheta}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\theta}\int_1^\infty G_n(r, \tilde{r}, \kappa(\vartheta, \tau))\tilde{r}^2\widehat{w}_0(\tilde{r}, n, \vartheta)d\tilde{r}d\vartheta,$$

where the kernel G_n is symmetric w.r.t $r, \tilde{r}$ and, for $r \geq \tilde{r}$, it is given by

$$\begin{aligned} G_n(r, \tilde{r}, \kappa) &= \frac{\pi}{2\mathrm{i}}(r\tilde{r})^{-\frac{1}{2}}\Big(J_n(\tilde{r}\kappa) - \frac{J_n(\kappa)}{H_n(\kappa)}H_n(\tilde{r}\kappa)\Big)H_n(r\kappa), \\ &= \frac{\pi}{4\mathrm{i}}(r\tilde{r})^{-\frac{1}{2}}\Big(\overline{H}_n(\tilde{r}\kappa) - \frac{\overline{H}_n(k)}{H_n(k)}H_n(\tilde{r}\kappa)\Big)H_n(r\kappa). \end{aligned} \tag{4.2}$$

Here $J_n(z) = \frac{1}{2}(H_n(z) + \overline{H}_n(z))$ denotes the Bessel function and $\kappa(\vartheta, \tau) := \sqrt{\tau^2 - \vartheta^2}$. As n is an integer, $H_{-n}(z) = (-1)^n H_n(z)$, therefore the kernel aslo satisfies $G_n = G_{-n}$.
Taking $w_0 = \delta_{Q_0}$ with $Q_0 = (s, 0, 0)$ and $s \leq r$ yields $\tilde{r} = s$ and

$$\mathcal{R}(Q, Q_0, \tau) = s^2\int_{\mathbb{R}} e^{\mathrm{i}z\vartheta}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\theta}G_{|n|}(r, s, \kappa(\vartheta, \tau))d\vartheta.$$

Lemma 4.1. *Fix* $0 < h_0 < 1$ *small enough and let* $h \leq h_0$. *Let* $\chi \in C_0^\infty(1/2, 2)$ *valued in* $[0, 1]$ *and equal to* 1 *on* $[\frac{3}{4}, \frac{3}{2}]$. *There exist a uniform constant* $C > 0$ *such that for all* $1 \leq s \leq r \leq \sqrt{2}$ *and all* t, *we have*

$$I(Q, Q_0, \tau) := \int_0^\infty e^{\mathrm{i}t\tau}\chi(h\tau)\mathcal{R}(Q, Q_0, \tau)d\tau \leq C\frac{\|\chi\|_{L^1}}{h^2 t}. \tag{4.3}$$

Moreover, (4.3) *remains true for all* z *and* $r, s > 1$ *satisfying* $\frac{|z|}{(s+r-2)} \gg 1$.

Proof. Let $\kappa=\kappa(\vartheta,\tau)=\sqrt{\tau^2-\vartheta^2}$ and set

$$G^+_n(r,s,\kappa)=\frac{\pi}{4i\sqrt{rs}}\overline{H}^{(1)}_n(s\kappa)H^{(1)}_n(r\kappa),G^-_n(r,s,\kappa):=\frac{\pi}{4i\sqrt{rs}}\frac{\overline{H}^{(1)}_n(\kappa)}{H^{(1)}_n(\kappa)}H^{(1)}_n(s\kappa)H^{(1)}_n(r\kappa). \tag{4.4}$$

Substitute (4.4) in (4.2) and denote $\mathcal{R}^\pm$ and $I^\pm(Q,Q_0,\tau)$ the corresponding contributions, respectively, so that $I=I^++I^-$. We study the outgoing part I^-; the case of the incoming part I_+ is similar and left to the reader. Let $\chi_0\in C^\infty_0(-2,2)$ valued in $[0,1]$ and equal to 1 on $[-1,1]$ and let $\epsilon>0$ small enough, to be chosen later. Let first $|n|\geq 1$ and define

$$I^\pm_{\chi_1,\chi_2}:=\int_0^\infty e^{it\tau}\chi(h\tau)\sum_{n\neq 0}e^{in\theta}\int e^{iz\vartheta}\chi_1\big(\frac{\varepsilon\sqrt{\tau^2-\vartheta^2}}{|n|}\big)\chi_2\big(\frac{\varepsilon|n|}{\sqrt{\tau^2-\vartheta^2}}\big)s^2G^\pm_{|n|}(r,s,\kappa(\vartheta,\tau))d\vartheta d\tau, \tag{4.5}$$

where $\chi_1,\chi_2\in\{\chi_0,1-\chi_0,1\}$. We have $I^\pm=I^\pm_{1-\chi_0,1}+I^\pm_{\chi_0,1}$ and first we will prove that $I^-_{1-\chi_0,1}$ is bounded by $C/(h^2t)$ as in (4.3). We let $\vartheta=\tau\gamma$, then for all n, the phase functions in the sum defining $I^-_{1-\chi_0,1}$ are all equal to $\tau\phi^-_{1-\chi_0}$, with $\phi^-_{1-\chi_0}:=t+z\gamma+(r+s-2)\sqrt{1-\gamma^2}$. From (6.3) it follows that, for each $n\neq 0$, the symbol of the integrals in (4.5) with factor $e^{in\theta}$, that we denote $\tau J^-_{1-\chi_0,n}$ (and where τ comes from $\vartheta\to\gamma\tau$), is of the form

$$J^-_{1-\chi_0,n}=\frac{s}{r}\Big(1-\chi_0\big(\frac{\varepsilon\tau\sqrt{1-\gamma^2}}{|n|}\big)\Big)\frac{\sigma(r\tau\sqrt{1-\gamma^2})\sigma(s\tau\sqrt{1-\gamma^2})}{\sqrt{1-\gamma^2}},$$

where $\sigma(\cdot)=1+O(1/\cdot)$ (see (6.3)). With χ_0 as before and $\varepsilon_0=\sqrt{2}-1$, decompose $1=\chi_0(z/\varepsilon_0)+(1-\chi_0)(z/\varepsilon_0)$ and let $I^{-,\chi_*}_{1-\chi_0,1}:=\chi_*(z/\varepsilon_0)\sum_{n\neq 0}e^{in\theta}\int e^{i\tau\phi^-_{1-\chi_0}}\chi(h\tau)J^-_{1-\chi_0,n}d\gamma d\tau$. Consider first $I^{-,1-\chi_0}_{1-\chi_0,1}$: the phase function $\phi^-_{1-\chi_0}$ has the critical point $\gamma_c=\frac{z}{\sqrt{z^2+(r+s-2)^2}}$: let $\gamma=\gamma_c\Gamma$, then the phase is non-stationary for Γ outside a small neighborhood of 1 and for each n, the contribution of each integral is $O((h/|z|)^\infty)$, as in this case $|\partial_\gamma\phi^-_{1-\chi_0}|\gtrsim|z|$ and on the support of $(1-\chi_0)(z/\varepsilon_0)$ we can perform repeated integrations by parts with large parameter $\sim 1/h$. As on the support of the symbol $J^-_{1-\chi_0,n}$ we have $|n|\leq 2\epsilon\tau\sqrt{1-\gamma^2}\leq\epsilon/h$ it follows that the sum over n of all these contributions remains $O((h/|z|)^\infty)$. If $|t|>4|z|$, the phase is non-stationary w.r.t. τ and we obtain a contribution $O((h/|t|)^\infty)$ as $|\partial_\tau\phi^-_{1-\chi_0}|\geq t$; for $|t|\lesssim|z|$ we obtain the same contribution as $(h/|z|)\lesssim(h/|t|)$. Near the critical point $\Gamma=1$, the second derivative equals $|\partial^2_\Gamma\Phi|_{\Gamma=1}|=\frac{z^2}{(r+s-2)^2}(z^2+(r+s-2)^2)^{1/2}$. For $1\leq s<r\leq 1+\varepsilon_0$ and z on the support of $(1-\chi_0)(z/\varepsilon_0)$, we have $|\partial^2_\Gamma\Phi|_{\Gamma=1}|>\varepsilon_0$ and we may apply the stationary phase for Γ near 1 ; this yields $I^{-,1-\chi_0}_{1-\chi_0,1}=\sum_{n\neq 0}e^{in\theta}I^{n,-,1-\chi_0}_{1-\chi_0,1}+O((h/|t|)^\infty)$, where

$$I^{n,-,1-\chi_0}_{1-\chi_0,1}=\int e^{i\tau[t+\sqrt{z^2+(r+s-2)^2)}]}\chi(h\tau)\tau^{1-1/2}\Big[\tilde{J}^-_{1-\chi_0,n}(1-\chi_0)(\frac{z}{\varepsilon_0})+O\Big(\big(\frac{(r+s-2)^2/z^2}{\tau\sqrt{z^2+(r+s-2)^2}}\big)^\infty\Big)\Big]d\tau, \tag{4.6}$$

for some new symbols $\tilde{J}^-_{1-\chi_0,n}$ with main contribution $J^-_{1-\chi_0,n}|_{\gamma_c}\times\frac{(r+s-2)/|z|}{(z^2+(r+s-2)^2)^{1/4}}$. To deal with the sum of $I^{n,-,1-\chi_0}_{1-\chi_0,1}$, we notice that its phase is non-stationary for $\frac{-t}{\sqrt{z^2+(r+s-2)^2}}\notin[1/2,2]$ and for such values repeated integrations by parts yield, as before, $O((h/|t|)^{-\infty})$. For $|t|\sim\sqrt{z^2+(r+s-2)^2}$ we bound from above the sum over n as follows

$$|I^{-,1-\chi_0}_{1-\chi_0,1}|\leq\frac{1}{h^2t}\Big[h^{\frac12}\frac{(r+s-2)}{(z^2+(r+s-2)^2)^{1/4}}\Big]\|\chi\|_{L^1}. \tag{4.7}$$

We now turn to the integral for $|z|\leq 2\varepsilon_0$. when the stationary phase doesn't apply anymore. The phase is stationary w.r.t. τ only if $t\leq 4\varepsilon_0$; when this isn't the case we obtain a $O((h/|t|)^\infty)$ contribution (using that $|n|<1/h$ to sum over n). For small $|t|\leq 4\varepsilon_0$ we bound

$$|I^{-,\chi_0}_{1-\chi_0,1}|\lesssim\int_0^1\int_0^\infty\chi_0(h\tau)\frac{\tau\varepsilon\sqrt{1-\gamma^2}}{\sqrt{1-\gamma^2}}d\gamma d\tau\leq\frac{1}{h^2}\varepsilon\int_0^\infty h^2\frac{\tilde\tau}{h}\chi(\tilde\tau)d\frac{\tilde\tau}{h}=\frac{1}{h^2}\varepsilon\|\chi\|_{L^1}.$$

For $I^+_{1-\chi_0,1}$ one can follow the same path (with $r+s-2\leq 2\varepsilon_0$ replaced by $r-s\leq 2\varepsilon_0$). When r,s are large but $|z|\gg r+s-2$ the proof follows as in the case of $I^{\pm,1-\chi_0}_{1-\chi_0,1}$.

We are left with $I^{\pm}_{\chi_0,1}$ that we split as follows $I^{\pm}_{\chi_0,1} = I^{\pm}_{\chi_0,\chi_0} + I^{\pm}_{\chi_0,1-\chi_0}$. We first deal with the main part $I^{\pm}_{\chi_0,\chi_0}$ which corresponds to the regime $\frac{\varepsilon}{2} \leq \frac{\sqrt{\tau^2-\vartheta^2}}{|n|} \leq \frac{2}{\varepsilon}$. We further separate the regimes $\sqrt{\tau^2-\vartheta^2}/|n| < 1-\tilde{\varepsilon}$, $\sqrt{\tau^2-\vartheta^2}/|n| \in [1-\tilde{\varepsilon}, 1+\tilde{\varepsilon}]$ and $\sqrt{\tau^2-\vartheta^2}/|n| > 1+\tilde{\varepsilon}$ for some $\tilde{\varepsilon} > 2\varepsilon_0$. In order to do that we set $\chi_{\pm}(\ell) := (1-\chi_0(\ell))1_{\pm\ell>0}$ and we write

$$I^{\pm}_{\chi_0,\chi_0} = \sum_{*\in\{0,\pm\}} I^{\pm,\chi_*}_{\chi_0,\chi_0}, \quad I^{\pm,\chi_*}_{\chi_0,\chi_0} = \sum_{n\in\mathbb{N}\setminus\{0\}} (e^{in\theta}+e^{-in\theta}) I^{\pm,\chi_*}_{\chi_0,\chi_0,n},$$

where $I^{\pm,\chi_*}_{\chi_0,\chi_0}$ has the same form as $I^{\pm}_{\chi_0,\chi_0}$ but with additional cut-off $\chi_*((\frac{\sqrt{\tau^2-\vartheta^2}}{n}-1)/\tilde{\varepsilon})$. Consider first $I^{\pm,\chi_+}_{\chi_0,\chi_0}$ with cut-off $\chi_+((\frac{\sqrt{\tau^2-\vartheta^2}}{n}-1)/\tilde{\varepsilon})$ supported for $\frac{\sqrt{\tau^2-\vartheta^2}}{n} > 1+\tilde{\varepsilon}$, $n \geq 1$.
Let $\vartheta = \tau\gamma$, $\rho = \frac{\tau\sqrt{1-\gamma^2}}{n}$ and $\tilde{\rho} \in \{\rho, r\rho, s\rho\}$. With Φ_+ given in (6.2) we get from (6.4)

$$H_n(n\tilde{\rho}) = 2e^{-\frac{i\pi}{3}}\Big(\frac{4\tilde{\zeta}(\tilde{\rho})}{1-\rho^2}\Big)^{\frac14} n^{-\frac13} A_+(n^{\frac23}\tilde{\zeta}(\tilde{\rho}))\Big[\sum_{j\geq 0}\Big(a_j + n^{-\frac43}\Phi_+(n^{\frac23}\tilde{\zeta}(\tilde{\rho}))b_j\Big)(-n^{\frac23}\tilde{\zeta}(\tilde{\rho}))^{-3j/2}\Big], \tag{4.8}$$

$$A_+(n^{\frac23}\tilde{\zeta}(\tilde{\rho})) = n^{-\frac16}\big(-\tilde{\zeta}(\tilde{\rho})\big)^{-\frac14} e^{-in\frac23(-\tilde{\zeta}(\tilde{\rho}))^{\frac32}}\Big(1+O\big((-n^{\frac23}\tilde{\zeta}(\tilde{\rho}))^{-1}\big)\Big), \text{ if } n^{\frac23}\tilde{\zeta}(\tilde{\rho}) > 2.$$

On the support of the cut-offs of $I^{\pm,\chi_+}_{\chi_0,\chi_0}$, the symbol of $G^{\pm}_n$ becomes

$$J^{\pm,\chi_+}_{\chi_0,\chi_0,n}(r,s,\rho) := n^{-2\times\frac13-2\times\frac16}\frac{s^2}{(rs)^{1/2}}\Big((r\rho)^2-1\Big)^{-\frac14}\Big((s\rho)^2-1)\Big)^{-\frac14}\Sigma_{\pm}(r,s,\rho,n) \tag{4.9}$$

where the symbols $\Sigma_{\pm}$ are asymptotic expansion with small parameter n^{-1} and with main contribution obtained as a product of a_0 in (6.4) and σ_0 in (6.1) hence elliptic The phase functions of $I^{\pm,\chi_+}_{\chi_0,\chi_0,n}$, denoted $\phi^{\pm}_n$, read as

$$\tau\phi^{\pm}_n := t\tau + z\gamma\tau - n\Big(f_0(r,\rho) \mp f_0(s,\rho)\Big), \quad f_0(r,\rho) := \tfrac23\big(-\tilde{\zeta}(r\rho)\big)^{\frac32} - \tfrac23\big(-\tilde{\zeta}(\rho)\big)^{\frac32}, \tag{4.10}$$

where we recall $\rho = \frac{\tau\sqrt{1-\gamma^2}}{n}$. The phases $\phi^{\pm}_n$ of $I^{\pm,\chi_+}_{\chi_0,\chi_0,n}$ are stationary when $\nabla_{\tau,\gamma}(\tau\phi^{\pm}_n) = 0$,

$$\partial_\tau(\tau\phi^{\pm}_n) = t + z\gamma - \frac{n}{\tau}\big(f_1(r,\rho) \mp f_1(s,\rho)\big), \quad \tau\partial_\gamma\phi^{\pm}_n = \tau\big(z + \frac{\gamma}{\sqrt{1-\gamma^2}}\frac{(f_1(r,\rho) \mp f_1(s,\rho))}{\rho}\big), \tag{4.11}$$

where $f_1(r,\rho) := \sqrt{(r\rho)^2-1} - \sqrt{\rho^2-1}$, $\rho = \frac{\tau\sqrt{1-\gamma^2}}{n}$, and where we have used Lemma 2.5 to obtain the derivative of f_0. We introduce the cut-off functions $\psi_0 \in C_0^\infty(1-\frac18, 1+\frac18)$ and $\psi \in C_0^\infty(\frac12, 2)$, both equal to 1 near 1, and such that

$$\begin{aligned}\psi_0(\sqrt{1-\gamma^2}) + \sum_{j\geq 1}\psi(2^j\sqrt{1-\gamma^2}) = 1, \quad I^{-,\chi_+}_{n,\chi_0,\chi_0} = \sum_{j\geq 0} I^{-,\chi_+,j}_{\chi_0,\chi_0,n},\\ I^{\pm,\chi_+,j}_{\chi_0,\chi_0,n} = \int e^{i\tau\phi^{\pm}_n}\chi(h\tau)\tau J^{\pm,\chi_+}_{\chi_0,\chi_0,n}(r,s,\rho)\chi_0\Big(\varepsilon\rho\Big)\chi_+\Big(\frac{\rho-1}{\tilde{\varepsilon}}\Big)\psi(2^j\sqrt{1-\gamma^2})d\gamma d\tau,\end{aligned} \tag{4.12}$$

and for $j = 0$, in $I^{\pm,\chi_+,j=0}_{\chi_0,\chi_0,n}$ the last cut-off is $\psi_0(\sqrt{1-\gamma^2})$. In (4.12) we have removed the cut-off $\chi_0(\frac{\varepsilon}{\rho})$ as it is identically equal to 1 on the support of $\chi_+\Big(\frac{\rho-1}{\tilde{\varepsilon}}\Big)$. In the following we deal with the case $j \geq 1$, as the integrals $I^{\pm,\chi_*,j=0}_{\chi_0,\chi_0,n}$ and $I^{\pm,j=0}_{\chi_0,1-\chi_0,n}$ may be dealt with in a similar way for each j. When $j = 0$ then $\gamma < 1/2$ and we may proceed as for $j = 1$ but without the change of variable below.

Lemma 4.2. *There exists $C = C(\varepsilon, \tilde{\varepsilon})$ such that for all $\sqrt{2} = 1+\varepsilon_0 \geq r \geq s \geq 1$ the following holds $\sum_{n\geq 1, j\geq 1}|I^{\pm,\chi_+,j}_{\chi_0,\chi_0,n}| \leq \frac{C}{h^2|t|}$. For $r \geq s$ with $r \geq \sqrt{2}$ and for $j(r,z,h)$ such that*

$$2^{-3j(r,z,h)}r \leq Mh \text{ and } 2^{-4j(r,z,h)}|z| \leq Mh \text{ for some large, fixed } M \gg 1, \tag{4.13}$$

we also have $\sum_{n\geq 1, j\geq j(r,z,h)}|I^{\pm,\chi_+,j}_{\chi_0,\chi_0,n}| \leq \frac{1}{h^2|t|}$.

Proof. We focus on $I^{-,\chi_+,j}_{\chi_0,\chi_0,n}$. Consider first $j\geq 1$. We let $\varphi:=2^j\sqrt{1-\gamma^2}$, then $\varphi\sim 1$ on the support of $\psi(2^j\sqrt{1-\gamma^2})$ and $|\gamma|\geq 1/4$. Let $\phi^-_{n,j}:=\phi^-_n|_{\gamma=\sqrt{1-2^{-2j}\varphi^2}}$ for $\varphi\sim 1$.

Consider first $1<s\leq r\leq 1+\varepsilon_0=\sqrt{2}$. If $2^{-2j}|z|\geq 1$, then $\tau\partial_\varphi\phi^-_{n,j}=\tau\frac{\partial\varphi}{\partial\gamma}\partial_\gamma\phi^-_{n,j}=\tau 2^{-2j}(|z|+O(2^jr))\sim\tau 2^{-2j}|z|\geq\tau$ (as $|z|\geq 2^{2j}$) and we perform repeated integrations by parts and obtain $O(\tau^{-N}(2^{-2j}|z|)^{-N})$ for all $N\geq 1$. If $|t|/|z|\in[1/2,2]$, then we obtain a factor 2^{-2j} as $\frac{\partial\gamma}{\partial\varphi}=-2^{-2j}\frac{\gamma}{\varphi}$, $|\frac{\gamma}{\varphi}|\sim 1$; as Σ_+ is bounded and $|\frac{\overline{H}^{(1)}_n}{H^{(1)}_n}(\rho)|=1$, after the change of variable $\tau=\tilde\tau/h$, we may bound for all $N\geq 1$

$$|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2}\times\frac{2^{-2j}}{n}\times\frac{s^2}{(rs)^{1/2}}\frac{(\tau^{-N}(2^{-2j}|z|)^{-N}}{((r\rho)^2-1)^{1/4}((s\rho)^2-1)^{1/4}},\tag{4.14}$$

where for small $r\geq s$, the factor depending on r,s,ρ in (4.14) is uniformly bounded by a constant depending only on $\varepsilon,\tilde\varepsilon,\varepsilon_0$: indeed, as $\rho\geq 1+\tilde\varepsilon$ and $r\geq s\geq 1$ we have $((r\rho)^2-1)^{-1/4}\leq((s\rho)^2-1)^{-1/4}\leq(2\tilde\varepsilon)^{-2}$ and $s^2/(rs)^{1/2}\lesssim 1$ as $s\leq r\leq 1+\varepsilon_0$. As the sum over n is bounded by $\log\Big(\frac{1}{\varepsilon(1+\tilde\varepsilon)}\Big)$, we take $N=2$ which gives $\sum_j 2^{-2j}(2^{2j}/|z|)^2=\frac{1}{|z|^2}\sum_{2^{2j}\leq|z|}2^{2j}\sim 1/|z|\sim 1/|t|$ and end up with $\sum_{n,j}|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{O(h^\infty)}{h^2|t|}$. If $|t|/|z|\notin[1/2,2]$, then repeated integrations by parts yield the same kind of bounds.
Let now $2^{-2j}|z|\leq 1$ then if $|t|\sim|z|$ we bound, using (4.9) and setting $\tau=\tilde\tau/h$,

$$|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2}\frac{2^{-2j}}{n}\times\frac{s^2}{(rs)^{1/2}}\frac{1}{((r\rho)^2-1)^{1/4}((s\rho)^2-1)^{1/4}}\tag{4.15}$$

and proceed as in the case $2^{2j}\leq|z|$. If $\frac{|t|}{|z|}\notin[\frac12,2]$, we integrate by parts in τ.

Let now $r\geq s\geq 1+\varepsilon_0=\sqrt{2}$ such that $\tau 2^{-3j}r\leq M$ for large $M\gg 1$; since the phase is stationary w.r.t. γ when $|z|\sim 2^jr$, it follows that, if $\tau 2^{-4j}|z|\geq M$, we may integrate by parts in φ (in which case the remainders may be dealt with as before) to conclude. Let therefore $\tau 2^{-4j}|z|\leq M$. We notice that if $|t|\geq 4(|z|+2^jr)$ then the phase $\tau\phi^-_n$ is not stationary in τ : indeed, we have $|z\gamma|\in[|z|/4,|z|]$ and, using (4.11) and $r\geq s$, we also have $\frac{|n|}{\tau}(f_1(r,\rho)+f_1(s,\rho))=\frac{(f_1(r,\rho)+f_1(s,\rho))}{\rho}\sqrt{1-\gamma^2}\sim 2^{-j}r$, where we used that $\frac{f_1(s,\rho)}{\rho}\leq\frac{f_1(r,\rho)}{\rho}=\sqrt{r^2-1/\rho^2}-\sqrt{1-1/\rho^2}$ with $r\geq 1+\varepsilon_0$ and $\rho\geq 1+\tilde\varepsilon$. Hence in this case we integrate by parts in τ and obtain an upper bound of the form

$$|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2}\frac{2^{-2j}}{n}\times\frac{s^2}{(rs)^{1/2}}\frac{O((h/|t|)^N)}{\rho(r^2-1/\rho^2)^{1/4}(s^2-1/\rho^2)^{1/4}},\ \text{for all } N\geq 1.$$

Taking $N=1$ gives $\sum_{n,j\geq1,\rho\in[1+\tilde\varepsilon,2/\varepsilon]}|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{h}{h^2t}\sum\frac{2^{-2j}}{n}\leq\frac{h}{h^2t}\log(\frac{1}{\varepsilon(1+\tilde\varepsilon)})$.

Let now $|t|\leq 4(|z|+2^jr)$, when the same bounds as in (4.15) hold. As $r\geq s$, the factor depending on r,s,ρ is always bounded by $r^{2-1-1}=O(1)$ and we are left to estimate the sum over $j\geq 1$ satisfying $\tau 2^{-4j}|z|,\tau 2^{-3j}r\leq M$. If moreover $2^{-2j}|z|\leq M$, then, for $\tau 2^{-4j}(|z|+2^jr)\leq M$, $\tau\sim 1/h$, $\frac{1}{|z|+2^jr}\leq 1/|t|$ and $\rho\sim 2^{-j}hn\in[1+\tilde\varepsilon,2/\varepsilon]$, we have

$$\sum_{n\geq1,j\geq1}|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\sum_{2^{-3j}\leq\frac{hM}{(|z|+2^jr)},1+\tilde\varepsilon\leq\frac{2^{-j}}{hn}}\frac{2^{-2j}}{h^2n}\leq\frac{1}{h^2}\frac{M}{|t|}\log\Big(\frac{1}{\varepsilon(1+\tilde\varepsilon)}\Big).$$

When $2^{-2j}|z|\geq M$ with M large enough we apply the stationary phase (and keep track of the remainders) as follows : the critical point w.r.t φ satisfies $-z=\frac{\gamma}{\sqrt{1-\gamma^2}}(f_1(r,\rho)+f_1(s,\rho))$ and the second order derivative of $\phi^-_{n,j}$ reads as

$$\partial^2_\varphi\phi^-_{n,j}|_{\partial_\varphi\phi^-_{n,j}=0}=\Big(\frac{\partial\gamma}{\partial\varphi}\Big)^2\partial^2_\gamma\phi^-_n|_{\varphi=2^j\sqrt{1-\gamma^2},\partial_\gamma\phi^-_n=0},\tag{4.16}$$

$$\partial^2_\gamma\phi^-_n=\frac{1}{\sqrt{1-\gamma^2}^3}\frac{(f_1(r,\rho)+f_1(s,\rho))}{\rho}+\frac{\tau}{n}\frac{\gamma^2}{(1-\gamma^2)}\frac{2}{\rho^2}\Big(\frac{2}{\sqrt{\rho^2-1}}-\frac{1}{\sqrt{(r\rho)^2-1}}-\frac{1}{\sqrt{(s\rho)^2-1}}\Big),$$

where we recall the notation $\rho=\frac{\tau\sqrt{1-\gamma^2}}{n}$. As the sum in brackets is positive for all $r,s\geq 1$ we may bound from below

$$\partial^2_\gamma\phi^-_n|_{\varphi=2^j\sqrt{1-\gamma^2},\partial_\gamma\phi^-_n=0}\geq\frac{1}{\sqrt{1-\gamma^2}^3}\frac{(f_1(r,\rho)+f_1(s,\rho))}{\rho}|_{\varphi=2^j\sqrt{1-\gamma^2},\partial_\gamma\phi^-_n=0}=\frac{(-z/\gamma)}{1-\gamma^2}\sim 2^{2j}|z|,$$

which implies $\partial^2_\varphi\phi^-_{n,j}|_{\partial_\varphi\phi^-_{n,j}=0}\gtrsim(2^{-2j})^2\times 2^{2j}|z|=2^{-2j}|z|\sim 2^{-j}r$. As $\tau 2^{-3j}(|z|+2^jr)\geq M$, then $\tau 2^{-2j}(|z|+2^jr)\geq 2^jM$; the stationary phase applies and gives

$$I^{-,\chi_+,j}_{\chi_0,\chi_0,n}=\int e^{i\tau\phi^-_n}\chi(h\tau)\frac{\tau^{1/2}}{\sqrt{\partial^2_\varphi\phi^-_{n,j}|_{\partial_\varphi\phi^-_{n,j}=0}}}\Big(\tilde J^{-,\chi_+}_{\chi_0,\chi_0,n}(r,s,\frac{\tau}{2^jn})+\frac{1}{n}O((\tau 2^{-2j}|z|)^{-\infty})\Big)d\tau, \tag{4.17}$$

where $\tilde J^{-,\chi_+}_{\chi_0,\chi_0,n}(r,s,\frac{\tau}{2^jn})$ is the symbol with main contribution $J^{-,\chi_+}_{\chi_0,\chi_0,n}$. The sum for j such that $2^{2j}\leq|z|$ of the remainder terms is bounded by $O(M^{-\infty}/(|z|n))$ (as in the case of small r,s) and we conclude as the sum over n is bounded by $\log(\frac{1}{\varepsilon(1+\tilde\varepsilon)})$. It remains to evaluate the main contributions in (4.17). Using (4.11), a simple computation shows that the phase ϕ^-_n is stationary in both τ,γ when $t=-z/\gamma$, hence the critical value of $\phi^-_{n,j}$ (w.r.t.φ) at the critical point φ_c will be stationary w.r.t. τ when $t=-z/\sqrt{1-2^{-2j}\varphi_c^2}$. For $t/|z|\notin[1/2,2]$, repeated integrations by parts in τ yields a $O((h/|t|)^\infty)$ contribution. For $t\sim|z|$ and with $1+\tilde\epsilon\lesssim\rho\sim\frac{2^{-j}}{hn}\leq\frac{2}{\varepsilon}$ we obtain for $2^{-4j}|z|\lesssim hM$ (that we use as $|z|^{1/2}\leq M^{1/2}h^{1/2}2^{2j}$)

$$|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2|t|}\times\frac{h^{1/2}|z|^{1/2}2^{-2j+j}}{n}\times\frac{\frac{s^2}{(rs)^{1/2}}}{((r\rho)^2-1)^{1/4}((s\rho)^2-1)^{1/4}}\leq\frac{M^{1/2}}{h^2|t|}\times\frac{h^{1/2+1/2}2^j}{n}, \tag{4.18}$$

$$\sum_{n,j\geq 1}|I^{-,\chi_+,j}_{\chi_0,\chi_0,n}|\leq\frac{M^{1/2}}{h^2|t|}\times\frac{h^{1/2+1/2}}{n^2h},$$

where we used that $2^j\leq 1/(hn)$ on the support of the symbol. In the same way one may deal with $I^{+,\chi_+,j}_{\chi_0,\chi_0,n}$ and obtain similar bounds. The proof of the Lemma is achieved.

□

Next, we turn to $I^{\pm,\chi_0}_{\chi_0,\chi_0}=\sum_{n\geq1}(e^{in\theta}+e^{-in\theta})I^{\pm,\chi_0}_{\chi_0,\chi_0,n}$, supported for $\frac{\sqrt{\tau^2-\vartheta^2}}{n}=\frac{\tau\sqrt{1-\gamma^2}}{n}\in[1-2\tilde\varepsilon,1+2\tilde\varepsilon]$ where $\vartheta=\tau\gamma$ and where we recall that $\tau\sim 1/h$. We decompose as before $I^{\pm,\chi_0}_{\chi_0,\chi_0,n}=\sum_j I^{\pm,\chi_0,j}_{\chi_0,\chi_0,n}$, setting $1-\gamma^2=2^{-2j}\varphi^2$, $\varphi\sim 1$ and using the partition (4.12). For each $j\geq 1$, it will be convenient to take $\tau 2^{-j}\varphi=n+n^{1/3}w$: on the support of the symbol of $I^{\pm,\chi_0,j}_{\chi_0,\chi_0,n}$ we now have $wn^{-2/3}\in[-2\tilde\varepsilon,+2\tilde\varepsilon]$, where $n\sim 2^{-j}/h\geq 1$ (notice in particular that n may be small if j is large enough). We deal separately with the cases $w>1$, $|w|\leq 2$ and $w<-1$ by inserting in the integral the cut off functions $\sum_{*\in\{0,\pm\}}\chi_*(w)$, where $\chi_\pm(\ell)=(1-\chi_0)(\ell)1_{\pm\ell>0}$ and denote $I^{\pm,\chi_0,*,j}_{\chi_0,\chi_0,n}$ the corresponding integrals.

Lemma 4.3. *For $1<s\leq r\leq\sqrt2$ we have $\sum_{n\geq1,j\geq1}|I^{\pm,\chi_0,*,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2|t|}$, $*\in\{0,+\}$. For $r\geq s$ with $r\geq\sqrt2$ and $j(r,z,h)$ as in (4.13) we have $\sum_{n\geq1,j\geq j(r,s,h)}|I^{\pm,\chi_0,*,j}_{\chi_0,\chi_0,n}|\lesssim\frac{1}{h^2|t|}$, $*\in\{0,+\}$.*

On the support of $\chi_+(w)$ we may proceed in a similar way as in Lemma 4.2 as the same asymptotic expansions holds for the Airy factors ; as the computations look very much the same, we skip the details for this part. Notice that this case contains the situation $r,s\geq\sqrt2$ hence in the following we consider only $1<s\leq r\leq\sqrt2$ and $I^{\pm,\chi_0,0,j}_{\chi_0,\chi_0,n}$ with symbol $\chi_0(w)$ and focus on the $-$ sign. The expansion (4.8) still holds, but, when $n^{2/3}\tilde\zeta(\rho)<2$, the Airy factors don't oscillate they may be brought into the symbol. In this case the phase of $I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}$ equals $\tau(t-z\sqrt{1-\gamma^2})$, and after the changes of coordinates $\tau=\frac{2^j}{\varphi}(n+n^{1/3}w)$ we are reduced to obtaining bounds for

$$\frac{s^2}{(rs)^{1/2}}\int e^{i\frac{2^j}{\varphi}(n+n^{1/3}w)(t-z\sqrt{1-2^{-2j}\varphi^2})}\chi(h\frac{2^j}{\varphi}(n+n^{1/3}w))\frac{2^j}{\varphi}(n+n^{1/3}w)\psi(\varphi)\chi_0(w)n^{-2/3+1/3}\frac{2^{-2j+j}}{\varphi}d\varphi dw, \tag{4.19}$$

where the factors $2^{-2j}\frac{2^j}{\varphi}n^{1/3}$ come from $\gamma\to\varphi$, $\tau\to w$. The phase is stationary when

$$2^j(n+n^{1/3}w)\Big[-\frac{(t-z\sqrt{1-2^{-2j}\varphi^2})}{\varphi^2}+\frac{2^{-2j}z}{\sqrt{1-2^{-2j}}}\Big]=0,\quad 2^jn^{1/3}(t-z\sqrt{1-2^{-2j}\varphi^2})=0.$$

Notice that on the support of χ we have $2^jn\sim\frac{1}{h}$, then $2^jn^{1/3}\sim 2^{2j/3}/h^{1/3}\geq\frac{1}{h^{1/3}}$ and $2^j\leq\frac{1}{h}$. Using $h^{1/3}2^{j/3}\leq 1$, for bounded t, the sum of (4.19) may be bounded as follows

$$\sum_{j,n}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq\frac{s^2}{(rs)^{1/2}}\sum_{j,n\sim 2^{-j}/h}n^{2/3}\sim\frac{s^2}{(rs)^{1/2}}\sum_{j,h^{1/3}2^{j/3}\leq 1}(2^{-j}/h)^{5/3}\leq\frac{1}{h^2}. \tag{4.20}$$

For large $|t|>1$ satisfying $|t|\geq 2|z|$, the phase is non-stationary w.r.t. w ; integrations by parts with the large parameter $2^jn^{1/3}$ yield a contribution $O((2^jn^{1/3}/|t|)^{-N})$ for all $N\geq 1$ and we conclude. For $1\leq|t|\leq 2|z|$ we have $1/|z|\leq\frac{4}{|t|}$ we check whether the stationary phase applies in both w,φ : as $\partial^2_w\phi^-_n=0$, then the determinant of the Hessian matrix equals $(\partial^2_{w,\varphi}\phi^-_n)^2\sim(2^{-j}n^{1/3}|z|)^2$ for $\varphi\sim 1$. If $2^{-j}n^{1/3}|z|\geq h^{-\epsilon}$ for some small $\epsilon>0$, then the stationary phase in w and φ gives, for small r,s,

$$\sum_{j,n\geq 1}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq\frac{s^2}{(rs)^{1/2}}\sum_{j,n\sim 2^{-j}/h}\frac{n^{2/3}}{2^{-j}n^{1/3}|z|}\lesssim\sum_{j,h2^j\leq 1}\frac{2^j}{|z|}\times(2^{-j}/h)^{4/3}(1+O(h^\infty))\leq\frac{1}{h^2|t|}.$$

If $2^{-j}n^{1/3}|z|\leq h^{-\epsilon}$ then we may bound the sum as in (4.20) and conclude using that $n^{1/3}\sim 2^{-j/3}/h^{1/3}\leq 2^jh^{-\epsilon}/|z|$ which gives $2^{-4j/3}\leq h^{1/3-\epsilon}/|z|\leq 4h^{1/3-\epsilon}/|t|$.

Let now $\sqrt{2}\geq r$ such that $n^{2/3}\tilde\zeta(r\rho)>1$, then we either have $n^{2/3}\tilde\zeta(s\rho)\leq 2$ or $n^{2/3}\tilde\zeta(s\rho)>1$; in the second case, both Airy factors $A(n^{2/3}\tilde\zeta(r\rho))$, $Ai(n^{2/3}\tilde\zeta(s\rho))$ do oscillate and we may proceed as with $\chi_+(w)$ (the only differences with the case χ_+ are the absence of the phase functions of $\overline{H}_n(n\rho)/H_n(n\rho)$, which means replacing $f_1(r,\rho)$ by $\sqrt{(r\rho)^2-1}$, and also the fact that the factor depending on r,s,ρ in (4.15) may not be bounded but at most $n^{1/3}$). Let therefore $n^{2/3}\tilde\zeta(s\rho)\leq 2$, then $|H_n(ns\rho)|\sim n^{-1/3}$, while $H_n(nr\rho)$ has the expansion (4.8) with an oscillating Airy factor. On the support of the cut-offs of $I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}$ the symbol of G^-_n becomes

$$J^{-,\chi_0,0}_{\chi_0,\chi_0,n}(r,s,\rho):=n^{-2\times\frac{1}{3}-\frac{1}{6}}\frac{s^2}{(rs)^{1/2}}\Big((r\rho)^2-1\Big)^{-\frac{1}{4}}\Sigma_0(r,s,\rho,n) \tag{4.21}$$

where the elliptic symbol Σ_0 is an asymptotic expansion with small parameter n^{-1} and with main contribution obtained as a product of a_0 in (6.4), σ_0 in (6.1) and $(H_n(ns\rho)n^{-1/3})\times\frac{\overline{H}_n(n\rho)}{H_n(n\rho)}$, where the last three Hankel factors are not oscillating as $\rho=\tau\sqrt{1-\gamma^2}/n=1+n^{-2/3}w$ with $|w|\leq 2$ and $s\leq 1+2n^{-2/3}$. In (4.21), the factor $((r\rho)^2-1)^{-1/4}$ may be large when r is close to $1+2n^{-2/3}$ and we have $\frac{s^2}{(rs)^{1/2}}\Big((r\rho)^2-1\Big)^{-\frac{1}{4}}\leq n^{1/6}$. The changes of variables $\eta\to\tau\gamma$, $\gamma\to\varphi$, $\tau\to w$ yield an additional factor $2^jn\times 2^{-2j}\times 2^jn^{1/3}/\varphi$.

Let r be such that $(r\rho)^2-1-\gamma^2\geq 1$ when $(r\rho)^2-2\geq\gamma^2\geq\frac{1}{16}$: we show that in this case the stationary phase applies in w,φ. The phase of $I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}$ equals

$$\tau\phi^-_n:=\tau(t+z\gamma)-\frac{2}{3}n(-\tilde\zeta(r\rho))^{3/2},\quad\gamma=\sqrt{1-2^{-2j}\varphi^2},\quad\rho=\frac{\tau\sqrt{1-\gamma^2}}{n}=1+n^{-2/3}w.$$

The phase is stationary w.r.t. τ and φ when we have, respectively,

$$\partial_\tau(\tau\phi^-_n)=t+z\gamma-\frac{1}{\sqrt{1-\gamma^2}}\frac{\sqrt{(r\rho)^2-1}}{\rho}=0,\quad\partial_\gamma(\tau\phi^-_n)=\tau(z+\frac{\gamma}{\sqrt{1-\gamma^2}}\frac{\sqrt{(r\rho)^2-1}}{\rho})=0,$$

and $\partial_w\phi^-_n=\frac{\partial\tau}{\partial w}\partial_\tau(\tau\phi^-_n)$, $\partial_\varphi\phi^-_n=\frac{\partial\gamma}{\partial\varphi}\partial_\gamma(\tau\phi^-_n)+\frac{\partial\tau}{\partial\varphi}\partial_\tau(\tau\phi^-_n)$, $\frac{\partial\tau}{\partial w}=\frac{2^jn^{1/3}}{\varphi}$, $\frac{\partial\gamma}{\partial\varphi}=-\frac{2^{-2j}\varphi}{\sqrt{1-2^{-2j}\varphi^2}}$. At the critical points we have $\det\mathrm{Hess}_{w,\varphi}(\phi^-_n)=(\frac{\partial\tau}{\partial w})^2(\frac{\partial\gamma}{\partial\varphi})^2\det\mathrm{Hess}_{\tau,\gamma}(\tau\phi^-_n)$ and

$$\det\mathrm{Hess}_{\tau,\gamma}(\tau\phi^-_n)=|\partial^2_{\tau,\tau}(\tau\phi^-_n)\partial^2_{\gamma,\gamma}(\tau\phi^-_n)-(\partial^2_{\tau,\gamma}(\tau\phi^-_n))^2|.$$

As $\partial^2_{\tau,\tau}(\tau\phi_n^-)=-\frac{1}{n\rho}\frac{1}{\sqrt{(r\rho)^2-1}}<0$ and as $\partial^2_{\gamma,\gamma}(\tau\phi_n^-)=\frac{\tau}{\sqrt{1-\gamma^2}^3}\frac{((r\rho)^2-1-\gamma^2)}{\rho\sqrt{(r\rho)^2-1}}$, it follows that when $(r\rho)^2-1-\gamma^2\geq 1$ both terms have same sign and we have a bound from below

$$\det \mathrm{Hess}_{w,\varphi}(\tau\phi_n^-)\gtrsim (2^jn^{1/3})^2\times 2^{-4j}\times\frac{2^{2j}((r\rho)^2-2)}{((r\rho)^2-1)}=n^{2/3}\frac{((r\rho)^2-2)}{((r\rho)^2-1)},$$

where the first two factors come from the changes of variables and where we used that $|\partial^2_{\tau,\tau}(\tau\phi_n^-)\partial^2_{\gamma,\gamma}(\tau\phi_n^-)|\geq\frac{\tau}{n}\frac{2^{3j}((r\rho)^2-2)}{(r\rho)^2-1)}\sim 2^{3j-j}\frac{((r\rho)^2-2)}{((r\rho)^2-1)}$ as $\rho\sim\frac{\tau 2^{-j}}{n}\sim 1$. At the critical points we have $z\sim 2^j\sqrt{(r\rho)^2-1}$ and $|t|\sim|z|+2^j\sqrt{(r\rho)^2-1}$. It follows that, when $n^{2/3}\geq h^{-\epsilon}$ for some $\epsilon>0$, the stationary phase applies and yields

$$\sum_{j\geq 1,n\sim 2^{-j}/h}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq r^{-1}\sum_{2^j\sim|z|/r,n\sim 2^{-j}/h}n^{-2/3-1/6}\times 2^{j-2j+j}n^{1+1/3-1/3}\leq\frac{1}{h^2}\frac{r}{r|z|}\times h^{5/6},$$

where we used $\frac{s^2}{(rs)^{1/2}}\Big((r\rho)^2-1\Big)^{-\frac14}\lesssim 1$. We have also used that $n\sim 2^{-j}/h$ on the support of $\chi(h\tau)$ to bound $\sum_{n\sim 2^{-j}/h}n^{1/6}\leq(2^{-j}/h)^{7/6}$ and that $2^j\sim|z|/r$ to obtain $r^{-1}\sum_j 2^{-7j/6}\leq r^{-1}\frac{r}{|z|}\sum_{j\geq 1}2^{-j/6}\leq 1/|z|\sim 1/|t|$. Let $n^{2/3}\leq h^{-\epsilon'}$ for all $\epsilon'>0$; if $|t|\geq 4(|z|+2^jr)$, we integrate by parts in w and conclude. Let $|t|\leq 4(|z|+2^jr)$: if $|z|\geq 2^{j+2}r$ (or $2^jr\geq 4|z|$), the phase is non-stationary in φ and if, moreover, $n2^{-j}|z|\geq h^{-\epsilon}$ (or if $nr\geq h^{-\epsilon}$, respectively) for some $\epsilon>0$, then we conclude by integrations by parts. If $|z|\geq 2^{j+2}r$ and $n2^{-j}|z|\leq h^{-\epsilon'}$ for all $\epsilon'>0$ (or if $2^jr\geq 4|z|$ and $|n|r\leq h^{-\epsilon'}$ for all $\epsilon'>0$), we obtain an upper bound of the form $|t|\leq 4(|z|+2^jr)\leq 2^{j+4}/n$ and

$$\sum_{j\geq 1,n\sim 2^{-j}/h}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq r^{-1}\sum_{n\sim 2^{-j}/h}n^{-2/3-1/6}\times 2^{j-2j+j}n^{1+1/3}\leq\sum_{2^{-j}/h\geq 1}\frac{2^{j+4}}{|t|}(2^{-j}/h)^{1/2}\leq\frac{1}{h|t|}.$$

When $n^{2/3}\leq h^{-\epsilon'}$, $|t|\leq 4(|z|+2^jr)$ and $|z|\sim 2^jr$, ϕ_n^- may be stationary in φ. We have

$$\sum_{j\geq 1,n\sim 2^{-j}/h}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq r^{-1}\sum_{2^j\sim|z|/r,n\sim 2^{-j}/h}n^{-2/3-1/6}\times 2^{j-2j+j}n^{1+1/3}\leq r^{-1}(\frac{2^{-j}}{h})^{3/2}\leq\frac{1}{h^2|t|}.$$

Let now r be sufficiently close to 1, so that $n^{2/3}\tilde\zeta(r\rho)>1$ but $(r\rho)^2-1\leq 1+\gamma^2$ then the determinant of the Hessian matrix can no longer be bounded from below. In this case we have $\frac{s^2}{(rs)^{1/2}}\Big((r\rho)^2-1\Big)^{-\frac14}\leq n^{1/6}$ and if z is such that $\tau 2^{-j}|z|\sim 2^jn\times 2^{-j}|z|\sim n|z|\geq h^{-\epsilon}$ for some $\epsilon>0$ we proceed by integrations by parts in φ as the phase is not stationary and its parameter is large. If $n|z|\leq h^{-\epsilon'}$ for all $\epsilon'>0$, then for $|t|\geq 4|z|$ we integrate by parts, while for $|t|\leq 4|z|$ we have

$$\sum_{j\geq 1,n\sim 2^{-j}/h}|I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}|\leq r^{-1}\sum_{n\sim 2^{-j}/h\leq h^{-\epsilon'}/|z|}n^{-2/3-1/6+1/6}\times 2^{j-2j+j}n^{1+1/3}\leq\frac{h^{4/3-\epsilon'}}{h^2|z|}<\frac{1}{h^2|t|}.$$

□

Lemma 4.4. *For $1<s\leq r\leq\sqrt2$ we have $\sum_{|n|\geq 1,j\geq 1}|I^{\pm,\chi_0,-,j}_{\chi_0,\chi_0,n}|\leq\frac{1}{h^2|t|}$. For $r\geq s$ with $r\geq\sqrt2$ and $j(r,z,h)$ given in (4.13), we also have $\sum_{|n|\geq 1,j\geq j(r,z,h)}|I^{\pm,\chi_0,-,j}_{\chi_0,\chi_0,n}|\leq\frac{1}{h^2|t|}$.*

Proof. Recall that $\rho=\tau\sqrt{1-\gamma^2}/|n|$, on the support of $\chi_0(\varepsilon\rho)\chi_0(\varepsilon/\rho)$ we had $\rho\in[\varepsilon/2,2/\varepsilon]$ and we have added a cutoff $\chi_0((\rho-1)/\varepsilon)$ to localize at $\rho\in[1-\tilde\varepsilon,1+\tilde\varepsilon]$. We set $1-\gamma^2=2^{-2j}\varphi^2$, with $\varphi\sim 1$ on the support of ψ, and also $\rho=1+|n|^{-2/3}w$: on the support of $I^{\pm,\chi_0,-,j}_{\chi_0,\chi_0,n}$ we have $w\leq -1$, that is $\rho\in[1-\tilde\varepsilon,1-|n|^{-2/3}]$. It will be convenient to use the representation of G_n in terms of Bessel functions J_n instead of Henkel functions H_n, hence the first line in (4.2). We need to estimate

$$\begin{aligned}&\frac{s^2}{(rs)^{1/2}}\sum_{|n|\geq 1}e^{in\theta}\sum_{j\geq 1}\int e^{i\frac{2^j(n+n^{\frac13}w)}{\varphi}(t-z\sqrt{1-2^{-2j}\varphi^2})}\chi\Big(\frac{2^jh(|n|+|n|^{\frac13}w)}{\varphi}\Big)\frac{J_n(n(1+n^{-\frac23}w))}{H_n(n(1+n^{-\frac23}w))}\\&\psi(\varphi)\chi_-(w)H_n(nr(1+n^{-\frac23}w))H_n(ns(1+n^{-\frac23}w))\frac{2^j}{\varphi}(n+n^{1/3}w)2^{-2j+j}n^{\frac13}dwd\varphi.\end{aligned}\tag{4.22}$$

The Bessel function $J_n(n\rho)$ satisfies (6.5). The factor J_n/H_n corresponds to $\frac{Ai}{A_+}(n^{\frac{2}{3}}\tilde{\zeta}(\rho)) = e^{-2i\pi/3} + e^{-\frac{4}{3}|n|\tilde{\zeta}(\rho)^{\frac{3}{2}}}$ (see Lemma 6.1). On the support of the cut-offs of $I^{-,\chi_0,-,j}_{\chi_0,\chi_0,n}$ its symbol has the form

$$J^{-,\chi_0,-}_{\chi_0,\chi_0,n} := \frac{s^2}{(rs)^{1/2}}|n|^{-2/3}\Big(\frac{4\tilde{\zeta}(r\rho))}{1-(r\rho)^2}\Big)^{1/4}\Big(\frac{4\tilde{\zeta}(s\rho))}{1-(s\rho)^2}\Big)^{1/4}A_+(n^{2/3}\tilde{\zeta}(r\rho))A_+(n^{2/3}\tilde{\zeta}(s\rho))e^{-\frac{4}{3}|n|\tilde{\zeta}(\rho)^{\frac{3}{2}}}\Sigma_-, \tag{4.23}$$

for some symbol Σ_- of order 0. In the case of $I^{-,\chi_0,-,j}_{\chi_0,\chi_0,n}$ one should replace $A_+(n^{2/3}\tilde{\zeta}(s\rho))$ by $Ai(n^{2/3}\tilde{\zeta}(s\rho))$ and remove the exponential decreasing factor. When $n^{2/3}\tilde{\zeta}(r\rho), n^{2/3}\tilde{\zeta}(s\rho) < 2$ we can proceed exactly as for $I^{-,\chi_0,0}_{\chi_0,\chi_0,n}$ with $r, s \leq 1+2|n|^{-2/3}$ small ; when $n^{2/3}\tilde{\zeta}(s\rho) \geq 2 > n^{2/3}\tilde{\zeta}(r\rho)$, we can proceed as for $I^{-,\chi_0,0}_{\chi_0,\chi_0,n}$ with $r \geq 1+2|n|^{-2/3} \geq s \geq 1$. Assume therefore $n^{2/3}\tilde{\zeta}(s\rho) \geq n^{2/3}\tilde{\zeta}(s\rho) \geq 1$, in which case

$$J^{+,\chi_0,-}_{\chi_0,\chi_0,n} := \frac{s^2}{(rs)^{1/2}}\frac{|n|^{-\frac{1}{3}-\frac{1}{6}}}{(1-(r\rho)^2)^{\frac{1}{4}}}\frac{|n|^{-\frac{1}{3}-\frac{1}{6}}}{(1-(s\rho)^2)^{\frac{1}{4}}}e^{-\frac{4}{3}|n|\tilde{\zeta}(\rho)^{\frac{3}{2}}+\frac{2}{3}|n|\tilde{\zeta}(s\rho)^{\frac{3}{2}}+\frac{2}{3}|n|\tilde{\zeta}(r\rho)^{\frac{3}{2}}}\Sigma_-, \tag{4.24}$$

As we are assuming $\tilde{\zeta}(r\rho), \tilde{\zeta}(s\rho) > 0$, we have, using Lemma 2.5, $\frac{2}{3}\tilde{\zeta}(s\rho)^{3/2} - \frac{2}{3}\tilde{\zeta}(\rho)^{3/2} = -\int_\rho^{s\rho}\frac{\sqrt{1-w^2}}{w}dw \leq 0$. The phase function of $I^{\pm,\chi_0,-}_{\chi_0,\chi_0,n}$ is $\tau(t+z\gamma)$ and we can proceed as done before as the factor $\frac{s^2}{(rs)^{1/2}}(1-(s\rho)^2)^{-\frac{1}{4}}(1-(r\rho)^2)^{-\frac{1}{4}}$ is at most $n^{1/3}$ when $1-s\rho \sim 1-r\rho \sim |n|^{-2/3}$, while for $r, s \geq 2$ this term is uniformly bounded by 1. From now one can proceed as in the case of $I^{-,\chi_0,0,j}_{\chi_0,\chi_0,n}$ as on the support of $\chi(h\tau)$ we still have $|n| \sim 2^{-j}/h$, the phase is stationary for $|t| \sim |z|$ and for $|n|2^{-j}|z| \geq h^{-\epsilon}$ we can integrate by parts, while for $|n|2^{-j}|z| \leq h^{-\epsilon}$ we conclude as done previously. We are left with $I^{\pm,\chi_-}_{\chi_0,\chi_0}$ and $I_{\chi_0,1-\chi_0}$.

Lemma 4.5. *For $1 < s \leq r \leq \sqrt{2}$ we have $\sum_{|n|\geq 1, j\geq 1}|I^{\pm,\chi_-,j}_{\chi_0,\chi_0,n}| \leq \frac{1}{h^2|t|}$. For $r \geq s$ with $r \geq \sqrt{2}$ and $j(r,z,h)$ as in* (4.13)*, we also have $\sum_{|n|\geq 1, j\geq j(r,s,h)}|I^{\pm,\chi_-,j}_{\chi_0,\chi_0,n}| \leq \frac{1}{h^2|t|}$.*

On the support of $I^{-,\chi_-,j}_{\chi_0,\chi_0,n}$ we have $\rho = \frac{\tau\sqrt{1-\gamma^2}}{n} \in [\varepsilon/2, 1-\tilde{\varepsilon}]$ with $1 > \tilde{\varepsilon} \geq 2\varepsilon_0 = 2(\sqrt{2}-1)$, $\tau \sim 1/h$, $1-\gamma^2 = 2^{-2j}\varphi^2$, $\varphi \sim 1$. The symbol of $I^{-,\chi_-,j}_{\chi_0,\chi_0,n}$ has also the form (4.23). For small $r, s \leq \varepsilon_0$ and $\tilde{\varepsilon} > 2\varepsilon_0$, we write $1-r\rho = 1-r+r(1-\rho)$ to deduce that, if $\rho \leq 1-\tilde{\varepsilon}$, then the symbol (4.23) takes the form (4.24) where the factor $\frac{s^2}{(rs)^{1/2}}(1-(s\rho)^2)^{-\frac{1}{4}}(1-(r\rho)^2)^{-\frac{1}{4}}$ is uniformly bounded by a constant depending only on $\tilde{\varepsilon}$ and we conclude as before. When r, s are large (and $\tau 2^{-2j}|z| \leq h^{-\epsilon'}$, $\tau 2^{-j} r \leq h^{-\epsilon'}$ for all $\epsilon' > 0$), we separate the possible situations : the only new one is the case $n^{2/3}(1-r\rho), n^{2/3}(1-s\rho) \geq 1$ and r sufficiently large but such that $r < 1/\rho \leq 1/(1-\tilde{\varepsilon})$, in which case $\frac{s^2}{(rs)^{1/2}}(1-(s\rho)^2)^{-\frac{1}{4}}(1-(r\rho)^2)^{-\frac{1}{4}} \leq rn^{1/3} \leq n^{1/3}/(1-\tilde{\varepsilon})$. In this case we have additional decay from the exponential factors and conclude as before.

Lemma 4.6. *Let $I^{\pm}_{\chi_0,1-\chi_0} := \sum_{n\geq 1}(e^{in\theta}+e^{-in\theta})I^{\pm}_{\chi_0,1-\chi_0,n}$, $I^{\pm}_{\chi_0,1-\chi_0,n} = \sum_{j\geq 0}I^{\pm,j}_{\chi_0,1-\chi_0,n}$ using the decomposition* (4.12)*. For $1 < s \leq r \leq \sqrt{2}$ we have $\sum_{n\geq 1, j\geq 1}|I^{\pm,j}_{\chi_0,1-\chi_0,n}| \leq \frac{1}{h^2|t|}$. Let $r \geq s$, $r \geq \sqrt{2}$ and $j(r,z,h)$ as in* (4.13)*, then the sum for $j \geq j(r,z,h)$ of $|I^{\pm,j}_{\chi_0,1-\chi_0,n}|$ is also uniformly bounded by $\frac{1}{h^2|t|}$ for all t.*

Proof. In this case we use the formulas (6.6) and take advantage of the exponential decay. The proof is left to the reader. □

We have only considered here the case $|n| \geq 1$. For $n = 0$ we can proceed exactly in the same way as for $n = 1$ but with much less cases to consider : in fact in (4.5) we have to replace $\chi_{1,2}$ by $\chi_1(\varepsilon\sqrt{\tau^2-\vartheta^2})$ and $\chi_2(\frac{\varepsilon}{\sqrt{\tau^2-\vartheta^2}})$ and then use the expansions of J_0, H_0.

□

5. Small frequency case

Let $\tau \leq 1/h_0$ for some fixed $h_0 > 0$, small enough. We use again the parametrix in terms of Bessel functions introduced in section 4 and keep the same notations. We split $I = I^+ + I^-$, and for $|n| \geq 1$, $I^{\pm} = I^{\pm}_{1-\chi_0,1} + I^{\pm}_{\chi_0,1}$, with $I^{\pm}_{\chi_1,\chi_2}$ introduced in (4.5). Taking $\varepsilon = h_0$, we notice that there is no $n \neq 0$ satisfying $\varepsilon\tau\sqrt{1-\gamma^2}/n \geq 1$ and we are left with $I^{\pm}_{\chi_0,1} = I^{\pm}_{\chi_0,\chi_0} + I^{\pm}_{\chi_0,1-\chi_0}$ supported for $|n| \geq \varepsilon\tau\sqrt{1-\gamma^2}/2$. We aim at proving that $|I^{\pm}_{\chi_0,1}| \lesssim C(h_0)/t$.

- On the support of $I^{\pm}_{\chi_0,\chi_0}$ we have moreover $|n| \leq 2\tau\sqrt{1-\gamma^2}/\varepsilon < 2/\varepsilon = 2/h_0$; as $\sqrt{1-\gamma^2} \sim 2^{-j}$ and $\tau \leq 1/h_0$, only a finite number of j such that $2^j \leq 1/(h_0\varepsilon) = 1/h_0^2$ may contribute (as otherwise $\tau 2^{-j}/\varepsilon < 1$). For each j, n on this finite set, the symbols of $I^{\pm}_{\chi_0,\chi_0}$ are bounded and their phase may oscillate only for large t or z. If t is bounded then if r or z are larger than $\max\{4|t|, M\}$ for some $M > 1$ large enough, integrations by parts allow to conclude (using that the sum is finite); if $|z|, r \leq 4t$ each integral is bounded and we conclude that $|I^{\pm}_{\chi_0,\chi_0}| \lesssim C(h_0)$. Let now t be sufficiently large, then if $|t|/(|z| + 2^j r) \notin [1/8, 8]$, integrations by parts yields a contribution $O(1/t^N)$ for each pair (j, n) on the support of $I^{\pm}_{\chi_0,\chi_0}$) and we conclude. If $|t|/(|z| + 2^j r) \in [1/8, 8]$, we separate the cases $2^{-2j}|z| \geq M$ for some large M, when we apply the stationary phase in $\varphi = 2^j\sqrt{1-\gamma^2}$ and we conclude as in (4.18) or $2^{-2j}|z| \leq M$, when we bound directly as in (4.15).
- On the support of $I^{\pm}_{\chi_0,1-\chi_0}$ we have $|n| \geq \tau\sqrt{1-\gamma^2}/\varepsilon$, hence the sum over n is unbounded but as $n \gg \tau\sqrt{1-\gamma^2}$ we may use (6.6) and conclude.

6. Appendix

6.1. **Airy functions.** Define for $w \in \mathbb{C}$, $A(w) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}(s^3/3+sw)}ds$ and $A_\pm(w) := A(e^{\mp 2\mathrm{i}\pi/3}w)$. We have $A_-(w) = \overline{A}_+(\overline{w})$ and $A(w) = e^{\mathrm{i}\pi/3}A_+(w) + e^{-\mathrm{i}\pi/3}A_-(w)$. Moreover, $A_\pm(w), A'_\pm(w)$ are not zero for any $w \in \mathbb{R}$, while all the zeros of $A(w)$ and $A'(w)$ are real and negative.

Lemma 6.1. *Let $\Sigma(w) := (A_+(w)A_-(w))^{1/2}$, then $\Sigma(z) = |A_+(w)| = |A_-(w)|$ is real, monotonic increasing in w and nowhere vanishing. We let $\mu(w) := \frac{1}{2\mathrm{i}}\log(\frac{A_-(w)}{A_+(w)})$ on the set $\mathcal{S} := \{w \in \mathbb{C} : \mathrm{Re}\,(w) \leq \frac{1}{2}\mathrm{Re}\,(e^{2\pi\mathrm{i}/3}\omega_1)\}$, where ω_1 is the first zero of $A(w)$. Then $A_\pm(w) = \Sigma(w)e^{\mp\mathrm{i}\mu(w)}$.*
- *For $w \in (-\infty, -1] \subset \mathcal{S} \cap \mathbb{R}$, the following asymptotic expansions hold*

$$\Sigma(w) \sim (-w)^{-\frac{1}{4}}\sum_{j\geq 0}\sigma_j(-w)^{-\frac{3j}{2}},\ \mu(w) \sim \frac{2}{3}(-w)^{\frac{3}{2}}\sum_{j\geq 0}e_j(-w)^{-\frac{3j}{2}},\ \sigma_0 = \frac{1}{2\sqrt{\pi}},\ e_0 = 1. \tag{6.1}$$

- *For $w \in \mathcal{S}$, the Airy quotient $\Phi_+(w) = \frac{A'_+(w)}{A_+(w)} = \frac{\Sigma'(w)}{\Sigma(w)} - \mathrm{i}\mu'(w)$ satisfies everywhere $\Phi'_+(w) = w - \Phi_+^2(w)$. In particular $\Phi'_+(w)$ is bounded for any $w \in \mathcal{S} \cap \mathbb{R}$ and*

$$\Phi_+ \sim (-w)^{\frac{1}{2}}\sum_{j\geq 0}d_j(-w)^{-\frac{3j}{2}},\ d_0 = 1, \forall\, w \leq -1. \tag{6.2}$$

- *For $w \geq 1$, the solutions $A_\pm(w)$ grow exponentially $A_\pm(w) = \Sigma_\pm(w)e^{\frac{2}{3}w^{3/2}}$, where $\Sigma_\pm$ are classical symbols of order $-1/4$ and*

$$\frac{A_-(w)}{A_+(w)} + e^{2\mathrm{i}\pi/3} = O(w^{-\infty}) \text{ when } w \to \infty, \quad \frac{A_-(w)}{A_+(w)} \sim e^{2\mathrm{i}\mu(w)} \text{ when } w \to -\infty.$$

Moreover the Airy function $A(w)$ decays exponentially for $w \geq 1$, $A(w) \sim |w|^{-\frac{1}{4}}e^{-\frac{2}{3}w^{3/2}}$.

6.2. **Bessel and Hankel functions.** The Hankel function $H_n(z)$ is a solution to the Bessel's equation $z^2H''(w) + wH'(w) + (w^2 - n^2) = 0$. As we consider cylindrical coordinates we deal with $n \in \mathbb{Z}$. The couple $\{H_n(w), \overline{H}_n(w)\}$ is a fundamental system of solutions for the Bessel equations. The real and imaginary part of $H_n(w)$, denoted $J_n(w)$ and $Y_n(w)$ respectively, are the usual Bessel function of the first and second type. We list below some properties of the Bessel and Hankel functions that are used in the paper. When $|w| \gg |n|$, we have ([1, 9.2])

$$H_n(w) = \sqrt{\frac{2}{\pi w}}e^{\mathrm{i}(z-\frac{\pi n}{2}-\frac{\pi}{4})}\sigma(w), \tag{6.3}$$

where $\sigma(w) = 1 + O(\frac{1}{w})$. For large order n and fixed argument $w = n\rho$, the Hankel functions have the following *uniform* asymptotic expansion (see [1, (9.3.35), (9.3.37)])

$$H_n(n\rho) = 2e^{-\frac{\mathrm{i}\pi}{3}}\Big(\frac{4\tilde{\zeta}(\rho)}{1-\rho^2}\Big)^{\frac{1}{4}}\Big(n^{-\frac{1}{3}}A_+(n^{\frac{2}{3}}\tilde{\zeta}(\rho))\sum_{j\geq 0}a_jn^{-2j} + n^{-\frac{5}{3}}A'_+(n^{\frac{2}{3}}\tilde{\zeta}(\rho))\sum_{j\geq 0}b_jn^{-2j}\Big), \tag{6.4}$$

and $\tilde{\zeta}(\rho)$ given in Lemma 2.5. The Bessel function satisfies a similar formula

$$J_n(n\rho) = 2e^{-\frac{i\pi}{3}}\Big(\tfrac{4\tilde{\zeta}(\rho)}{1-\rho^2}\Big)^{\frac{1}{4}}\Big(n^{-\frac{1}{3}}Ai(n^{\frac{2}{3}}\tilde{\zeta}(\rho))\sum_{j\geq 0} a_j n^{-2j} + n^{-\frac{5}{3}}Ai'(n^{\frac{2}{3}}\tilde{\zeta}(\rho))\sum_{j\geq 0} b_j n^{-2j}\Big). \tag{6.5}$$

When the order is much larger than the argument $|n| \gg |w|$, we have (see [1, (9.3.1)])

$$J_n(w) = \sqrt{\frac{1}{2\pi n}}\Big(\frac{ew}{2n}\Big)^n\Big(1 + O(\frac{|w|}{|n|})\Big), \quad Y_n(w) \sim -\sqrt{\frac{1}{2\pi n}}\Big(\frac{ew}{2n}\Big)^{-n}\Big(1 + O(\frac{|w|}{|n|})\Big), \quad n \geq 1. \tag{6.6}$$

When $n = 0$ we have $J_0(w) = \frac{2}{\pi}\int_1^\infty \frac{\sin(w\beta)}{\sqrt{\beta^2-1}}d\beta$, $H_0(w) = -\frac{2i}{\pi}\int_1^\infty \frac{e^{iw\beta}}{\sqrt{\beta^2-1}}d\beta$. Taking $\beta = 1/\cos\omega$ gives $H_0(w) = \frac{2i}{\pi}\int_{\cos\omega=0}^{\cos\omega=1}\frac{e^{\frac{iw}{\cos\omega}}}{\cos\omega}d\omega$ which has an unique critical point $\omega = 0$, which implies that for large w we have $H_0(w) = \sqrt{\frac{2}{\pi w}}e^{i(w-\frac{\pi}{4})}(1 + O(\frac{1}{w}))$.

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